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Aims and target group of the conference:

The conference **MITAV 2024** is the 11th annual MITAV conference. It should attract in particular teachers of all types of schools and is devoted to the most recent discoveries in mathematics, informatics, and other sciences as well as to the teaching of these branches at all kinds of schools for any age groups, including e-learning and other applications of information technologies in education. The organizers wish to pay attention especially to the education in the areas that are indispensable and highly demanded in contemporary society. The goal of the conference is to create space for the presentation of results achieved in various branches of science and at the same time provide the possibility for meeting and mutual discussions of teachers from different kinds of schools and orientation. We also welcome presentations by (diploma and doctoral) students and teachers who are just beginning their careers, as their novel views and approaches are often interesting and stimulating for other participants.

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Now, in autumn 2024, this **post-conference proceedings** were published, containing extended versions of selected MITAV 2024 contributions. The proceedings are published in English and contain extended versions of 15 selected conference papers. Published articles have been chosen from 21 conference papers and every article was once more reviewed.

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SIMPSON'S RULE IN SCHOOL MATHEMATICS

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Abstract: *The article is aimed at the theory of teaching of Mathematics at secondary schools and universities. It contains possible use of Simpson's rule for counting volumes of bodies of rotation. The article shows the derivation of the formula and its usage in specifying expressions for volumes of some bodies of rotation.*

Keywords: Volume of bodies, Simpson's rule, definite integral, Cavalieri's principle.

INTRODUCTION

In this article we will deal with deducing Simpson's rule for the calculation of volumes of bodies of rotation and its use for defining relations for volumes of simple bodies of rotation in school Mathematics. This method is closely related to integral calculus and its introduction into secondary and tertiary school Mathematics teaching is desirable mainly due to the fact that it enables deducing some relations of bodies of rotation volumes which are often presented without proofs. Using Simpson's rule we can calculate the volume of a body of rotation without the use of a definite integral either exactly or with an error for which we know the expression.

1 DEDUCTION

Now let us devote to the formulation of Simpson's rule and its deduction. This rule is usually incorporated into the program of numerical methods and it is as follows:

„ If the solid of revolution has two parallel bases and every section parallel with them in the height of x has an area which is the value of a whole rational function in variable x of the 3rd degree at the utmost, then for the volume of such solid there applies:

$$V = \frac{v}{6}(S_1 + 4S + S_2), \tag{1}$$

where $\boldsymbol{v}$ *is the height of the solid,* $\boldsymbol{S_1}$, $\boldsymbol{S_2}$ *are areas of bases and* $\boldsymbol{S}$ *is the volume of the middle section."*

Let us show the deduction of the relation (1), which is accessible to secondary school students. Let us start from the general formula for the volume of a body of rotation

$V = \int_a^b S(x)dx$, where the bases lie in planes $x = a$, $x = b$, $S(x)$ is the area of the plane section parallel with the bases in the height x $(a < x < b)$. Let us solve this integral numerically.

Let us assume that $S(x)$ is a function continuous on the interval $\langle a, b\rangle$. The general solution is given by Newton-Cotes formulas of degree k. The general theory is the part of numerical methods, and because of the extent of the article, it is not given here. Let us only note that

number k is significant for dividing the interval $\langle a, b\rangle$, on the base of which the interpolation polynomials are the dividing points. For $k = 1$ this method is called trapezoidal rule, for $k = 2$ it is Simpson‘s rule. For further simplification, let us restrict ourselves to closed Simpson‘s rule, while dividing the interval $\langle a, b\rangle$ into two disjunctive half-intervals of the same lengths. Thus we obtain points $[a, S(a)]$, $[\frac{a+b}{2}, S(\frac{a+b}{2})]$, $[b, S(b)]$. Let us enter these points into the second degree Lagrange interpolating polynomial and after its modification we will get:

$$V = \int_a^b S(x)dx = \frac{b-a}{6}\left[S(a) + 4S(\frac{a+b}{2}) + S(b)\right]. \qquad (2)$$

When denoting $S(a) = S_1$, $S(\frac{a+b}{2}) = S$, $S(b) = S_2$, $b - a = v$, the relation (2) will change into relation (1). The calculation and modifications are not too complicated but they are rather lengthy. Their presentation to students can be a motivating factor for their further study of numerical methods while using IT, especially for the general theory of numerical solving of definite integrals. The essential problem is the estimate of the error which happens while calculating definite integral using relation (1) (we used an interpolation while deriving it). It can be proved that the error does not exceed the value $\frac{(b-a)^5}{2880} M_4$, where M_4 equals the maximum level of the fourth derivatives of function $S(x)$ on the interval $\langle a, b\rangle$. Concurrently, let us assume that the fourth derivative of function $S(x)$ is bounded on $\langle a, b\rangle$.

From the stated above, it is evident that if $S(x)$ is a polynomial function of a maximum degree of three, its fourth derivative is equal zero everywhere on the given interval and the formula (1) gives the value of the volume exactly. Let us note that the formula (1) can be used also in the case when $S(x)$ is not a polynomial function of a maximum degree of three (let us assume merely its continuity). However, we have to take into account a certain error. In conclusion, let us note that the detailed deductions of both Simpson´s rule and general Newton-Cotes formulas of degree k are available in books about numerical mathematics, e.g. [3], [4], [5], [6], [7], [8], including the deduction of the upper estimate of the possible error. Relations for the calculation of geometrical bodies can be found in secondary school textbooks, as the whole in [1].

2 EXAMPLES

Now we will show several examples how to use relation (1) in school mathematics. We will derive several formulas for the volume of bodies of rotation which are discussed at secondary schools. Let us mention the notation $b - a = v$; from methodological reasons we will assume that both numbers a, b are positive.

Example 1: The volume of a right-circular cylinder with the radius of the base r and the height v. There holds $S(x) = \pi r^2$. After substitution to (1) we will get: $V = \frac{v}{6}(\pi r^2 + 4\pi r^2 + \pi r^2) = \pi r^2 v$. It is evident that we do not calculate the volume of the cylinder in this way. However, the aim of this example was to show that the formula (1) holds also for simple regular bodies.

Example 2: The volume of a right-circular cone with the radius of the base r and the height v. There holds $S(a) = 0$, $S(b) = \pi r^2$. We can easily deduce that the radius of the middle section is

0,5 r, that means $S\left(\frac{a+b}{2}\right) = 0,25\,\pi r^2$. There holds $b - a = v$. The function expressing the area of the section parallel to the base in the height $x\ (a<x<b)$ is defined by the formula $S(x) = \pi\frac{r^2}{(b-a)^2}(x-a)^2$. This function is quadratic, so according to (1) there exists an exact representation

$$V = \frac{v}{6}(0 + 4.\ 0{,}25\pi r^2 + \pi r^2) = \frac{1}{3}\pi r^2 v.$$

Example 3: The volume of a solid sphere with radius *r*. We will place the centre of the sphere in a point [*r, 0*] of Cartesian coordinate system, so the equation of the main semi-circle in the 1st and 2nd quadrant is $(x-r)^2 + y^2 = r^2$. Let us denote $a = 0$, $b = 2r$, then the required areas are $S(a) = S(b) = 0$, $S(r) = \pi r^2$. For the function $S(x)$ there generally applies $S(x) = -\pi x^2 + 2\pi x r$, which is the 2nd degree function. According to (1) there exactly holds

$$V = \frac{b-a}{6}(0 + 4\,\pi r^2 + 0) = \frac{4}{3}\pi r^3$$ (there evidently holds $b - a = 2r$).

Example 4: The volume of a truncated cone with height *v* and the radii of bases *R, r* ($R > r$). Let $S(a) = \pi R^2$, $S(b) = \pi r^2$, $S\left(\frac{a+b}{2}\right) = \pi\left(\frac{R+r}{2}\right)^2$. Let us denote $b - a = v$. The radius of the parallel section in the point *x* ($a \le x \le b$) equals $r_x = -\frac{R-r}{v}x + R$, $S(x) = \pi r_x^2$, so again we get a quadratic function. According to (1) there holds

$$V = \frac{v}{6}\left[\pi R^2 + 4\pi\left(\frac{R+r}{2}\right)^2 + \pi r^2\right] = \frac{1}{3}\pi v\,(R^2 + Rr + r^2).$$

Example 5: The volume of a spherical layer with bases radii *R, r* ($R > r$). According to the spherical layer definition in secondary school mathematics let us consider the spherical layer as a subset of one of the hemispheres composing the sphere with radius ρ, which means $r < R < \rho$. Now we need to calculate the radius of the middle section and the area of the middle section of the given spherical layer. We know that $S(a) = \pi R^2$, $S(b) = \pi r^2$. After a rather lengthy modification, although not very difficult one, we will get $S\left(\frac{a+b}{2}\right) = \frac{\pi}{4}(2R^2 + 2r^2 + v^2)$, where $v = b - a$. After substituting to (1) we will get an exact representation $V = \frac{\pi v}{6}(3R^2 + 3r^2 + v^2)$.

Note: In previous examples there was shown how to apply the formula (1) when we get exact results. Two following examples will show how to use (1) even if the calculated value is burdened with an error.

Example 6: The volume of a solid of revolution which is created by the rotation of a parabola $y = x^2$ around the axis *x* for $x \in \langle 0, 1\rangle$. It is evident that $S(0) = 0$, $S(1) = \pi$, $S(0{,}5) = \frac{\pi}{16}$. The height of this solid of revolution equals one. Then according to (1) there holds

$V = \frac{1}{6}(0 + \frac{\pi}{4} + \pi) = \frac{5}{24}\pi$.

However, $S(x) = \pi x^4$ (4^{th} grade function), so the calculated value is burdened by an error. We will estimate the value of the error. There holds $S^{(4)}(x) = 24\pi$, therefore also $M_4 = 24\pi$. The maximum value of the error is $\frac{1}{2880} 24\pi = \frac{1}{120}\pi$. Now let us calculate the volume with the help of an integral: $V = \pi \int_0^1 (x^2)^2 \, dx = \frac{\pi}{5}$. The volume calculated with the help of (1) is greater exactly by $\frac{1}{120}\pi$ than the one determined using the integral, which approximately equals *0,02618*. Evidently, the use of (1) in this case is debatable in relation to the error value.

Example 7: The volume of a solid of revolution which is created by the rotation of the part of an exponential curve $y = e^x$ around the axis x for $x \in \langle 0, 1 \rangle$. There applies $S(0) = \pi$, $S(0,5) = \pi e$, $S(1) = \pi e^2$. After substituting to (1) we will get

$$V = \frac{1}{6}(\pi + 4\pi e + \pi e^2) = 10,079 \ldots$$

There holds $S(x) = \pi e^{2x}$, the 4^{th} derivation of function $S(x)$ equals $16\pi e^{2x}$, and its maximum value for $x \in \langle 0, 1 \rangle$ is $M_4 = 16\pi e^2$. The error does not exceed the value $\frac{16\pi e^2}{2880}$, which approximately equals 0,13. For the sake of the comparison let us calculate the volume with the help of the integral: $V = \pi \int_0^1 e^{2x} \, dx = 0,5\, \pi (e^2 - 1)$, which approximately equals *10, 028*. Again, it depends on circumstances if the value determined using Simpson´s rule is for us accurate enough.

3 CAVALIERI´S PRINCIPLE

Now let us introduce another possibility how to derive the volume of solid bodies – Cavalieri´s principle which was discovered by an Italian mathematician Bonaventura Cavalieri (1598–1647). Although the formulation of this principle is quite simple, a number of students are not acquainted with it.

„Two solid bodies with the same base areas and the same heights have the same volumes, if their cross-sections parallel with bases at the same distance have the same volume.

From the theoretical point of view this principle is the corollary of Fubini´s theorem giving the conditions under which it is possible to compute a double integral by using an iterated integral. The details are to be found in mathematical analysis textbooks, e.g. [2].

Cavalieri´s principle can be used for calculating a sphere´s volume using elementary tools. Firstly, let us show that a hemisphere with radius R has the same volume as a cylinder of the base with radius R and height R from which an upturned cone has been cut out – in the way as is shown in Figure 1. Bases and heights at both solid bodies are equal and the same applies for the cross-section areas in any height v above the base. The hemisphere´s cross-section forms a circle, its radius is $r = \sqrt{R^2 - v^2}$, so its volume is $S_k = \pi r^2 = \pi (R^2 - v^2)$. The cut-off cylinder has a cross-section in the form of an annulus with the area of $S_v = \pi R^2 - \pi v^2 = \pi (R^2 - v^2)$, which equals the area of the hemisphere´s cross-section S_k. Therefore, the supposition of

Cavalieri´s principle holds here, and it means that both solid bodies have the same volume. The volume of the cut-off cylinder equals the difference between the cylinder´s and cone´s volumes: $V = \pi R^3 - \frac{1}{3}\pi R^3 = \frac{2}{3}\pi R^3$. The volume of the whole sphere is double, so $V = \frac{4}{3}\pi R^3$.

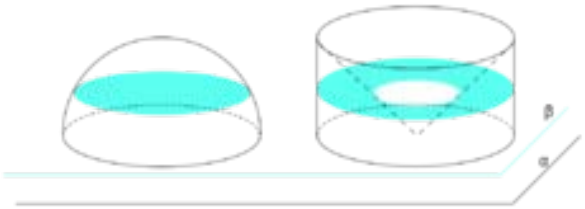

Fig.1: The middle cross-sections of both solid bodies.
Source: Cavalieriův princip – Wikipedie (wikipedia.org)

CONCLUSION

From the above given examples it is obvious that Simpson´s rule (1) represents a useful tool for calculating volumes of simple solid bodies of rotation and for deriving formulas for their volumes. Certainly, the relation (1) can be used in some other cases (e.g. if the "classical" calculation using an integral seems too complicated). Nevertheless, it is inevitable to verify the continuity of the function *S(x)* and determine the error which we have made. During lessons showing students how to derive Simpson´s rule (1) is suitable because students mostly learn the formula by heart, and they miss the connection with numerical solving of definite integrals. Therefore, it is desirable to introduce this rule not only to students but also to their future mathematics teachers. Cavalieri´s principle which is discussed at the end of this article adds to the suitability of the introduction of numerical methods into their teaching.

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NON-DELAYED LINEAR PLANAR DISCRETE SYSTEMS WITH CONSTANT COEFFICIENTS EQUIVALENT TO LINEAR PLANAR DISCRETE DELAYED SYSTEMS WITH CONSTANT COEFFICIENTS

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Abstract: *Planar linear discrete delayed systems*

$$w(k+1) = Aw(k) + Bw(k-m),\ k = 0,1,\ldots$$

are considered, where A and B are constant matrices and m is a positive integer denoting a delay. Recently, formulas have been derived for the general solution of such systems provided that one eigenvalue of A is nonzero while the other is zero and that matrices A and B satisfy the conditions known for so-called weakly delayed systems. The conditions characterizing weakly delayed systems imply that there are only two roots of the relevant quasi-characteristic equation. The present paper is concerned with the problem of a possible replacement of the delayed systems in question with non-delayed planar ones

$$x(k+1) = Cx(k),\ k = 0,1,\ldots$$

having an identical general solution. It is shown that this is possible if a transient interval is passed and possible forms of the matrix C are found. Some relationships with previously known results are commented. Results are illustrated by examples

Keywords: planar delayed system, weak delay, initial problem, equivalent discrete system, general solution.

INTRODUCTION

Consider a planar system of delayed discrete equations

$$w(k+1) = Aw(k) + Bw(k-m),\ k = 0,1,\ldots \tag{1}$$

where $A = \{a_{ij}\}_{i,j=1}^{2}$ and $B = \{b_{ij}\}_{i,j=1}^{2}$ are 2×2 constant matrices, m is a positive fixed integer called delay, and $w = (w_1, w_2)^T$ (the superscript "T" denoting the transpose) is an unknown vector function. Throughout the paper we assume that one of the eigenvalues of A is nonzero while the other is zero. Assume also that the matrix B is a non-zero matrix. Such systems

have recently been considered in [9], where, under the assumption of the entries of matrices A and B satisfy conditions characterizing so-called weakly delayed systems, the general solutions are found. As the quasi-characteristic equations of such (two-dimensional) systems have only two roots, a principle was applied in [2, 3] based on the possibility of weakly delayed systems being replaced by systems with no delay having identical general solutions. This led to series of results formulated in Theorem 6.1 - Theorem 6.7 in [2] and in Theorem 13 - Theorem 16 in [3]. In [2, 3] it is assumed that the matrix of the non-delayed terms in the system considered has nonzero eigenvalues. For reasons explained below it is not possible to extend these results to systems whose matrix of non-delayed terms has one or two zero eigenvalues. The purpose of the present paper is to consider a problem of the equivalence of non-delayed and delayed systems in the case of system (1) while constructing

$$x(k+1) = Cx(k),\ k = 0, 1, \ldots, \tag{2}$$

where C is a 2×2 constant matrix such that their general solutions coincide with the general solutions to system (1). Obviously, the matrices A and C should be different. It is explained that this equivalence cannot occur for all $k = 0, 1, \ldots$, as there is a transient interval, characterized by values $k = 1, \ldots, m+1$, where equivalent systems of the type (2) do not exist. For $k = m+2, \ldots$, the general solutions constructed are equivalent if the matrix C is of a special form, derived in the paper.

1 PRELIMINARIES

Throughout the paper we assume that matrices A and B satisfy conditions

$$\mathrm{tr}B = \det B = \det(A+B) = 0. \tag{3}$$

These conditions are, in general, related with conditions characterizing the so-called weakly delayed systems and were derived in [9] under the assumption that one eigenvalue of the matrix of linear terms is nonzero while the other is zero.

Weakly delayed systems were considered in [4]. Adapting Definition 1.1 of weakly delayed system given in [4] to a system

$$w(k+1) = \mathscr{A}w(k) + \mathscr{B}w(k-m),\ k = 0, 1, \ldots \tag{4}$$

with general constant 2×2 matrices $\mathscr{A}$ and $\mathscr{B}$, equation (4) is said to be weakly delayed if the characteristic polynomial of the matrix $\mathscr{A}$ coincides with the characteristic quasi-polynomial of the matrix $\mathscr{A} - \lambda^{-m}\mathscr{B}$ for every nonzero λ. That is, if

$$\det(\mathscr{A} - \lambda E) = \det(\mathscr{A} - \lambda E - \lambda^{-m}\mathscr{B})$$

for every (complex) λ except for the value $\lambda = 0$, where E is the unit 2×2 matrix. Thus, such a definition is not suited for the case of one of eigenvalues of $\mathscr{A}$ equaling zero or both eigenvalues being zeros. Nevertheless, as was shown in [9], the coefficient criteria, derived in this general case for matrices $\mathscr{A}$ and $\mathscr{B}$, we refer to [4, Theorem 1.3], can be "ad hoc" used even in this "singular" case, which is very helpful in the process of deriving the general solution. Then, for system (1), we derive conditions (3). As it is shown in [9], conditions (3) remain invariant under an arbitrary regular transformation

$$w(k) = Tu(k),\ k = 0, 1, \ldots, \tag{5}$$

where T is a regular 2×2 matrix and $u(k)=(u_1(k),u_2(k))^T$. This means that the matrices A^*, B^* of the system

$$u(k+1)=A^*u(k)+B^*u(k-m),\ k=0,1,\ldots, \tag{6}$$

where $A^*=T^{-1}AT$, $B^*=T^{-1}BT$ and T^{-1} is the inverse of T, satisfy the conditions

$$\mathrm{tr}B^*=\det B^*=\det(A^*+B^*)=0 \tag{7}$$

as well.

2 GENERAL SOLUTIONS TO SYSTEM (1)

To identify discrete non-delayed systems (2) with the general solutions identical to systems (1) we will introduce results, proved in [9]. Assume that a suitable transformation (5) transforms the system (1) into a system (6) with the matrix A^* being the Jordan form $J(A)$ of the matrix A, that is

$$A^*=J(A)=\begin{pmatrix}\lambda & 0\\ 0 & 0\end{pmatrix}$$

where λ is the nonzero eigenvalue of A. Below we consider a system

$$u(k+1)=J(A)u(k)+B^*u(k-m),\ k=0,1,\ldots, \tag{8}$$

where $J(A)=T^{-1}AT$ and $B^*=T^{-1}BT$. Along with system (1), consider, following initial problem for this system

$$w(k)=\psi(k)=(\psi_1(k),\psi_2(k))^T,\ k=-m,\ldots,0, \tag{9}$$

where ψ is a given vector function. Initial problem (9) for system (1) can be transformed by transformation (5) into an initial problem

$$u(k)=\psi^*(k)=(\psi_1^*(k),\psi_2^*(k))^T=T^{-1}\psi(k),\ k=-m,\ldots,0 \tag{10}$$

for the system (8). Each fixed initial problem (9) defines a unique solution to system (1) as a function $w=w(k)$ defined for each $k=-m,\ldots$, satisfying (1) for $k=0,\ldots$, and $w(k)=\psi(k)$ for $k=-m,\ldots,0$. For the matrices $A^*=J(A)$ and B^*, conditions (3) have the form

$$b_{11}^*+b_{22}^*=0,\ \begin{vmatrix}b_{11}^* & b_{12}^*\\ b_{21}^* & b_{22}^*\end{vmatrix}=0,\ \begin{vmatrix}\lambda+b_{11}^* & b_{12}^*\\ b_{21}^* & b_{22}^*\end{vmatrix}=0,$$

that is,

$$b_{11}^*=b_{22}^*=b_{12}^*b_{21}^*=0. \tag{11}$$

There are only two nontrivial possible cases implied by (11). These are covered by the following two theorems, taken from [9], providing, among others, formulas for the solution of system (8) as generated by the initial problem (9).

Theorem 1 *Let $b_{11}^*=b_{21}^*=b_{22}^*=0$, $b_{12}^*\neq 0$. Then, the solution of the initial problem* (1)*,* (9) *is given by the formula $w(k)=Tu(k)$, $k=1,\ldots$, where*

$$u_1(k)=\lambda^k\left(\psi_1^*(0)+b_{12}^*\sum_{r=0}^{k-1}\lambda^{-1-r}\psi_2^*(r-m)\right), \tag{12}$$

$$u_2(k) = 0 \tag{13}$$

if $k = 1, \ldots, m+1$ *and*

$$u_1(k) = \lambda^k \left(\psi_1^*(0) + b_{12}^* \sum_{r=0}^{m} \lambda^{-1-r} \psi_2^*(r-m) \right), \tag{14}$$

$$u_2(k) = 0 \tag{15}$$

if $k = m+2, \ldots$.

Theorem 2 *Let* $b_{11}^* = b_{12}^* = b_{22}^* = 0$, $b_{21}^* \neq 0$. *Then, the solution of the initial problem* (1), (9) *is given by the formula* $w(k) = Tu(k)$, $k = 0, 1, \ldots$, *where*

$$u_1(k) = \lambda^k \psi_1^*(0), \tag{16}$$

$$u_2(k) = b_{21}^* \psi_1^*(k-m-1) \tag{17}$$

if $k = 1, \ldots, m+1$ *and*

$$u_1(k) = \lambda^k \psi_1^*(0), \tag{18}$$

$$u_2(k) = b_{21}^* \lambda^{k-m-1} \psi_1^*(0) \tag{19}$$

if $k = m+2, \ldots$.

3 EQUIVALENT NON-DELAYED SYSTEMS

We will search for non-delayed discrete systems having the form of the system (2) with a suitable matrix C. The simplest way to do this seems to be using formulas in Theorem 1 and in Theorem 2. The principal idea is that the eigenvalues of the matrix C should coincide with the eigenvalues of the matrix A. Applying this, we should use general solutions described by formulas (12) - (15) in Theorem 1 and by formulas (16) - (19) in Theorem 2. However, the first groups of these formulas, i.e., formulas (12), (13) in Theorem 1 or formulas (16), (17) in Theorem 2 cannot be used as shown below. Tracing them carefully, we conclude the following. Formula (12) can be rewritten as

$$u_1(k) = R_1(k) \lambda^k, \tag{20}$$

where

$$R_1(k) := \psi_1^*(0) + b_{12}^* \sum_{r=0}^{k-1} \lambda^{-1-r} \psi_2^*(r-m)$$

is a function defined for $k = 1, \ldots, m+1$. Similarly, formula (17) can be rewritten as

$$u_2(k) = R_2(k), \tag{21}$$

where

$$R_2(k) := b_{21}^* \psi_1^*(k-m-1)$$

is a function, defined for $k = 1, \ldots, m+1$. As both $R_1(k)$ and $R_2(k)$ are functions rather than some fixed constants, it cannot be expected that the system (2) with a constant matrix C should have the same solutions as that given by the above formulas. This is possible only if the matrix C is variable. Therefore, for a "transient" interval $k = 1, \ldots, m+1$, there exists no system (2)

having the same solutions. Nevertheless, as it follows from (12), (14) and (17), (19), for $k = m+2,\ldots$, both $R_1(k)$ and $R_2(k)$ become constants and we have

$$R_1(k) = R_1(m+1) = \psi_1^*(0) + b_{12}^* \sum_{r=0}^{m} \lambda^{-1-r} \psi_2^*(r-m) = \text{const},\ k = m+2,\ldots,$$

and

$$R_2(k) = R_2(m+1) = b_{21}^* \psi_1^*(0) = \text{const},\ k = m+2,\ldots.$$

Formula (20) changes into formula (14), i.e.,

$$u_1(k) = R_1(m+1)\lambda^k$$

and formula (19) can be written using (21) as

$$u_1(k) = \left(\lambda^{-m-1} R_2(m+1)\right) \lambda^k.$$

These formulas together with the remaining formulas (15) and (18) in Theorem 1 and in Theorem 2, suggests that systems (2) with constant matrices C having the same solutions as the system (1) will exist. In view of the above reasoning, formulas (14), (15) in Theorem 1 and formulas (18), (19) in Theorem 2 also suggest that, for the eigenvalues μ_i, $i = 1,2$ of C, we would have $\mu_1 = \lambda$ and $\mu_2 = 0$.

3.1 EQUIVALENT NON-DELAYED SYSTEM FOR DELAYED SYSTEM (8)

First, we will construct an auxiliary system

$$z_1(k+1) = c_{11}^* z_1(k) + c_{12}^* z_2(k), \tag{22}$$
$$z_2(k+1) = c_{21}^* z_1(k) + c_{22}^* z_2(k), \tag{23}$$

i.e., a system

$$z(k+1) = C^* z(k),\ k = m+1,\ldots, \tag{24}$$

with a matrix $C^* = \{c_{ij}^*\}$, $i, j = 1,2$, having the same solutions as described by formulas (14), (15) and (18), (19).

Lemma 1 *System* (24) *with*

$$C^* = C_1^* := \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \tag{25}$$

where $k = m+1,\ldots$ has the general solution

$$z_1(k) = K\lambda^k, \tag{26}$$
$$z_2(k) = 0, \tag{27}$$

where $k = m+2,\ldots$ and K in (26) *is an arbitrary constant. This general solution is the same as that described by formulas* (14), (15)*, where $k = m+2,\ldots$.*

PROOF. Looking for a solution of system (24) in the form (26), (27), from its scalar form (22), (23), we get

$$K\lambda^{k+1} = c_{11}^* K\lambda^k + c_{12}^* \cdot 0,$$

$$0 = c_{21}^* K\lambda^k + c_{22}^* \cdot 0.$$

Then, obviously, $c_{11}^* = \lambda$ and $c_{21}^* = 0$, and the auxiliary system (22), (23) is

$$z_1(k+1) = \lambda z_1(k) + c_{12}^* z_2(k), \tag{28}$$
$$z_2(k+1) = c_{22}^* z_2(k). \tag{29}$$

Taking into account that the initial data for (22), (23), as it follows from formulas (12), (13), where $k = m+1$, i.e.,

$$u_1(m+1) = \lambda^{m+1}\left(\psi_1^*(0) + b_{12}^* \sum_{r=0}^{m} \lambda^{-1-r}\psi_2^*(r-m)\right) = \lambda^{m+1}R_1(m+1),$$
$$u_2(m+1) = 0$$

would coincide with the initial data for the system (28), (29), i.e.,

$$z_1(m+1) = \lambda^{m+1}R_1(m+1),$$
$$z_2(m+1) = 0,$$

we conclude that one can put $c_{12}^* = c_{22}^* = 0$. Finally, let us remark that the constant $R_1(m+1)$ can be considered arbitrary since it is a linear combination of arbitrary initial data $\psi_1^*(0), \psi_2^*(-m), \ldots, \psi_2^*(0)$ and, therefore, is equivalent to an arbitrary constant K. The case of $R_1(m+1) = 0$, as a trivial one, is omitted. □

Remark 1 *Analysing the system* (28)*,* (29)*, we conclude that the matrix C_1^* is not unique. Every matrix*

$$C_1^*(a,b) = \begin{pmatrix} \lambda & a \\ 0 & b \end{pmatrix},$$

where a and b are arbitrary fixed constants, has the same property.

Lemma 2 *System* (24) *with*

$$C^* = C_2^* := \begin{pmatrix} \lambda & 0 \\ b_{21}^*\lambda^{-m} & 0 \end{pmatrix}, \tag{30}$$

where $k = m+1, \ldots,$ has the general solution

$$z_1(k) = \psi_1^*(0)\lambda^k, \tag{31}$$
$$z_2(k) = b_{21}^*\psi_1^*(0)\lambda^{k-m-1}, \tag{32}$$

where $k = m+2, \ldots$. This general solution is the same as that described by formulas (18)*,* (19) *if $k = m+2, \ldots$.*

PROOF. Assume $\psi_1^*(0) \neq 0$. In the opposite case, the proof is trivial. Looking for a solution of system (24) in the form (31), (32), from its scalar form (22), (23), we get

$$\psi_1^*(0)\lambda^{k+1} = c_{11}^*\psi_1^*(0)\lambda^k + c_{12}^* b_{21}^*\psi_1^*(0)\lambda^{k-m-1}, \tag{33}$$
$$b_{21}^*\psi_1^*(0)\lambda^{k-m} = c_{21}^*\psi_1^*(0)\lambda^k + c_{22}^* b_{21}^*\psi_1^*(0)\lambda^{k-m-1}. \tag{34}$$

Simplifying (33), (34) we obtain

$$\lambda = c_{11}^* + c_{12}^* b_{21}^*\lambda^{-m-1}, \tag{35}$$

$$b_{21}^*\lambda^{-m} = c_{21}^* + c_{22}^* b_{21}^*\lambda^{-m-1}. \tag{36}$$

From (35) and (36), we have

$$c_{12}^* = (\lambda - c_{11}^*)\frac{1}{b_{21}^*}\lambda^{m+1}, \tag{37}$$

$$c_{21}^* = b_{21}^*\lambda^{-m-1}(\lambda - c_{22}^*). \tag{38}$$

Using (37) and (38), system (22), (23) can be written as

$$z_1(k+1) = c_{11}^* z_1(k) + (\lambda - c_{11}^*)\frac{1}{b_{21}^*}\lambda^{m+1} z_2^*(k), \tag{39}$$

$$z_2(k+1) = b_{21}^*\lambda^{-m-1}(\lambda - c_{22}^*)z_1(k) + c_{22}^* z_2(k). \tag{40}$$

Looking for eigenvalues μ_i, $i = 1,2$ of the matrix of linear terms of system (39), (40), we obtain an equation

$$\begin{vmatrix} c_{11}^* - \mu & (\lambda - c_{11}^*)\frac{1}{b_{21}^*}\lambda^{m+1} \\ b_{21}^*\lambda^{-m-1}(\lambda - c_{22}^*) & c_{22}^* - \mu \end{vmatrix} = \mu^2 - (c_{11}^* + c_{22}^*)\mu - \lambda^2 + (c_{11}^* + c_{22}^*)\lambda = 0$$

having the roots

$$\mu_1 = \lambda,\ \mu_2 = c_{11}^* + c_{22}^* - \lambda.$$

Therefore, the system (22), (23) has two linearly independent solutions $z^1(k)$, $z^2(k)$, where

$$z^1(k) = (z_{11}(k), z_{12}(k))^T,\ z_{11}(k) = \xi_{11}\lambda^k,\ z_{12}(k) = \xi_{12}\lambda^k,$$

$$z^2(k) = (z_{21}(k), z_{22}(k))^T,\ z_{21}(k) = \xi_{21}(c_{11}^* + c_{22}^* - \lambda)^k,\ z_{22}(k) = \xi_{22}(c_{11}^* + c_{22}^* - \lambda)^k$$

and ξ_{11}, ξ_{12} and ξ_{21}, ξ_{22} are the coordinates of the respective eigenvectors. Applying the first variant to (39), (40) we get

$$(c_{11}^* - \lambda)\xi_{11} - (c_{11}^* - \lambda)\frac{1}{b_{21}^*}\lambda^{m+1}\xi_{12} = 0,$$

$$b_{21}^*\lambda^{-m-1}(\lambda - c_{22}^*)\xi_{11} - (\lambda - c_{22}^*)\xi_{12} = 0$$

and one can set, e.g., $\xi_{11} = \psi_1^*(0)$, $\xi_{12} = b_{21}^*\psi_1^*(0)\lambda^{-m-1}$. This can be made independent of the situation where one sets $c_{11}^* = \lambda$ or $c_{22}^* = \lambda$. Because we are looking for systems (24) having only solutions coinciding with solutions of the system (8) and the latter system has no solutions other than that described by formulas (18), (19), we conclude that $z^2(k)$ must be the trivial solution, i.e., $c_{11}^* + c_{22}^* = \lambda$. Then, system (39), (40) has the form

$$z_1(k+1) = c_{11}^* z_1(k) + (\lambda - c_{11}^*)\frac{1}{b_{21}^*}\lambda^{m+1} z_2^*(k), \tag{41}$$

$$z_2(k+1) = b_{21}^*\lambda^{-m-1}c_{11}^* z_1(k) + (\lambda - c_{11}^*)z_2(k). \tag{42}$$

The choice $c_{11}^* = \lambda$ ends the proof. □

Remark 2 *Analysing the system* (41)*,* (42)*, we conclude that the matrix C_2^* is not unique. Every matrix*

$$C_2^*(a) = \begin{pmatrix} a & (\lambda - a)\frac{1}{b_{21}^*}\lambda^{m+1} \\ b_{21}^*\lambda^{-m-1}a & \lambda - a \end{pmatrix},$$

where a is a fixed arbitrary constant, has the same property.

3.2 EQUIVALENT NON-DELAYED SYSTEM FOR DELAYED SYSTEM (1)

Based on Lemmas 1 and 2, the following two theorems can be proved. Their proofs being simple we are omitting them.

Theorem 3 *Let $b_{11}^* = b_{21}^* = b_{22}^* = 0$, $b_{12}^* \neq 0$. Then, the delayed system* (1) *has the same general solution as the non-delayed system* (2)*, if*

$$C := TC_1^*T^{-1}$$

and $k = m+1, \ldots$. This solution is given by the formula

$$w(k) = Tz(k) = \lambda^k T(K \;\; 0)^T, \;\; k = m+2, \ldots,$$

where K is an arbitrary constant and $K = R_1(m+1)$.

Theorem 4 *Let $b_{11}^* = b_{12}^* = b_{22}^* = 0$, $b_{21}^* \neq 0$. Then, the delayed system* (1) *has the same general solution as the non-delayed system* (2) *if*

$$C := TC_2^*T^{-1}$$

and $k = m+1, \ldots$. This solution is given by the formula

$$w(k) = Tz(k) = \psi_1^*(0)\lambda^k T(1 \;\; \lambda^{-m-1})^T, \;\; k = m+2, \ldots.$$

4 EXAMPLES

Below we consider two examples to be treated by Theorem 1 and by Theorem 2.

Example 1 Assume that the system (1) is specified as follows. Let $m = 2$,

$$A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}, \; B = \frac{1}{3}\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix},$$

and let the initial data (9) be

$$\psi(-2) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \;\; \psi(-1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \;\; \psi(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}. \tag{43}$$

The eigenvalues of A are $\lambda = 3$ and $\lambda = 0$. System (1) can be written as

$$w_1(k+1) = 4w_1(k) - 2w_2(k) - \frac{2}{3}w_1(k-2) + \frac{4}{3}w_2(k-2), \tag{44}$$

$$w_2(k+1) = 2w_1(k) - \;\; w_2(k) - \frac{1}{3}w_1(k-2) + \frac{2}{3}w_2(k-2), \tag{45}$$

where $k = 0, 1, \ldots$. Conditions (3) hold, we have

$$\mathrm{tr}B = b_{11} + b_{22} = -2 + 2 = 0, \quad \det B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix} = 0$$

and

$$\det(A+B)=\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}=\begin{vmatrix} 4-2/3 & -2+4/3 \\ 2-1/3 & -1+2/3 \end{vmatrix}=0.$$

To get the Jordan form $A^*=J(A)$ of the matrix A, let us transform the system (44), (45) by transformation (5), where

$$T=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ T^{-1}=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then

$$J(A)=T^{-1}AT=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}=\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$B^*=T^{-1}BT=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The system (44), (45) will then have the form of a system (8), i.e.

$$u(k+1)=J(A)u(k)+B^*u(k-2),\ k=0,1,,\dots$$

and

$$u_1(k+1)=3u_1(k)+u_2(k-2), \tag{46}$$
$$u_2(k+1)=0, \tag{47}$$

where $k=0,1,\dots$. The initial data for system (46), (47), transformed from (43) by transformation (5), (10) are as shown below

$$u(k):=\psi^*(k)=T^{-1}\psi(k)=\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix},\ k=-2,-1,0$$

and

$$\psi^*(-2)=\begin{pmatrix} 0 \\ 2 \end{pmatrix},\ \psi^*(-1)=\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \psi^*(0)=\begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{48}$$

Theorem 1 can be applied and the solution of the system (44), (45), defined by initial data (43) can be expressed by the transformation

$$w(k)=Tu(k)=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix},\ k=1,2,\dots, \tag{49}$$

where, for $k=1,2,3$, (we refer to formulas (12), (13)),

$$u_1(1)=\lambda\left(\psi_1^*(0)+b_{12}^*\lambda^{-1}\psi_2^*(-m)\right)=3\cdot 3^{-1}2=2,\ u_2(1)=0,$$

$$u_1(2)=\lambda^2\left(\psi_1^*(0)+b_{12}^*(\lambda^{-1}\psi_2^*(-m)+\lambda^{-2}\psi_2^*(1-m)\right)=3^2(3^{-1}2+3^{-2})=7,\ u_2(2)=0,$$

$$u_1(3)=\lambda^3\left(\psi_1^*(0)+b_{12}^*(\lambda^{-1}\psi_2^*(-m)+\lambda^{-2}\psi_2^*(1-m)+\lambda^{-3}\psi_2^*(2-m)\right)$$
$$=3^3(3^{-1}2+3^{-2}+3^{-3}(-1))=20,\ u_2(2)=0,$$

and (we refer to formulas (14), (15))

$$u_1(k) = \lambda^k \left(\psi_1^*(0) + b_{12}^* \sum_{r=0}^{m} \lambda^{-1-r} \psi_2^*(r-m) \right) = 3^k (3^{-1}2 + 3^{-2} + 3^{-3}(-1)) = \frac{20}{27} \cdot 3^k,$$
$$u_2(k) = 0$$

if $k = 4, \ldots$. By Theorem 3, the delayed system (44), (45) has the same general solution as the non-delayed system (2) if

$$C := TC_1^* T^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A,$$

where C_1^* is defined by (25) and $k = 3, \ldots$. This solution is given by the formula (for z we refer to (26), (27))

$$w(k) = Tz(k) = \lambda^k T (K \;\; 0)^T = \lambda^k T (R_1(m+1) \;\; 0)^T, \; k = m+2, \ldots,$$

where $K = R_1(m+1)$ is an arbitrary constant or, in the case of initial data (48), by formulas (31), (32), K is specified as

$$K = R_1(m+1) = \psi_1^*(0) + b_{12}^* \sum_{r=0}^{m} \lambda^{-1-r} \psi_2^*(r-m) = \sum_{r=0}^{m} \lambda^{-1-r} \psi_2^*(r-m)$$
$$= \sum_{r=0}^{2} 3^{-1-r} \psi_2^*(r-2) = 3^{-1} \cdot 2 + 3^{-2} + 3^{-3}(-1) = \frac{20}{27}.$$

and

$$w(k) = Tz(k) = T \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda^k R_1(m+1) \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 20 \cdot 3^k/27 \\ 0 \end{pmatrix}. \qquad (50)$$

Formulas (49) and (50) give the same values for $k = 4, \ldots$.

Example 2 Assume that the system (1) is specified as follows. Let $m = 2$,

$$A = \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \; B = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix},$$

and let the initial data (9) be

$$\psi(-2) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \; \psi(-1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; \psi(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}. \qquad (51)$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = 0$. System (1) can be written as

$$w_1(k+1) = \frac{4}{5} w_1(k) + \frac{2}{5} w_2(k) + \frac{2}{5} w_1(k-2) + \frac{1}{5} w_2(k-2), \qquad (52)$$
$$w_2(k+1) = \frac{2}{5} w_1(k) + \frac{1}{5} w_2(k) - \frac{4}{5} w_1(k-2) - \frac{2}{5} w_2(k-2), \qquad (53)$$

where $k = 0, 1, \ldots$. Conditions (3) hold, we have

$$\operatorname{tr} B = b_{11} + b_{22} = \frac{2}{5} - \frac{2}{5} = 0, \quad \det B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \frac{1}{5} \begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} = 0$$

and

$$\det(A+B)=\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}=\frac{1}{5}\begin{vmatrix} 4+2 & 2+1 \\ 2-4 & 1-2 \end{vmatrix}=0.$$

To get the Jordan form $A^*=J(A)$ of the matrix A, let us transform the system (44), (45) by transformation (5), where

$$T=\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},\ T^{-1}=\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Then

$$J(A)=T^{-1}AT=\frac{1}{25}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$B^*=T^{-1}BT=\frac{1}{25}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The system (52), (53) will have the form of a system (8), i.e.

$$u(k+1)=J(A)u(k)+B^*u(k-2),\ k=0,1,,\dots$$

and

$$u_1(k+1)=u_1(k), \tag{54}$$
$$u_2(k+1)=u_1(k-2), \tag{55}$$

where $k=0,1,\dots$. The initial data for system (54), (55), transformed from (51) by transformation (5), (10) are as shown below

$$u(k):=\psi^*(k)=T^{-1}\psi(k)=\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \psi_1(k) \\ \psi_2(k) \end{pmatrix},\ k=-2,-1,0$$

and

$$\psi^*(-2)=\begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \psi^*(-1)=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \psi^*(0)=\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Theorem 2 can be applied and the solution of the system (52), (53), defined by initial data (43) can be expressed by the transformation

$$w(k)=Tu(k)=\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix},\ k=1,2,\dots, \tag{56}$$

where, for values $k=1,2,3$, (we refer to formulas (16), (17)),

$$u_1(1)=\lambda\psi_1^*(0)=-1,\ u_2(1)=b_{21}^*\psi_1^*(1-2-1)=2,$$

$$u_1(2)=\lambda^2\psi_1^*(0)=-1,\ u_2(2)=b_{21}^*\psi_1^*(2-2-1)=1,$$

$$u_1(3)=\lambda^3\psi_1^*(0)=-1,\ u_2(3)=b_{21}^*\psi_1^*(3-2-1)=-1$$

and (we refer to formulas (18), (19))

$$u_1(k)=\lambda^k\psi_1^*(0)=-1,$$

$$u_2(k) = b_{21}^* \lambda^{k-m-1} \psi_1^*(0) = -1$$

if $k = 4, \dots$. By Theorem 4, the delayed system (52), (53) has the same general solution as the non-delayed system (2) if

$$C := TC_2^* T^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix},$$

where C_2^* is defined by (30) and $k = 3, \dots$. This solution is given by the formula (for z we refer to (31), (32)

$$w(k) = T \begin{pmatrix} z_1(k) \\ z_2(k) \end{pmatrix} = T \begin{pmatrix} \psi_1^*(0)\lambda^k \\ b_{21}^* \psi_1^*(0) \lambda^{k-m-1} \end{pmatrix} = T \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \ k = 4, \dots, \tag{57}$$

Formulas (56) and (57) give the same values for $k = 4, \dots$.

CONCLUSION

The paper considers a singular case of weakly delayed discrete planar systems with constant matrices assuming that one eigenvalue of the matrix of non-delayed terms is nonzero while the other is zero. New non-delayed discrete planar systems are constructed such that the general solutions of both the delayed and the non-delayed systems coincide. Unfortunately, this coincidence does not hold on the whole interval of the delayed system domain, i.e., not for $k = 1, \dots$ but only on the interval with $k = m+2, \dots$, after a "transient" interval characterized by the values $k = 1, \dots, m+1$ is passed. The case of both eigenvalues being nonzero was considered in detail in [2,3]. However, the methods and results of [2,3] (and well as those of [5–8,12,21]) cannot be applied here due to the presence of a zero eigenvalue. Let us note, that weakly-delayed systems appear, e.g., in in [1, 13, 15, 16] and a close investigation can be found in [18] as well. For rudiments of theory of difference equations we refer to [10, 11, 14, 17, 19, 20]. The results are illustrated by examples.

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LINEAR DISCRETE TWO-DIMENSIONAL SYSTEMS WITH CONSTANT COEFFICIENTS - CONTRUCTION OF GENERAL SOLUTIONS IN THE CASE OF BOTH EIGENVALUES OF THE MATRIX OF NONDELAYED TERMS BEING ZEROS WITH THE CONDITIONS CHARACTERIZING WEAKLY DELAYED SYSTEMS SATISFIED

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Abstract: *Linear discrete two-dimensional systems*

$$y(n+1) = Gy(n) + My(n-r),\ n \geq 0$$

are considered, where the 2 *by* 2 *constant matrices G and M satisfy the conditions known for so-called weakly delayed systems. The system has a single delay represented by a positive integer r, n is an independent variable and y in an unknown two dimensional vector function defined for all* $n = -r, -r+1, \ldots$. *Considered is also a new case of both eigenvalues of G equaling zero. Formulas are derived for solutions of initial problems with two illustrative examples presented. Some relations to previously known results are discussed.*

Keywords: weakly delayed system, discrete equation, single delay, initial problem, zero eigenvalues, general solution.

INTRODUCTION

For a given integer s, define a set of integers $D_s := \{s, s+1, \ldots\}$. The paper considers a two-dimensional system of discrete equations

$$y(n+1) = Gy(n) + My(n-r) \tag{1}$$

with the following assumed. The domain of the independent variable n is the set D_0. The unknown two-dimensional vector function $y = (y_1, y_2)^T$ in (1) is real defined on the set D_{-r}, where r is a positive fixed integer describing the delay (the superscript "T" denotes the transpose of a vector). A solution of (1) is defined as a function $y = y^* : D_{-r} \to \mathbb{R}^2$ such that (1) with $y = y^*$ is satisfied for every $n \in D_0$.

In the paper, both eigenvalues of the 2 by 2 constant matrix $G = \{g_{ij}\}_{i,j=1}^2$ are assumed to be zeroes. Moreover, we assume that the matrix G and the nonzero 2 by 2 matrix $M = \{m_{ij}\}_{i,j=1}^2$

satisfy the conditions characterizing weakly delayed discrete systems. Next, formulas are derived for solutions of the initial problem in this case, that is, for the general solution of the system (1). A brief overview of previously known results is presented. The results are illustrated by two examples.

1 WEAKLY DELAYED DISCRETE SYSTEMS

Weakly delayed linear discrete systems are considered in [2,3,5–7,9]. In a pioneering paper [9], a weakly delayed system is defined with its definition shown below. If system (1) is considered (lifting temporarily the above formulated restrictions on G assuming that it is a general matrix) this definition is saying the following: the system is weakly delayed if the values of the determinants

$$\det(G-\lambda E) \quad \text{and} \quad \det(G-\lambda E-\lambda^{-r}M) \tag{2}$$

are identical for every $\lambda \in \mathscr{C} \setminus \{0\}$, where E is a 2 by 2 unit matrix and $\mathscr{C}$ stands for the set of all complex numbers. This definition can be presented algebraically, in terms of entries of matrices G and M. In [9], Theorem 1.3, it is proved that the system (1) is weakly delayed if and only if the matrices F and G satisfy $\operatorname{tr} M = \det M = 0$ and $\det(G+M) = \det G$, or, in terms of their entries, if and only if

$$m_{11}+m_{22}=0, \tag{3}$$

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = 0 \tag{4}$$

and

$$\begin{vmatrix} g_{11} & g_{12} \\ m_{21} & m_{22} \end{vmatrix} + \begin{vmatrix} m_{11} & m_{12} \\ g_{21} & g_{22} \end{vmatrix} = 0. \tag{5}$$

In the above mentioned papers, it is shown that weakly delayed systems can be solved more easily than the non-weakly delayed ones. In [9], for every admissible Jordan form of the matrix G, an ingenious specific method is used to solve the system.

In [2] and in [3] explicit general solutions are derived of planar linear discrete systems with constant coefficients and weak delays for systems with two delays and several delays, respectively. The main approach to deriving general solutions consists in applying the well-known method of steps. In [5], the results of the paper [3] are used to study conditional stability and asymptotic behaviour of solutions of two dimensional linear multi-delayed weakly delayed discrete systems.

A unified and universal systematic approach, based on the $\mathscr{Z}$-transform, is developed in [6, 7]. Note that these methods are applicable only if the values of the determinants in (2) are identical for every complex λ with the exception of $\lambda = 0$. This motivated the interest in investigating the case of conditions (3)–(5) holding even if one of the eigenvalues of the matrix G equals zero. In the conference contribution [8] such a case is considered with the results generalized in [12] to linear discrete systems of the form

$$y(n+1) = \alpha(n)Gy(n) + \beta(n)My(n-r),$$

where $\alpha(n)$ and $\beta(n)$ are nonzero functions defined on D_0 such that $\alpha(n)\beta(n) \neq 0$ for every $n \in D_0$.

In the present contribution, the case is considered of both eigenvalues of the matrix G equaling zero. Such a case has not yet been considered. We will consider two subcases taking into account the geometrical multiplicity $m_g(G,0)$ of the zero eigenvalue. The first one is characterized by the geometrical multiplicity $m_g(G,0) = 2$ while the second one by the geometrical

multiplicity of the zero eigenvalue $m_g(G,0) = 1$. The Jordan forms $\mathscr{J}(G)$ of the matrix G, used below are

$$\mathscr{J}(G) = G_2 = \Theta := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

if $m_g(G,0) = 2$ while

$$\mathscr{J}(G) = G_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

if $m_g(G,0) = 1$.

2 PRELIMINARIES

A solution to system (1) is defined as a function $y\colon D_{-r} \to \mathbb{R}^2$ such that $y(n)$ satisfies system (1) for an arbitrary $n \in D_0$ in (1). Let a discrete real vector function

$$\varphi(n) = \begin{pmatrix} \varphi_1(n) \\ \varphi_2(n) \end{pmatrix}, \ \forall n \in D_{-r} \setminus D_1$$

be given. An initial (starting) problem for system (1) is defined as a set of given initial data

$$y(n) := \begin{pmatrix} \varphi_1(n) \\ \varphi_2(n) \end{pmatrix}, \ \forall n \in D_{-r} \setminus D_1. \tag{6}$$

Initial values (6) determine exactly one solution $y = y(n)$, where $n \in D_{-r}$ is arbitrary, to system (1). For the theory of discrete equations, we refer to [10] and to [14, 20].

If a system (1) with a matrix G having non-zero eigenvalues is weakly delayed, then its arbitrary regular linear transformation transforms (1) into a system that is weakly delayed as well. This means that the conditions (2) as well as conditions (3)–(5) will be satisfied if matrices G and M and their entries are replaced with the transformed matrices and their entries.

If one of G's eigenvalues is zero while the other is non-zero, then the conditions (2) are not satisfied and it is not clear whether conditions (3)–(5) will be helpful. Nevertheless, as it is proved in [8], conditions (3)–(5) remain in force for the new matrices after every regular transformation. Thanks to this, the general solution to system was found in [8].

If both eigenvalues of G are zeroes, conditions (2) cannot be used either. However, as it follows from an analysis of the proof in [8], conditions (3)–(5) remain in force for the new matrices after every regular transformation.

3 THE CASE OF $\mathbf{m_g(G,0) = 2}$ AND GENERAL SOLUTION TO (1)

Assume that the initial-value problem (1), (6) is transformed by a regular transformation

$$y(n) = \mathscr{T} z(n), \ n \in D_{-r}, \tag{7}$$

where $\mathscr{T}$ is a regular 2 by 2 matrix and $z\colon D_{-r} \to \mathbb{R}^2$ is a new unknown function, into a new linear system with the Jordan form of matrix of linear non-delayed terms G_2. Then, the system (1) can be written as

$$z(n+1) = G_2 z(n) + M^* z(n-r), \ n \in D_0, \tag{8}$$

where $G_2 = \mathscr{T}^{-1} G \mathscr{T} = \Theta$ and $M^* = \mathscr{T}^{-1} M \mathscr{T} = \{m^*_{ij}\}_{i,j=1}^2$. Since $G_2 = \Theta$, obviously, the matrix G must be the zero matrix as well. Then, it is not necessary to use any regular transformation (7) as it is sufficient to solve immediately the initial system (1) assuming $G = \Theta$. In

such a case, neither is it necessary to transform the initial problem (6). Conditions (3)–(5) are reduced to conditions (3), (4) only, that is to

$$m_{11}+m_{22}=0,\ \ m_{11}m_{22}=m_{12}m_{21}. \tag{9}$$

Taking into account that $M\neq\Theta$, equations (9) imply that at least one of the values m_{12}, m_{21} is nonzero. Therefore, we consider the following possible three subcases:

i) $m_{12}=0$, $m_{21}\neq 0$,
ii) $m_{12}\neq 0$, $m_{21}=0$,
iii) $m_{12}m_{21}\neq 0$.

3.1 THE SUBCASE OF $\mathbf{m_{12}=0}$, $\mathbf{m_{21}\neq 0}$

Here, $m_{11}=m_{22}=0$ and the system (1) reduces to

$$y_1(n+1)=0, \tag{10}$$
$$y_2(n+1)=m_{21}y_1(n-r), \tag{11}$$

where $n\in D_0$. The solution of system (10), (11) determined by initial data (6) is

$$y_1(n)=0, \tag{12}$$
$$y_2(n)=m_{21}\varphi_1(n-r-1) \tag{13}$$

if $n\in n\in D_1\setminus D_{r+2}$, and

$$y_1(n)=0, \tag{14}$$
$$y_2(n)=0 \tag{15}$$

if $n\in D_{r+2}$.

3.2 THE SUBCASE OF $\mathbf{m_{12}\neq 0}$, $\mathbf{m_{21}=0}$

Here, $m_{11}=m_{22}=0$ and the system (1) reduces to

$$y_1(n+1)=m_{12}y_2(n-r), \tag{16}$$
$$y_2(n+1)=0, \tag{17}$$

where $n\in D_0$. The solution of system (16), (17) determined by initial data (6) is

$$y_1(n)=m_{12}\varphi_2(n-r-1), \tag{18}$$
$$y_2(n)=0 \tag{19}$$

if $n\in n\in D_1\setminus D_{r+2}$ and

$$y_1(n)=0, \tag{20}$$
$$y_2(n)=0 \tag{21}$$

if $n\in n\in D_{r+2}$.

3.3 THE SUBCASE OF $\mathbf{m_{12}m_{21} \neq 0}$

Here, all entries m_{11}, m_{12}, m_{21}, m_{22} are non-zeros and the system (1) reduces to

$$y_1(n+1) = m_{11}y_1(n-r) + m_{12}y_2(n-r), \tag{22}$$
$$y_2(n+1) = m_{21}y_1(n-r) + m_{22}y_2(n-r), \tag{23}$$

where $n \in D_0$. A linear combination

$$m_{22}y_1(n+1) - m_{12}y_2(n+1)$$
$$= (m_{11}m_{22} - m_{21}m_{12})y_1(n-r) + (m_{12}m_{22} - m_{22}m_{12})y_2(n-r) = 0,\ n \in D_0,$$

leads to (condition (9) is used)

$$m_{22}y_1(n+1) = m_{12}y_2(n+1)\ \forall n \in D_0. \tag{24}$$

Applying (24) in (22), we get

$$y_1(n+1) = m_{11}y_1(n-r) + m_{12}y_2(n-r) = m_{11}\frac{m_{12}}{m_{22}}y_2(n-r) + m_{12}y_2(n-r) = 0,$$

where $n \in D_{r+1}$ and, in much the same way, applying (24) in (23), we get

$$y_2(n+1) = m_{21}y_1(n-r) + m_{22}y_2(n-r) = m_{21}\frac{m_{12}}{m_{22}}y_2(n-r) + m_{22}y_2(n-r) = 0,$$

where $n \in D_{r+1}$. Finally, the solution of system (22), (23) determined by initial data (6) is

$$y_1(n) = m_{11}\varphi_1(n-r-1) + m_{12}\varphi_2(n-r-1), \tag{25}$$
$$y_2(n) = m_{21}\varphi_1(n-r-1) + m_{22}\varphi_2(n-r-1), \tag{26}$$

if $n \in D_1 \setminus D_{r+2}$ and

$$y_1(n) = 0, \tag{27}$$
$$y_2(n) = 0 \tag{28}$$

if $n \in n \in D_{r+2}$.

4 THE CASE OF $\mathbf{m_g(G,0) = 1}$ AND GENERAL SOLUTION TO (1)

Assume that the problem (1), (6) is transformed by a regular transformation

$$y(n) = \mathscr{T}z(n),\ n \in D_{-r}, \tag{29}$$

where $\mathscr{T}$ is a regular 2 by 2 matrix and $z\colon D_{-r} \to \mathbb{R}^2$ is a new unknown function, into a linear system with the Jordan form of the matrix of linear non-delayed terms $\mathscr{J}(G) = G_1$. Then, system (1) can be written as

$$z(n+1) = G_1z(n) + M^*z(n-r),\ n \in D_0, \tag{30}$$

where $G_1 = \mathscr{T}^{-1}G\mathscr{T}$ and $M^* = \mathscr{T}^{-1}M\mathscr{T} = \{m^*_{ij}\}^2_{i,j=1}$. Conditions (3), (4), where the entries of matrices G and M are replaced with the elements of matrices G_1 and M^*, reduce to

$$m^*_{11} + m^*_{22} = 0,\ m^*_{11}m^*_{22} = m^*_{12}m^*_{21} \tag{31}$$

with condition (5) reducing to

$$m_{21}^* = 0. \tag{32}$$

Then, by (31) we have

$$m_{11}^* = m_{22}^* = 0.$$

Because the matrix M is non zero and, consequently, so is the matrix M^*, we have

$$m_{12}^* \neq 0,$$

and the transformed system (30) takes the form

$$z_1(n+1) = z_2(n) + m_{12}^* z_2(n-r), \tag{33}$$
$$z_2(n+1) = 0, \tag{34}$$

where $n \in D_0$. The initial data for system (33), (34) transformed from (6) by transformation (29) are the following

$$z(n) := \varphi^*(n) = \begin{pmatrix} \varphi_1^*(n) \\ \varphi_2^*(n) \end{pmatrix} = \mathscr{T}^{-1}\varphi(n) = \mathscr{T}^{-1}\begin{pmatrix} \varphi_1(n) \\ \varphi_2(n) \end{pmatrix}, \ \forall n \in D_{-r} \setminus D_1. \tag{35}$$

Consider equation (33) for $n \in D_0 \setminus D_{r+1}$. If $n = 0$, then

$$z_1(1) = \varphi_2^*(0) + m_{12}^* \varphi_2^*(-r), \tag{36}$$
$$z_2(1) = 0, \tag{37}$$

and, if $n \in D_1 \setminus D_{r+1}$ in (33), by (34), we have,

$$z_1(n) = m_{12}^* \varphi_2^*(n-1-r), \tag{38}$$
$$z_2(n) = 0, \tag{39}$$

where $n \in D_2 \setminus D_{r+2}$. By (34), system (33), (34) can be written as

$$z_1(n+1) = 0,$$
$$z_2(n+1) = 0$$

if $n \in D_{r+1}$, and, therefore,

$$z_1(n) = 0, \tag{40}$$
$$z_2(n) = 0, \tag{41}$$

where $n \in D_{r+2}$.

5 FINAL RESULTS

The below two theorems finalize all formulas derived in Parts 3 and 4.

Theorem 1 *Let the Jordan form of matrix G be* $\mathscr{J}(G) = G_2$ *and let the entries of matrix M satisfy conditions* (3), (4). *Then, the solution of the problem* (1), (6) *is given by the formulas*

$$y_1(n) = m_{11}\varphi_1(n-r-1) + m_{12}\varphi_2(n-r-1), \tag{42}$$
$$y_2(n) = m_{21}\varphi_1(n-r-1) + m_{22}\varphi_2(n-r-1), \tag{43}$$

if $n \in D_1 \setminus D_{r+2}$ *and by the formulas*

$$y_1(n) = 0, \tag{44}$$
$$y_2(n) = 0 \tag{45}$$

if $n \in D_{r+2}$.

PROOF. It is easy to see that, if $n \in D_1 \setminus D_{r+2}$, formulas (42), (43) cover formulas (12), (13) in Part 3.1, formulas (18), (19) in Part 3.2 as well as formulas (25), (26) in Part 3.3. Similarly, if $n \in D_{r+2}$, formulas (44), (45) coincide with formulas (14), (15) in Part 3.1, formulas (20), (21) in Part 3.2 as well with formulas (27), (28) in Part 3.3. □

Theorem 2 *Let the Jordan form of matrix G be* $\mathscr{J}(G) = G_1 = \mathscr{T}^{-1} G \mathscr{T}$ *and let the entries of the matrix M satisfy conditions* (3)–(5)*. Then, the solution of the problem* (1)*,* (6) *is given by the formula*

$$y(n) = \mathscr{T} z(n), \ n \in D_1,$$

where, for $n = 1$*,*

$$z_1(1) = \varphi_2^*(0) + m_{12}^* \varphi_2^*(-r), \tag{46}$$
$$z_2(1) = 0, \tag{47}$$

and, for $n \in D_2 \setminus D_{r+2}$*,*

$$z_1(n) = m_{12}^* \varphi_2^*(n-1-r), \tag{48}$$
$$z_2(n) = 0. \tag{49}$$

Finally, for $n \in D_{r+2}$*, we have*

$$z_1(n) = 0, \tag{50}$$
$$z_2(n) = 0. \tag{51}$$

The vector components φ^* *are defined by formula* (35) *and* m_{12}^* *is the corresponding entry of the matrix* $M^* = \mathscr{T}^{-1} M \mathscr{T}$.

PROOF. If $n \in D_1 \setminus D_{r+2}$, formulas (46)–(49) copy formulas (36)–(39) in Part 4. If $n \in D_{r+2}$, formulas (50), (51) coincide with formulas (40), (41) in Part 4. □

6 EXAMPLES

Below we illustrate the conclusions derived by considering two examples. The discrete system considered in Example 1 serves as an illustration of how Theorem 1 can be applied while, to solve the discrete system in Example 2, Theorem 2 is necessary.

Example 1 Assume that the system (1) is specified as follows. Let $G = \Theta$, $r = 2$,

$$M = \begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix}$$

and let the initial data be

$$\varphi(-2)=\begin{pmatrix}4\\8\end{pmatrix},\ \varphi(-1)=\begin{pmatrix}2\\4\end{pmatrix},\ \varphi(0)=\begin{pmatrix}1\\2\end{pmatrix}. \tag{52}$$

System (1) can now be written as

$$y_1(n+1)=6y_1(n-2)-4y_2(n-2), \tag{53}$$
$$y_2(n+1)=9y_1(n-2)-6y_2(n-2), \tag{54}$$

where $n\in D_0=\{0,1,\dots\}$. Since

$$m_{11}+m_{22}=6-6=0,\quad \begin{vmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{vmatrix}=\begin{vmatrix}6 & -4\\ 9 & -6\end{vmatrix}=0,$$

Theorem 1 is applicable and, by formulas (42), (43), the solution of the system (53), (54), defined by initial problem (52), can be expressed, if $n\in D_1\setminus D_4=\{1,2,3\}$, as

$$y_1(1)=6\varphi_1(-2)-4\varphi_2(-2)=-8,$$
$$y_2(1)=9\varphi_1(-2)-6\varphi_2(-2)=-12,$$

$$y_1(2)=6\varphi_1(-1)-4\varphi_2(-1)=-4,$$
$$y_2(2)=9\varphi_1(-1)-6\varphi_2(-1)=-6,$$

and

$$y_1(3)=6\varphi_1(0)-4\varphi_2(0)=-2,$$
$$y_2(3)=9\varphi_1(0)-6\varphi_2(0)=-3.$$

If $n\in D_4=\{4,5,\dots\}$ then, by formulas (44), (45), the solution of the system (53), (54), defined by initial problem (52) becomes a zero solution, that is,

$$y_1(n)=y_2(n)=0,\ n=4,5,\dots.$$

Example 2 Assume that the system (1) is specified as follows. Let $r=2$,

$$G=\begin{pmatrix}0.5 & -0.5\\ 0.5 & -0.5\end{pmatrix},\ M=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix},$$

and let the initial data be

$$\varphi(-2)=\begin{pmatrix}4\\8\end{pmatrix},\ \varphi(-1)=\begin{pmatrix}2\\4\end{pmatrix},\ \varphi(0)=\begin{pmatrix}1\\2\end{pmatrix}. \tag{55}$$

System (1) can be written as

$$y_1(n+1)=0.5y_1(n)-0.5y_1(n)+y_1(n-2)-y_2(n-2), \tag{56}$$
$$y_2(n+1)=0.5y_1(n)-0.5y_1(n)+y_1(n-2)-y_2(n-2), \tag{57}$$

where $n\in D_0=\{0,1,\dots\}$. We have

$$m_{11}+m_{22}=1-1=0,\quad \begin{vmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{vmatrix}=\begin{vmatrix}1 & -1\\ 1 & -1\end{vmatrix}=0.$$

To get the Jordan form of the matrix $\mathscr{J}(G) = G_1$, let us transform the system (56), (57) by transformation (29), where

$$\mathscr{T} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \ \mathscr{T}^{-1} = \begin{pmatrix} 0 & 1 \\ 0.5 & -0.5 \end{pmatrix}.$$

Then

$$G_1 = \mathscr{T}^{-1} G \mathscr{T} = \begin{pmatrix} 0 & 1 \\ 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} 0.5 & -0.5 \\ 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$M^* = \mathscr{T}^{-1} M \mathscr{T} = \begin{pmatrix} 0 & 1 \\ 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

The system (56), (57) can be written as a system (30), i.e.

$$z(n+1) = G_1 z(n) + M^* z(n-2), \ n \in D_0$$

and

$$z_1(n+1) = z_2(n) + 2z_2(n-2), \tag{58}$$
$$z_2(n+1) = 0, \tag{59}$$

where $n \in D_0 = \{0, 1, \dots\}$. The initial data for system (58), (59), transformed from (55) by transformation (29), are as shown below (we refer to (35))

$$z(n) := \varphi^*(n) = \mathscr{T}^{-1} \varphi(n) = \begin{pmatrix} 0 & 1 \\ 0.5 & -0.5 \end{pmatrix} \begin{pmatrix} \varphi_1(n) \\ \varphi_2(n) \end{pmatrix}, \ \forall n \in D_{-2} \setminus D_1 = \{-2, -1, 0\}$$

and

$$\varphi^*(-2) = \begin{pmatrix} 8 \\ -2 \end{pmatrix}, \ \varphi^*(-1) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \ \varphi^*(0) = \begin{pmatrix} 2 \\ -0.5 \end{pmatrix}.$$

Theorem 2 can be applied and the solution of the system (56), (57), defined by initial data (52) can be expressed by the transformation

$$y(n) = \mathscr{T} z(n) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \ n \in D_1, \tag{60}$$

where, for $n = 1$, (we refer to formulas (46), (47)),

$$z_1(1) = \varphi_2^*(0) + m_{12}^* \varphi_2^*(-2) = -0.5 + 2(-2) = -4.5,$$
$$z_2(1) = 0,$$

for $n = 2$, (we refer to formulas (48), (49)),

$$z_1(2) = m_{12}^* \varphi_2^*(-1) = 2(-1) = -2,$$
$$z_2(2) = 0,$$

and, for $n = 3$, (we refer to formulas (48), (49) again),

$$z_1(3) = m_{12}^* \varphi_2^*(0) = 2(-0.5) = -1,$$
$$z_2(3) = 0.$$

Finally, for $n \in D_4 = \{4,5,\dots\}$, we have, by formulas (50), (51),

$$z_1(n) = 0,$$
$$z_2(n) = 0.$$

By formula (60), we derive

$$y(n) = \mathscr{T} z(n) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} z(n) = \begin{pmatrix} z_1(n) + 2z_2(n) \\ z_1(n) \end{pmatrix} = \begin{pmatrix} z_1(n) \\ z_1(n) \end{pmatrix}, \ n \in D_1.$$

The solution of the initial system (56), (57), if $n = 1$, is the following,

$$y_1(1) = y_2(1) = z_1(1) = -4.5,$$

if $n = 2$, then

$$y_1(2) = y_2(2) = z_1(2) = -2,$$

if $n = 3$, then

$$y_1(3) = y_2(3) = z_1(3) = -1,$$

and, for $n \in D_4 = \{4,5,\dots\}$, we have

$$y_1(n) = z_1(n) = 0,$$

$$y_2(n) = z_1(n) = 0.$$

CONCLUSION

The paper investigates two-dimensional discrete delayed systems (1) for the case of both eigenvalues of the matrix G of the non-delayed system being zeros. It is assumed that the entries of matrices G and M satisfy conditions (3)–(5) characterizing so called discrete systems with weak delay (weakly delayed systems). Such an investigation has not yet been performed. The previously known investigations (e.g., [2, 3, 5–7, 9]) assume that both roots of such systems are nonzero and conditions (3)–(5) derived from (2) are satisfied. For further properties of weakly delayed systems we refer to [4, 21]. The papers [8, 12] consider the case of one eigenvalue being zero while the other one being nonzero. In our case, due to the presence of two zero eigenvalues, it is not possible to verify whether (2) holds. Consequently it is not possible to derive conditions (3)–(5). Nevertheless, assuming them to be satisfied "ad hoc", we obtain formulas of the general solutions for all possible cases of such systems. The results derived (Theorem 1 and Theorem 2) are saying the following. As it follows from formulas (44), (45) and (50), (51), all the solutions become zeros if $n \in D_{r+2}$. This is an important conclusion saying that if the "transient" interval $n \in D_1 \setminus D_{r+2}$ is passed, then the "history" of the process expressed by delayed initial data (6) or, equivalently, by (35), is eliminated and the previous "history" has no influence on the behaviour of the process in the future. Therefore, the process described by discrete systems (1) becomes a trivial one. The formulas derived can be useful in digital signal processing [11, 17, 19]. Finally, let us remark, that investigation of weakly delayed systems in continuous case, i.e., in the case of differential equations with delay being considered, has a practical meaning. We refer, e.g. to the papers [15, 16] and to the books [1, 13], where a three-dimensional weakly delayed linear differential system is considered. Some closely related problems are treated in [18] as well.

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USING FEYNMAN'S TECHNIQUE TO CALCULATE AN INTEGRAL MAPLE COULD NOT COPE WITH

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Abstract: *It is well known that the primitive function to some functions cannot be written with help of elementary functions. Yet in some cases, the definite integral over a certain interval, e.g.* $(0,\infty)$*, of such functions can be calculated precisely. In this paper we show an example of this type, using the so called Feynman's technique which combines integration and differentiation.*

Keywords: integral, error function, Maple

INTRODUCTION

Students and former students (now colleagues from different departments of our faculty) sometimes come to our department with interesting questions considering mathematical problems (e.g. [1]).

Not long ago, a former student asked the author of this paper for help with the following integral:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x, \quad \mu,\sigma \in \mathbb{R},\ \sigma > 0. \tag{1}$$

The second factor in the product may be familiar to the reader – it is the density function of the Gaussian probability distribution (omitting some constant). In fact, the main aim of the student was to study the divergence between probability distributions, see e.g. [2]. But here we will concentrate just on the calculation of this tricky integral.

The first idea of the student was, of course, to use some powerful mathematical software. But these experiments did not bring any result. E.g., Maple could calculate the most simple version with $\mu = 0$, but failed in case $\mu \neq 0$, see figure 1.

Here we show how this integral can be calculated with help of a trick, sometimes called Feynman's technique. This trick is based upon introducing a new variable into the integrand and then using differentiation and integration.

1 PRELIMINARIES

1.1 Gaussian Integral

In further calculations, we will use the value of the Gaussian integral

$$\int_{-\infty}^{\infty} \exp(-x^2)\,\mathrm{d}x = \sqrt{\pi}. \tag{2}$$

For a more general case, simple substitution $t = (x-b)\sqrt{a}$ leads to

$$\int_{-\infty}^{\infty} \exp(-a(x-b)^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}, \quad a,b \in \mathbb{R}, a > 0. \tag{3}$$

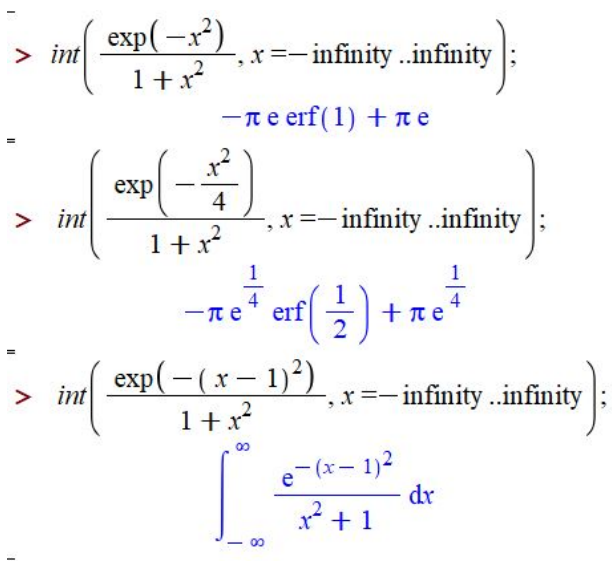

Figure 1: Results from Maple (Source: own)

1.2 Error Function

Although we will be able to calculate integral (1) in the end, elementary functions will not be sufficient. We will need the error function which is defined as

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)\,\mathrm{d}t, \quad z \in \mathbb{C}. \tag{4}$$

We will use the following properties of the error function (see, e.g. [3]):

$$\operatorname{erf}(-z) = -\operatorname{erf} z \tag{5}$$

$$\operatorname{erf}\bar{z} = \overline{\operatorname{erf} z} \tag{6}$$

$$(\operatorname{erf} z)' = \frac{2}{\sqrt{\pi}} \exp(-z^2) \tag{7}$$

Further, considering only real values of z,

$$\lim_{z\to\infty} \operatorname{erf} z = 1. \tag{8}$$

2 SOLUTION OF THE GIVEN PROBLEM

2.1 Case $\mu = 0, \sigma = \sqrt{2}/2$

This is the simplest case. We are interested in the value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp\left(-x^2\right) \mathrm{d}x. \tag{9}$$

The Feynman's technique is based upon introducing a new variable and then differentiating the function behind the integral sign.

Put

$$I(t) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp\left(-x^2\right) \exp(-t(1+x^2))\,\mathrm{d}x, \quad t \geq 0. \tag{10}$$

We can see that integral (9) is equal to $I(0)$. Further, integral $I(t)$ converges for every $t \geq 0$ and

$$\lim_{t\to\infty} I(t) = 0. \tag{11}$$

Now differentiate (10) with respect to t:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(t) &= \int_{-\infty}^{\infty} \frac{-(1+x^2)}{1+x^2} \exp\left(-x^2\right) \exp(-t(1+x^2))\,\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} \exp\left(-x^2-t-tx^2\right)\right)\,\mathrm{d}x = -\exp(-t)\int_{-\infty}^{\infty} \exp\left(-(1+t)x^2\right)\,\mathrm{d}x.\end{aligned}$$

Now, using (3) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = -\exp(-t)\sqrt{\frac{\pi}{1+t}}. \tag{12}$$

Computing the indefinite integral of the right hand side with respect to t using substitution $u = \sqrt{t+1}$ and (4), we get

$$\begin{aligned}-\sqrt{\pi}\int \frac{\exp(-t-1+1)}{\sqrt{1+t}}\,\mathrm{d}t = -2\mathrm{e}\sqrt{\pi}\int \exp(-u^2)\,\mathrm{d}u = \\ = -2\mathrm{e}\sqrt{\pi}\frac{\sqrt{\pi}}{2}\mathrm{erf}\,(u) = -\mathrm{e}\,\pi\,\mathrm{erf}\,(t+1)+c\end{aligned}$$

That gives

$$I(t) = -\mathrm{e}\,\pi\,\mathrm{erf}\,(t+1)+c \tag{13}$$

and using (8) and (11), we get

$$c = \mathrm{e}\,\pi.$$

Altogether we have

$$I(t) = -\mathrm{e}\,\pi\,\mathrm{erf}\,(t+1)+\mathrm{e}\,\pi$$

and thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp\left(-x^2\right)\,\mathrm{d}x = I(0) = \mathrm{e}\,\pi\,(1-\mathrm{erf}\,(1)) \tag{14}$$

which is the same result as we got from Maple, see figure 1.

2.2 Case $\mu \neq 0, \sigma = \sqrt{2}/2$

Now consider the more complicated case that could not be solved by Maple:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp\left(-(x-\mu)^2\right)\,\mathrm{d}x. \tag{15}$$

Similarly, put

$$I(t) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \exp\left(-(x-\mu)^2\right) \exp(-t(1+x^2))\,\mathrm{d}x, \quad t \geq 0. \tag{16}$$

We are interested in $I(0)$. As in the previous case, integral $I(t)$ converges for every $t \geq 0$ and

$$\lim_{t\to\infty} I(t) = 0. \tag{17}$$

Differentiate (16) with respect to t:

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = \int_{-\infty}^{\infty} \frac{-(1+x^2)}{1+x^2} \exp\left(-(x-\mu)^2\right) \exp(-t(1+x^2))\,\mathrm{d}x \tag{18}$$
$$= -\int_{-\infty}^{\infty} \exp\left(-x^2+2\mu x-\mu^2-t-tx^2\right)\,\mathrm{d}x.$$

Rearranging the argument of the exp function we get

$$-x^2+2\mu x-\mu^2-t-tx^2 = -((1+t)x^2-2\mu x)-\mu^2-t =$$
$$-(1+t)\left(x-\frac{\mu}{1+t}\right)^2+(1+t)\frac{\mu^2}{(1+t)^2}-\mu^2-t =$$
$$= -(1+t)\left(x-\frac{\mu}{1+t}\right)^2+\frac{\mu^2-(\mu^2+t)(1+t)}{1+t} =$$
$$= -(1+t)\left(x-\frac{\mu}{1+t}\right)^2+\frac{\mu^2-(\mu^2-1+1+t)(1+t)}{1+t} =$$
$$= -(1+t)\left(x-\frac{\mu}{1+t}\right)^2+\frac{\mu^2-(\mu^2-1)(1+t)-(1+t)^2}{1+t} =$$
$$= -(1+t)\left(x-\frac{\mu}{1+t}\right)^2-(1+t)+\frac{\mu^2}{1+t}-\mu^2+1. \tag{19}$$

Substituting this result back into (18) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = -\int_{-\infty}^{\infty} \exp\left(-(1+t)\left(x-\frac{\mu}{1+t}\right)^2-(1+t)+\frac{\mu^2}{1+t}-\mu^2+1\right)\mathrm{d}x$$
$$= -\exp\left(-(1+t)+\frac{\mu^2}{1+t}\right)\exp(-\mu^2+1)\int_{-\infty}^{\infty}\exp\left(-(1+t)\left(x-\frac{\mu}{1+t}\right)^2\right)\mathrm{d}x.$$

Finally, using (3) with $a=(1+t), b=\mu/(1+t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = -\exp\left(-(1+t)+\frac{\mu^2}{1+t}\right)\exp(-\mu^2+1)\sqrt{\frac{\pi}{1+t}}. \tag{20}$$

To determine $I(t)$, we have to find the integral

$$\int \exp\left(-(1+t)+\frac{\mu^2}{1+t}\right)\frac{1}{\sqrt{1+t}}\,\mathrm{d}t. \tag{21}$$

Using substitution $z=1+t$, we get

$$\int \exp\left(-z+\frac{\mu^2}{z}\right)\frac{1}{\sqrt{z}}\,\mathrm{d}z. \tag{22}$$

Remind that $t \geq 0$ and thus also $z>0$. Expression $1/\sqrt{z}$ can be rewritten as

$$\frac{1}{\sqrt{z}} = \frac{z}{\sqrt{z}^3} = \frac{z-\mathrm{j}\mu+z+\mathrm{j}\mu}{2\sqrt{z}^3}, \tag{23}$$

where j is the imaginary unit, $\mathrm{j}^2 = -1$. Further,

$$-z + \frac{\mu^2}{z} = -\left(\sqrt{z}^2 + \left(\frac{\mathrm{j}\mu}{\sqrt{z}}\right)^2 + 2\mathrm{j}\mu - 2\mathrm{j}\mu\right)$$

and thus

$$-z + \frac{\mu^2}{z} = -\left(\left(\sqrt{z} + \frac{\mathrm{j}\mu}{\sqrt{z}}\right)^2 - 2\mathrm{j}\mu\right), \tag{24}$$

and also

$$-z + \frac{\mu^2}{z} = -\left(\left(\sqrt{z} - \frac{\mathrm{j}\mu}{\sqrt{z}}\right)^2 + 2\mathrm{j}\mu\right). \tag{25}$$

Substituting (23), (24) and (25) into (22), we get

$$\int \exp\left(-z + \frac{\mu^2}{z}\right)\frac{1}{\sqrt{z}}\,\mathrm{d}z = \int \exp\left(-z + \frac{\mu^2}{z}\right)\frac{z - \mathrm{j}\mu}{2\sqrt{z^3}}\,\mathrm{d}z + \int \exp\left(-z + \frac{\mu^2}{z}\right)\frac{z + \mathrm{j}\mu}{2\sqrt{z^3}}\,\mathrm{d}z =$$
$$= \exp(2\mathrm{j}\mu)\int \exp\left(-\left(\sqrt{z} + \frac{\mathrm{j}\mu}{\sqrt{z}}\right)^2\right)\frac{z - \mathrm{j}\mu}{2\sqrt{z^3}}\,\mathrm{d}z +$$
$$+ \exp(-2\mathrm{j}\mu)\int \exp\left(-\left(\sqrt{z} - \frac{\mathrm{j}\mu}{\sqrt{z}}\right)^2\right)\frac{z + \mathrm{j}\mu}{2\sqrt{z^3}}\,\mathrm{d}z$$

Putting substitution for the exponent in both these integrals

$$\begin{aligned} u &= \sqrt{z} + \tfrac{\mathrm{j}\mu}{\sqrt{z}} & v &= \sqrt{z} - \tfrac{\mathrm{j}\mu}{\sqrt{z}} \\ \mathrm{d}u &= \left(\tfrac{1}{2\sqrt{z}} - \tfrac{\mathrm{j}\mu}{2\sqrt{z^3}}\right)\mathrm{d}z & \mathrm{d}v &= \left(\tfrac{1}{2\sqrt{z}} + \tfrac{\mathrm{j}\mu}{2\sqrt{z^3}}\right)\mathrm{d}z \\ \mathrm{d}u &= \tfrac{z - \mathrm{j}\mu}{2\sqrt{z^3}}\,\mathrm{d}z & \mathrm{d}v &= \tfrac{z + \mathrm{j}\mu}{2\sqrt{z^3}}\,\mathrm{d}z \end{aligned}$$

we get

$$\int \exp\left(-z + \frac{\mu^2}{z}\right)\frac{1}{\sqrt{z}}\,\mathrm{d}z =$$
$$\exp(2\mathrm{j}\mu)\int \exp\left(-u^2\right)\mathrm{d}u + \exp(-2\mathrm{j}\mu)\int \exp\left(-v^2\right)\mathrm{d}v =$$
$$\exp(2\mathrm{j}\mu)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{z} + \frac{\mathrm{j}\mu}{\sqrt{z}}\right) + \exp(-2\mathrm{j}\mu)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{z} - \frac{\mathrm{j}\mu}{\sqrt{z}}\right) =$$
$$\exp(2\mathrm{j}\mu)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{1+t} + \frac{\mathrm{j}\mu}{\sqrt{1+t}}\right) + \exp(-2\mathrm{j}\mu)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{1+t} - \frac{\mathrm{j}\mu}{\sqrt{1+t}}\right). \tag{26}$$

Returning to (20), we get

$$\begin{aligned} I(t) &= -\exp(-\mu^2+1)\sqrt{\pi}\int \exp\left(-(1+t)+\frac{\mu^2}{1+t}\right)\frac{1}{\sqrt{1+t}}\,\mathrm{d}t = \\ &= -\exp(-\mu^2+1)\sqrt{\pi}\left(\exp(2\mathrm{j}\mu)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{1+t}+\frac{\mathrm{j}\mu}{\sqrt{1+t}}\right)+\right. \\ &\qquad \left.\exp(-2\mathrm{j}\mu)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{1+t}-\frac{\mathrm{j}\mu}{\sqrt{1+t}}\right)\right)+c = \\ &= -\frac{\pi}{2}\exp(-\mu^2+1)\left(\exp(2\mathrm{j}\mu)\mathrm{erf}\left(\sqrt{1+t}+\frac{\mathrm{j}\mu}{\sqrt{1+t}}\right)\right. \\ &\qquad \left.+\exp(-2\mathrm{j}\mu)\mathrm{erf}\left(\sqrt{1+t}-\frac{\mathrm{j}\mu}{\sqrt{1+t}}\right)\right)+c \end{aligned} \tag{27}$$

Thanks to (17) and (8) we get

$$0 = -\frac{\pi}{2}\exp(-\mu^2+1)\left(\exp(2\mathrm{j}\mu)+\exp(-2\mathrm{j}\mu)\right)+c$$

and thus, using Euler's formula $\exp(\mathrm{j}x)=\cos x+\mathrm{j}\sin x$,

$$c = \frac{\pi}{2}\exp(-\mu^2+1)\left(\exp(2\mathrm{j}\mu)+\exp(-2\mathrm{j}\mu)\right) = \pi\exp(-\mu^2+1)\cos 2\mu. \tag{28}$$

The wanted integral (15) is equal to $I(0)$. Using (27) and (28),

$$\begin{aligned} I(0) &= -\frac{\pi}{2}\exp(-\mu^2+1)\left(\exp(2\mathrm{j}\mu)\mathrm{erf}\,(1+\mathrm{j}\mu)+\exp(-2\mathrm{j}\mu)\mathrm{erf}\,(1-\mathrm{j}\mu)\right)+ \\ &\quad +\pi\exp(-\mu^2+1)\cos 2\mu = \\ &= -\frac{\pi}{2}\exp(-\mu^2+1)\left(\exp(2\mathrm{j}\mu)\mathrm{erf}\,(1+\mathrm{j}\mu)+\exp(-2\mathrm{j}\mu)\mathrm{erf}\,(1-\mathrm{j}\mu)-2\cos 2\mu\right). \end{aligned}$$

Finally, using (6) and the well known identities for complex numbers

$$\exp\overline{z} = \overline{\exp z}, \qquad z+\overline{z} = 2\mathrm{Re}\,z, \qquad z\in\mathbb{C},$$

we get

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\exp\left(-(x-\mu)^2\right)\mathrm{d}x = \pi\exp(-\mu^2+1)(\cos 2\mu-\mathrm{Re}\,(\exp(2\mathrm{j}\mu)\mathrm{erf}\,(1+\mathrm{j}\mu)). \tag{29}$$

Notice the for $\mu=0$, (29) coincides with the solution from subsection 2.1:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\exp\left(-x^2\right)\mathrm{d}x = \pi\exp(1)(\cos 0-\mathrm{Re}\,(\exp(0)\mathrm{erf}\,(1)) = \pi\mathrm{e}(1-\mathrm{erf}\,1).$$

2.3 Case $\mu,\sigma\in\mathbb{R},\sigma>0$

Finally, consider the most general case

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x, \quad \mu,\sigma\in\mathbb{R},\ \sigma>0. \tag{30}$$

This time, put

$$I(t)=\int_{-\infty}^{\infty}\frac{1}{1+x^2}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\exp\left(-\frac{t(1+x^2)}{2\sigma^2}\right)\mathrm{d}x,\quad t\geq 0. \tag{31}$$

Again, we are interested in $I(0)$. As in the previous cases, integral $I(t)$ converges for every $t\geq 0$ and

$$\lim_{t\to\infty} I(t)=0. \tag{32}$$

Differentiate (31) with respect to t:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(t)&=-\frac{1}{2\sigma^2}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\exp\left(\frac{-t(1+x^2)}{2\sigma^2}\right)\mathrm{d}x\\&=-\frac{1}{2\sigma^2}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2\sigma^2}\left(x^2-2\mu x+\mu^2+t+tx^2\right)\right)\mathrm{d}x.\end{aligned}$$

Similarly as in (2.2), the argument of the exp function can be rearranged as

$$-\frac{1}{2\sigma^2}\left(x^2-2\mu x+\mu^2+t+tx^2\right)=-\frac{1}{2\sigma^2}\left((1+t)\left(x-\frac{\mu}{1+t}\right)^2+(1+t)-\frac{\mu^2}{1+t}+\mu^2-1\right).$$

This gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(t)=-\frac{1}{2\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left((1+t)-\frac{\mu^2}{1+t}\right)\right)\exp\left(-\frac{\mu^2-1}{2\sigma^2}\right)\times\\\int_{-\infty}^{\infty}\exp\left(-\frac{1+t}{2\sigma^2}\left(x-\frac{\mu}{1+t}\right)^2\right)\mathrm{d}x.\end{aligned}$$

Now, using (3) with $a=(1+t)/(2\sigma^2), b=\mu/(1+t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t)=-\frac{1}{2\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\left((1+t)-\frac{\mu^2}{1+t}\right)\right)\exp\left(-\frac{\mu^2-1}{2\sigma^2}\right)\sqrt{\frac{2\pi\sigma^2}{1+t}}. \tag{33}$$

Hence

$$I(t)=-\sqrt{\frac{\pi}{2\sigma^2}}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\int\exp\left(-\frac{1}{2\sigma^2}\left((1+t)-\frac{\mu^2}{1+t}\right)\right)\sqrt{\frac{1}{1+t}}\,\mathrm{d}t. \tag{34}$$

To calculate the last integral, we can use substitution $s=(1+t)/(2\sigma^2)$, i.e. $1+t=2\sigma^2 s$, which gives

$$\begin{aligned}I(t)&=-\sqrt{\frac{\pi}{2\sigma^2}}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\int\exp\left(-s+\frac{\mu^2}{4\sigma^4 s}\right)\sqrt{\frac{1}{s}}\sqrt{2\sigma^2}\,\mathrm{d}s\\&=-\sqrt{\pi}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\int\exp\left(-s+\frac{\mu^2}{4\sigma^4 s}\right)\sqrt{\frac{1}{s}}\,\mathrm{d}s\\&=-\sqrt{\pi}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\int\exp\left(-s+\frac{m^2}{s}\right)\sqrt{\frac{1}{s}}\,\mathrm{d}s,\end{aligned}$$

where $m=\mu/(2\sigma^2)$.

Comparing with (22) and (26), we can see that we have already solved this integral and the result is

$$I(t) = -\sqrt{\pi}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\left(\exp(2\mathrm{j}m)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{\frac{1+t}{2\sigma^2}}+\frac{\mathrm{j}m\sqrt{2\sigma^2}}{\sqrt{1+t}}\right)+\right. \tag{35}$$
$$\left.\exp(-2\mathrm{j}m)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{\frac{1+t}{2\sigma^2}}-\frac{\mathrm{j}m\sqrt{2\sigma^2}}{\sqrt{1+t}}\right)\right) =$$
$$-\sqrt{\pi}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\left(\exp\left(\frac{\mathrm{j}\mu}{\sigma^2}\right)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{\frac{1+t}{2\sigma^2}}+\frac{\mathrm{j}\mu}{\sqrt{2\sigma^2(1+t)}}\right)+\right.$$
$$\left.\exp\left(-\frac{\mathrm{j}\mu}{\sigma^2}\right)\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{\frac{1+t}{2\sigma^2}}-\frac{\mathrm{j}\mu}{\sqrt{2\sigma^2(1+t)}}\right)\right)+c.$$

Thanks to (32) and (8) we get

$$0 = -\frac{\pi}{2}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\left(\exp\left(\frac{\mathrm{j}\mu}{\sigma^2}\right)+\exp\left(-\frac{\mathrm{j}\mu}{\sigma^2}\right)\right)+c$$

and thus

$$c = \frac{\pi}{2}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\left(\exp\left(\frac{\mathrm{j}\mu}{\sigma^2}\right)+\exp\left(-\frac{\mathrm{j}\mu}{\sigma^2}\right)\right) = \pi\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\cos\frac{\mu}{\sigma^2}. \tag{36}$$

The aim was to determine $I(0)$. Substituting (36) for c and $t = 0$ into (35), we get

$$I(0) = \frac{\pi}{2}\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\left(2\cos\frac{\mu}{\sigma^2}-\exp\left(\frac{\mathrm{j}\mu}{\sigma^2}\right)\mathrm{erf}\left(\frac{1+\mathrm{j}\mu}{\sqrt{2\sigma^2}}\right)-\exp\left(-\frac{\mathrm{j}\mu}{\sigma^2}\right)\mathrm{erf}\left(\frac{1-\mathrm{j}\mu}{\sqrt{2\sigma^2}}\right)\right)$$

and thanks to (6) we get the final result

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\mathrm{d}x = I(0) = \tag{37}$$
$$\pi\exp\left(\frac{1-\mu^2}{2\sigma^2}\right)\left(\cos\frac{\mu}{\sigma^2}-\mathrm{Re}\left(\exp\left(\frac{\mathrm{j}\mu}{\sigma^2}\right)\mathrm{erf}\frac{1+\mathrm{j}\mu}{\sqrt{2\sigma^2}}\right)\right).$$

3 Graphs

In figure 2 we can see three examples of the integrated function a the corresponding results of integral (1).

CONCLUSION

With help of the so called Feynman technique, we are able to calculate some of integrals, that could not be found using traditional methods of integration. Here we showed one example but this method has a wider range of application.

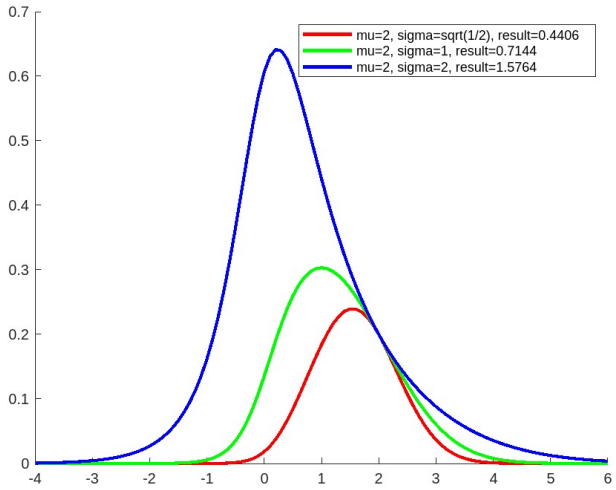

Figure 2: Graphs for $\mu = 2$ and various values of σ (Source: own)

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METHODOLOGY OF TEACHING THE PROPERTIES OF A TRIANGLE USING GEOGEBRA CLASSROOM

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Abstract: *The first part of the article deals with a new key competence in the stage of basic education. It records the setting of teaching in a specific primary school for the fulfilment of digital competences in the educational area of Mathematics and its applications. The practical part deals with the teaching of the triangle curriculum using dynamic mathematical software. It includes the methodology of creating individual applets in an activity that will be entered in GeoGebra Classroom. It also includes suggestions from teaching in mathematics lessons at primary school and possible evaluation of the pupil's activities.*

Keywords: GeoGebra Classroom, triangle, Geogebra.org, digital competence of teachers and pupils.

INTRODUCTION

Digital competences in primary schools are essential to prepare pupils for the demands of the modern digital world, which is crucial not only for their future professional success, but also for their personal development. These competencies cover a wide range of skills from basic computer operation to advanced use of information and communication technologies (ICT). Updating the school education programme is essential for integrating these skills into the school curriculum. The implementation of changes in curricular documents in all primary schools represents a major step forward in integrating modern technologies into the educational process and is essential for the effective teaching of digital competences.

1 TEACHING MATHEMATICS USING DIGITAL DEVICES

The following text provides an insight into the practical aspects of teaching in a particular primary school and can serve as an inspiration for staff rooms, members of subject committees for mathematics, ICT methodologies, as well as for school management. A number of tools and strategies that this school has implemented in order to effectively use digital competences during teaching are presented. The way in which applications and modern technologies are incorporated into the educational process in the field is described in detail. The use of these set tools, technologies and applications enables the fulfilment of the objectives set out in valid curricular documents [4] and supports modern approaches to teaching.

Within the educational area of Mathematics and its applications, systematic teaching using digital devices is introduced at school from the third grade. Pupils have computer labs with desktop PCs or iPads directly in their classrooms. Each student has their own personal access account to the school domain, which allows them to log in to the school's computer network, use Microsoft 365 cloud services, and communicate via school email. Pupils also use the school information system Škola OnLine.

From the fourth year onwards, students are equipped with the skills needed to use these platforms and applications independently. By the end of the fifth grade, pupils have acquired

skills such as assessing the completeness of data, searching for information, creating simple tables and graphs using the Excel platform. They draw data from websites such as the website of the Czech Statistical Office or other sources. To solve practical or verbal tasks and problems, they use standard operating system tools such as a calculator or special applications on iPads.
This integrated approach to digital education leads to the development of key digital competences that are essential for learners' success in an increasingly digitalised world.
In the second stage of primary school, pupils learn to effectively use digital tools such as Excel, the operating system calculator, as well as advanced symbolic calculation systems such as Microsoft Math Solver and Wolfram Alpha. These tools allow them to delve deeper into abstract mathematical concepts and solve mathematical problems effectively.
In the area of Dependencies, Relationships and Working with Data, pupils will learn to manipulate data using various spreadsheets, use data from state authorities, perform calculations using formulas and sort data according to various criteria. This skill is crucial for developing pupils' analytical skills and preparing them for further academic and professional use.
For the thematic field of Geometry in plane and space, the use of dynamic mathematical software GeoGebra is introduced, which allows students to interactively explore geometric concepts and solve problems corresponding to their year. In the field of spatial geometry, lessons are taught using modelling software such as Thinkercad or Sketchup, supplemented by the 3D Waterhouse module for practical modelling. These modern technologies allow pupils not only to better visualize and understand geometric shapes, but also to develop creative and project skills in the design and implementation of their own 3D models. As part of the research, pupils repeatedly participated in research surveys in this area. Research on pupils' information thinking recorded the state of pupils' cognition with the support of digital devices [3].

2 CURRICULUM TRIANGLE WITH THE APPLICATION OF DIGITAL COMPETENCES OF TEACHERS AND PUPILS

The practical part deals with specific teaching in the 6th grade at the 31st Elementary School in Pilsen in the thematic field of Geometry in plane and space in the educational area of Mathematics and its applications. According to [6], the age period of pupils in the 6th grade is crucial for acquiring basic knowledge and skills in the field of information and communication technologies (ICT). This chapter formulates specific requirements for teachers' pedagogical skills that are necessary for effective teaching of this subject. In addition to the theoretical foundations, there are specific requirements for the creation and modification of interactive applets, which are an integral part of the modern approach to geometry teaching. Furthermore, the chapter describes in detail the methodology that is suitable for involving pupils and making their learning process more effective. This is a good time to implement the outputs [1]. Special attention is paid to the practical implementation of teaching and the specific work of pupils, which allows for a better understanding of how to turn theoretical principles into practical skills.

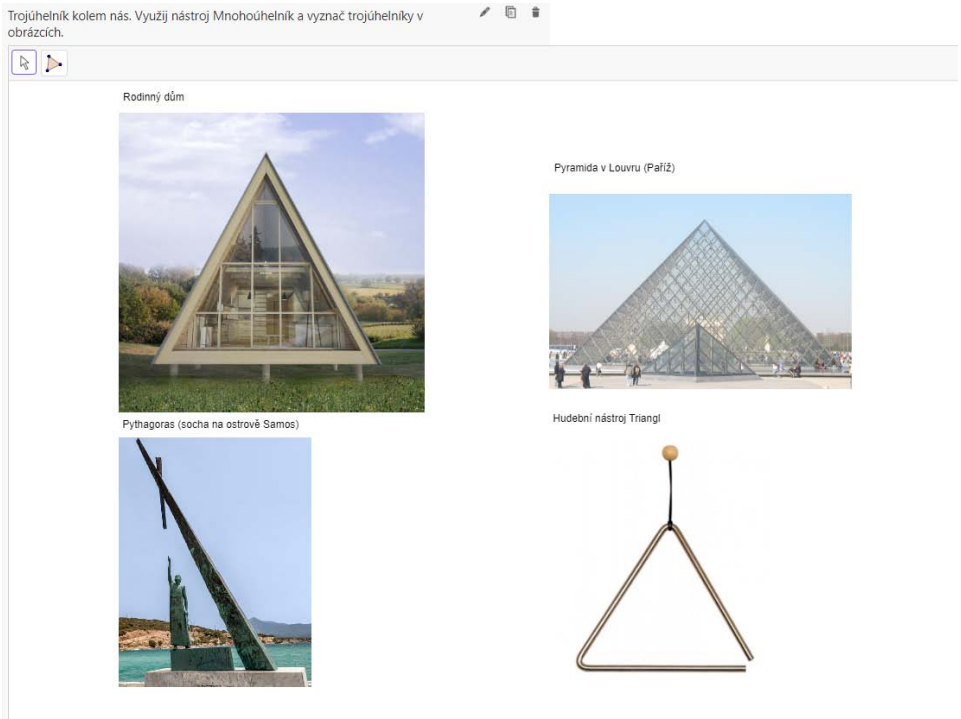
2.1 Digital Competence Framework for Educators

For the effective creation and modification of applets in dynamic mathematical software, it is necessary for mathematics teachers to have digital competencies. As a theoretical basis for integrating digital technologies into the educational process, the DigCompEdu framework has been developed by the European Commission, which identifies six key areas of digital competence that teachers should control. These competencies include the ability to create and

edit digital content, effective digital communication and collaboration, understanding the use of digital technologies and platforms with an emphasis on cybersecurity and data privacy, the creation of mathematical tasks and scenarios that support critical thinking and problem-solving, and finally the ability to use digital technologies to individualize teaching, track progress and provide feedback [2].
The DigCompEdu framework further defines competence levels, from novice (A1) to explorer (A2) and practitioner (B1) to expert (B2), leader (C1) and pioneer (C2), allowing educators to objectively assess and develop their digital skills. This audit can be carried out on the website RVP.CZ methodological portal under the name Profile Učitel 21, where teachers can identify their current competencies and set goals for their further development. Such an approach not only supports teachers' professional development, but also contributes significantly to a significant improvement in the quality of mathematics education using digital tools and technologies [1].
Each teacher can audit their own digital competences in a set scale on the website of the RVP.CZ methodological portal. – Profile Učitel 21.

2.2 Activity settings, creation and methodology of applets

Before starting classes using the GeoGebra Classroom platform, it is necessary to carefully prepare and set up individual activities so that they form a coherent set of tasks corresponding to the subject matter. Educators should actively use the Geogebra.org web platform to create, store, and share math projects and educational materials in the cloud.
To ensure the effective use of this platform, a demonstration activity for pupils was prepared, which included a series of graded tasks on the topic of triangle. These tasks were designed to be accessible to all pupils, while for more gifted students, extension examples were provided to provide additional challenges after completing the basic tasks.
The methodology of setting up these activities includes an emphasis on the use of dynamic software functions to support pupils' skills and knowledge as much as possible. A detailed description of tasks and activities allows the teacher to fully exploit the potential of GeoGebra Classroom and at the same time ensures that pupils acquire the key concepts and skills needed for successful mastery of the curriculum.
Task set1: Triangle Sorting
The introductory task includes four pictures from the world around us. The pupil has only one *Polygon tool* in the toolbar displayed. Using this tool, he marks the basic element of the picture – a triangle. Edits its colour. The task is supplemented by the Question section. An open question is a record of the state of the task for the pupil. In the text, the pupil must answer what geometric shape he or she has constructed.
Digital competence of the teacher for setting up the applet:
The teacher chooses suitable pictures from the world around us to create this applet. Selects an image under a Creative Commons (CC BY) license. Inserts objects into a file, anchors, fixes. In the advanced settings, it is advisable to consider the screen resolution (1920 x 1080), display the icon for resetting the structure and displaying the toolbar in which the toolbar is set and only the selected tools are selected. Add a question section, edit a question and select an open question.

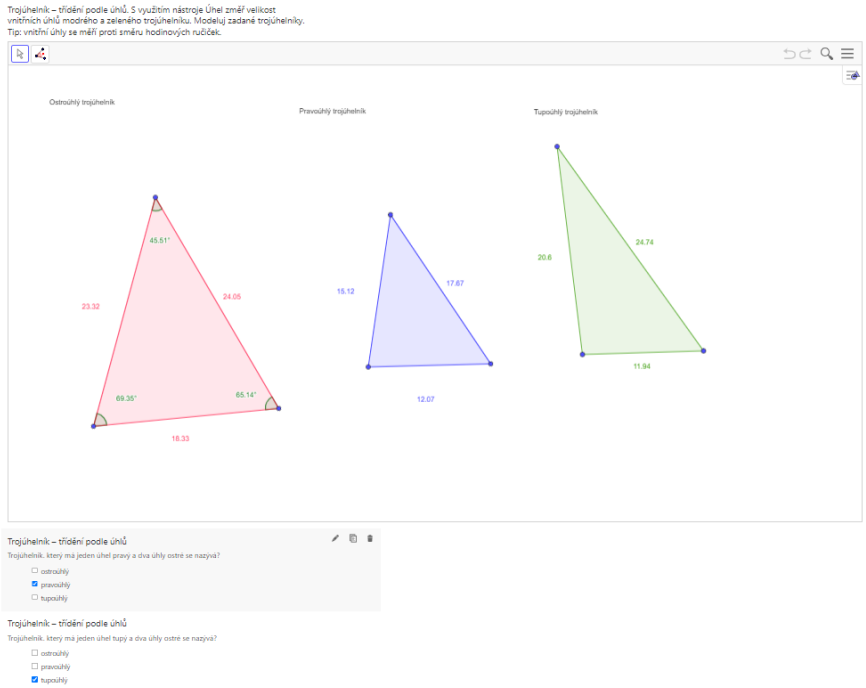


Picture. 1 Use the Polygon tool
Source: own

Note: The screen resolution is the same for all other tasks. It depends on the technical equipment of the computer room in which the teaching will take place.

Task set 2: Triangle – sorting by sides and angles

The next task includes a set of examples for sorting triangles. The problems culminate in the difficulty of the solution. In the first task, the pupils are assigned three triangles and the lengths of the sides of the triangles are displayed in them. In his task, the pupil models the given figure according to the assignment. Only *the Pointer* tool is available. In the second task, he models a triangle according to angles. For two triangles, it measures the magnitudes of the angles using the available Angle, respecting the recommendations in the assignment [5]. This occurs in this skill shift from manipulating objects to getting data about objects. The set ends with a set of questions that the pupil answers in the context of the acquired knowledge of the given tasks.



Picture. 2 Triangle – sorting by sides
Source: own

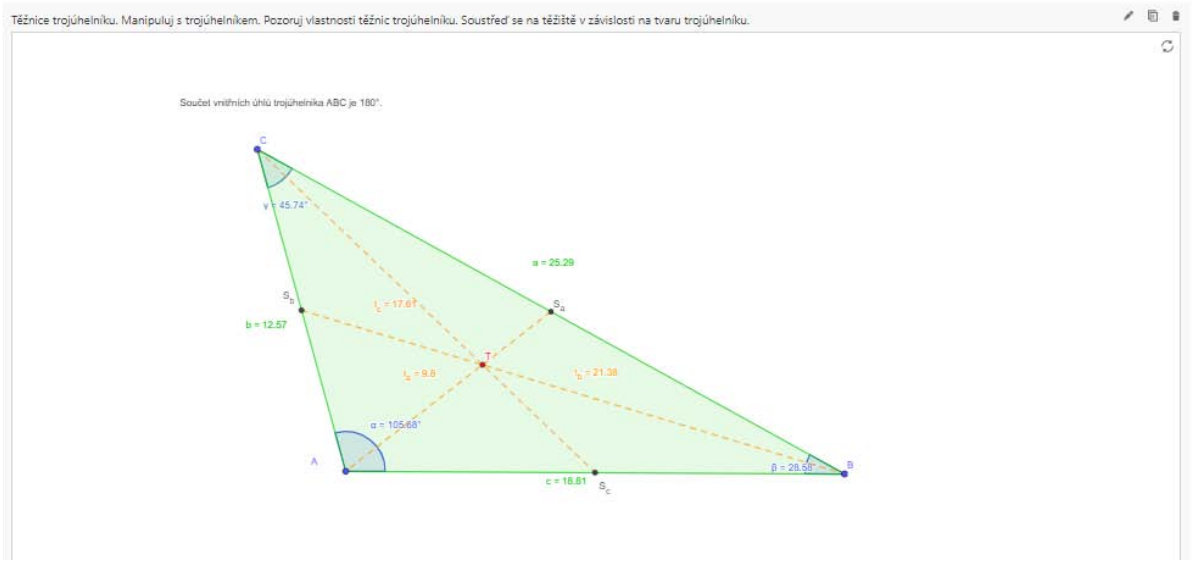
Digital competence of the teacher for setting up the applet:
To create these applets, the teacher creates a set of triangles with the required properties, where the lengths of the sides are displayed, the sizes of angles are displayed only for the first triangle. In the settings, it chooses the colours it uses as text markings in the assignment. It will only make available to pupils the tools that are necessary to solve the given task. In the advanced settings, it only displays the icon for resetting the structure in the first task. In the second task, the menu (tool limitation set in the applet). Adds a question section, edit question and choose open question.

Task set 3: Heights and centroids of a triangle

The next problem includes a set of problems for the heights and centroids of a triangle. In these tasks, the pupil always manipulates objects and obtains information about the properties of the given figure. The first tasks are for the properties of the heights of the triangle, the second tasks for the properties of the centroids of the triangle. The key is to observe the change in modelling the acute, right-angled, and obtuse-angled triangles. Subsequently, he creates his own figure using tools. In these tasks, the pupil has all the tools at his disposal. In the construction, he uses tools and procedures that correspond to traditional drawing from mathematics lessons and well-known drawing aids (triangle with a line, ruler, compass). Individually, the sets end with a set of questions, the answers to which are in the context of the knowledge gained by the knowledge of these tasks.

Digital competence of the teacher for setting up the applet:
To create the first applet of this task, the teacher constructs the heights (centroids) of the triangle. Appropriately edits their properties, designations, and types of lines so that they correspond to the assigned designation in the lesson. It only gives pupils the opportunity to use some tools. He/she uses his/her didactic prerequisites for teaching so that the figure is clear for the pupil, marked in the context of his/her knowledge.
In the advanced settings of the first task (heights, centroids), it only shows the reset of the structure. In the second applet, the pupil has prepared a triangle (construction of heights,

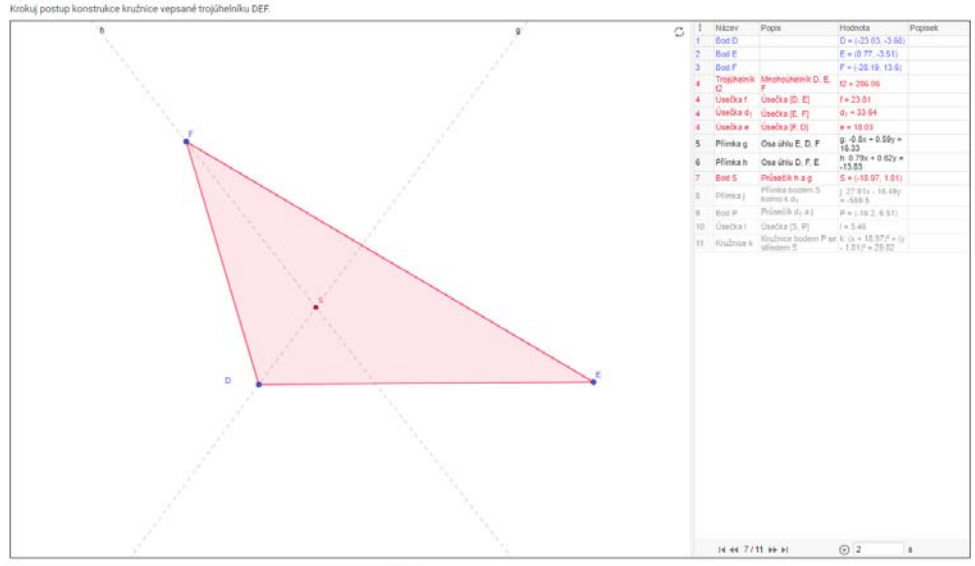
centroids) including interior angles. In this task, the student has a complete menu of tools at his disposal.



Picture. 3 Triangle – centre of gravity of the triangle, manipulation with the figure
Source: own

Task set 4: Circle inscribed and circumscribed by a triangle

The set includes a set of circumscribed and inscribed circles for modelling, the procedure for constructing both circles, and triangles for constructing a circumscribed and inscribed triangle circle. In the first task, the pupil has the Pointer tool and the ABC Text tool at his disposal. Using the above tools, he performs the first task. Other tasks escalate. The pupil has at his disposal the procedure of constructing a circle circumscribed (inscribed) of a triangle. It has the option to freely display the steps of the construction process. The acquired knowledge will be used in the actual design in the noon task. Here, using all tools, he constructs a circle circumscribed (inscribed) of a triangle into a triangle. The tasks are chosen by complementing the construction tasks that the pupils created using traditional drawing tools.



Picture. 4 Triangle – inscribed circle construction procedure
Source: own

Digital competence of the teacher for setting up the applet:
For the creation of the first applet of this task, the teacher constructs a circle inscribed (circumscribed) by a triangle. Appropriately edits their properties, designations, and types of lines so that they correspond to the assigned designation in the lesson. It only gives pupils the opportunity to manipulate the figures and determine the position of the centre of both circles depending on the shape of the triangles. He/she uses his/her didactic prerequisites for teaching so that the figure is clear for the pupil, marked in the context of his/her knowledge.
In the advanced settings of the second task (the construction progress of both circles), it only displays a menu to display the construction progress. In the last applet of this set, the pupil has triangles ready. In this task, the student has a complete menu of tools at his disposal.

3 CONDITIONS FOR IMPLEMENTATION AND VERIFICATION OF THE CURRICULUM

The case study was attended by pupils of the 6th grade (VI. B and VI. C) in a total of forty-three pupils. The pupils had prepared tasks in GeoGebra Classroom. The sharing link was stored on the school's network drive. The reader can use the link. After activation, it is possible to try out the described role of the pupil and perform the tasks https://www.geogebra.org/classroom/bzv8p3vf.
The role of the teacher in this phase of teaching, which is conducted in the computer classroom, is the role of an advisor and mentor. He monitors the work of pupils and is an advisor to partial tasks. The interactive board shows the activity of the pupils' work (it is recommended to hide the names at this stage).

3.1 Evaluation of pupils' activities, feedback

Based on the teaching experience, the work is evaluated in the next hour. The teacher uses a display device in the classroom and demonstrates to the pupils their activities, successfully solved tasks and mistakes they made in individual tasks. The activity is conducted with hidden names. At the same time, it immediately models the right solutions. The assessment is formative, leading to an overall view of the subject matter. It then generates a new Classroom for pupils. The link is made available to pupils through the school information system and for voluntary home practice.

CONCLUSION

The above example of good practice provides a view of specific solutions to tasks at the end of the triangle curriculum in the 6th grade of primary school using digital devices. It is a unique view of practical involvement in teaching in the Czech Republic. It can be stated that the teacher must be able to work effectively with digital technologies in order to fully exploit the potential of these tools in education. The integration of digital technologies into teaching then brings several benefits, such as student motivation, personalised learning, and preparation for digital skills essential for today's world of work. In this way, educators contribute to a better and more engaging learning environment for their pupils.

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ENLARGEMENT AND ALGEBRAIZATION OF THE SYSTEM OF CHAINS OF GROUPS OF DIFFERENTIAL NEURONS

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Abstract: *In the contribution there are used artificial neurons which are basic stones of neural networks. We treat certain properties of constructed algebraic structures of artificial neurons. In particular, we construct chains of differential neurons and hypergroupoids using proximity spaces. Moreover there is used the theory of presheaves which belongs to the important part of general topology.*

Keywords: interval arithmetic, artificial differential neuron, proximity space.

INTRODUCTION

We denote by $\mathfrak{I}_n(\mathbb{R})$ the system of all bounded open intervals (a,b) of the set $\mathbb{R}$ of all real numbers – cf. [8, 9, 12, 14, 16]. For a pair of intervals $(a,b),(c,d)\in\mathfrak{I}_n(\mathbb{R})$ we define the sum $(a,b)+(c,d)=(a+c,b+d)$ and the binary relation $\leq$ by $(a,b)\leq(c,d)$ whenever $a\leq c, b\leq d$. It is in paper we denote $\mathfrak{I}n_0(\mathbb{R})=\mathfrak{I}_n(\mathbb{R})\cup\{0\}$, with $0\in\mathbb{R}$. Then the structure $(Jn_0(\mathbb{R}),+,\leq)$ is an ordered additive commutative monoid – cf. [5, 9]. Properties of the structure can be described by using basic properties of the "Interval Arithmetic ".

We define a binary operations on the monoid $(\mathfrak{I}n_0(\mathbb{R}),+,\leq)$ using centers of intervals: Suppose $(a,b),(c,d)\in\mathfrak{I}n_0(\mathbb{R})$, where $(a,b)\neq 0\neq(c,d)$. We put $(a,b)\cdot(c,d)=(p,q)$, where

$$p=\min\left\{\frac{a+b}{2},\frac{c+d}{2}\right\}, q=\max\left\{\frac{a+b}{2},\frac{c+d}{2}\right\}$$

and $(a,b)\cdot(c,d)=0$ if $a+b=c+d$, i.e. if centers of intervals (a,b), (c,d) coincides. In the case if $q<p$, we define $(a,b)\cdot(c,d)=(q,p)$. Further we define $(a,b)\cdot 0=0=0\cdot(a,b)$ for each interval $(a,b)\in\mathfrak{I}n_0(\mathbb{R})$.

The binary operation "$\cdot$" is evidently commutative, however it is not associative, which follows from this example:

Example 1.

$$(10,12)\cdot((2,4)\cdot(8,16)) = (10,12)\cdot(3,12) = (7.5,12) \neq$$
$$\neq (7,12) = (6,8)\cdot(8,16) = ((10,12)\cdot(2,4))\cdot(8,16),$$

or

$$((2,4)\cdot(2,8))\cdot(4,8) = (3,5)\cdot(4,8) = (4,6) \neq$$
$$\neq (3,5.5) = (2,4)\cdot(5,6) = (2,4)\cdot((2,8)\cdot(4,8)).$$

The last fact can be verified in quite general setting.

Let us recall some basic terms concerning artificial differential neurons. As there is mentioned in papers from references artificial or formal neurons are basic simple structures for mathematically modeled neural networks – [2, 3, 4, 5, 6, 7, 8, 9, 10]. Artificial neural networks have the structure, derived from observations of biological neurons which are basic building blocks of biological neural networks.

In what follows we will consider a certain generalization of classical artificial neurons mentioned above such that inputs x_i and weight w_i will be functions of an argument t belonging into a linearly ordered (tempus) set T with the least element 0. As the index set we use the set $\mathbb{C}(J)$ of all continuous functions defined on an open interval $J \subset \mathbb{R}$. So, denote by W the set of all non-negative functions $w : T \to \mathbb{R}$ forming a subsemiring of the ring of all real functions of one real variable $x : \mathbb{R} \to \mathbb{R}$. Denote by $Ne(\vec{w}_r) = Ne(w_{r1}, \ldots, w_{rn})$ for $r \in \mathbb{C}(J)$, $n \in \mathbb{N}$ and the mapping

$$y_r(t) = \sum_{k=1}^{n} w_{r,k}(t)x_{r,k}(t) + b_r$$

which will be called the artificial neuron with the bias $b_r \in \mathbb{R}$. By $\mathbb{AN}(T)$ we denote the collection of all such artificial neurons.

Neurons are usually denoted by capital letters X, Y or X_i, Y_i, nevertheless we use also notion $Ne(\vec{w})$, where $\vec{w} = (w_1, \ldots, w_n)$ is the vector of weights.

We suppose - for the sake of simplicity - that transfer functions (activation functions) φ, σ (or f) are the same for all neurons from the collection $\mathbb{AN}(T)$ or the role of this function plays the identity function $f(y) = y$.

Now, similarly as in the case of the collection of linear differential operators above [6], we will construct a group of artificial neurons. Concerning the concept of a hypergroup, see e.g. [2, 3, 4].

Denote by δ_{ij} the so called Kronecker delta, $i, j \in \mathbb{N}$, i.e. $\delta_{ii} = \delta_{jj} = 1$ and $\delta_{ij} = 0$, whenever $i \neq j$.

Suppose $Ne(\vec{w}_r), Ne(\vec{w}_s) \in \mathbb{AN}(T)$, $r, s \in \mathbb{C}(J)$, $\vec{w}_r = (w_{r1}, \ldots, w_{r,n})$, $\vec{w}_s = (w_{s1}, \ldots, w_{s,n}), n \in \mathbb{N}$. Let $m \in \mathbb{N}$, $1 \leq m \leq n$ be a such an integer that $w_{r,m} > 0$. We define

$$Ne(\vec{w}_r) \cdot_m Ne(\vec{w}_s) = Ne(\vec{w}_u),$$

where

$$\vec{w}_u = (w_{u,1}, \ldots, w_{u,n}) = (w_{u,1}(t), \ldots, w_{u,n}(t)),$$
$$\vec{w}_{u,k}(t) = w_{r,m}(t)w_{s,k}(t) + (1 - \delta_{m,k})w_{r,k}(t), t \in T$$

and, of course, the neuron $Ne(\vec{w}_u)$ is defined as the mapping $y_u(t) = \sum_{k=1}^{n} w_k(t)x_k(t) + b_u, t \in T$, $b_u = b_r b_s$. Further for a pair $Ne(\vec{w}_r), Ne(\vec{w}_s)$ of neurons from $\mathbb{AN}(T)$ we put

$$Ne(\vec{w}_r) \leq_m Ne(\vec{w}_s), w_r = (w_{r,1}(t), \ldots, w_{r,n}(t)), w_s = (w_{s,1}(t), \ldots, w_{s,n}(t))$$

if $w_{r,k}(t) \leq w_{s,k}(t), k \in \mathbb{N}, k \neq m$ and $w_{r,m}(t) = w_{s,m}(t), t \in T$ and with the same bias.

1 Proximity spaces

Given a set P, a binary relation p for $expP$ will be called a proximity relation or a proximity for the set P if it satisfies these conditions:

(prox1) $\emptyset nonpP$ (the relation $(expP \times expP) - p$ is denoted by $nonp$),

(prox2) p is symmetric, i.e. XpY implies YpX,

(prox3) $X \subset P, Y \subset P, X \cap Y \neq \emptyset$ implies XpY,

(prox4) if $X_1 \subset P, X_2 \subset P$ then $(X_1 \cup X_2)pY$ if and only if X_1pY or X_2pY.

If a relation p is a proximity for a set P then the pair (P, p) is called a proximity space and if $X \subset P, Y \subset P$ and XpY, then X and Y are said to be proximal in the space (P, p) or under p. If $XnonpY$, then X and Y are said to be distant or non-proximal in (P, p).

Let us formulate and prove the following simple proposition [1, 25.A.3, p. 441]:

Proposition 1. *Suppose that the relation p is a proximity for a set P. Then*

(a) $X \subset Y \subset P, XpZ$ *implies* YpZ.

(b) If $\{X_i\}$ *and* $\{Y_j\}$ *are finite families in* $expP$ *such that* $(\bigcup_i \{X_i\})p(\bigcup_j \{Y_j\})$, *then* $X_i pY_j$ *for some* i *and* j.

Proof.

I. Statement (a) follows from (prox4); $XpZ, Y \subset P$ imply $(X \cup Y)pZ$ by (prox4), and $X \subset Y$ implies $X = X \cup Y$.

II. By induction it follows from (prox4) that, for each finite family $\{X_i\}$, $(\bigcup_i \{X_i\})pY$ implies $X_i pY$ for some i, and by (prox2)(the symmetry), $Yp(\bigcup_i \{X_i\})$ implies $X_i pY$ for some i. Hence if $\{X_i\}$ and $\{Y_j\}$ are finite families such that $(\bigcup_i \{X_i\})p(\bigcup_j \{Y_j\})$, then $(\bigcup_i \{X_i\})pY_j$ for some j and this implies $X_i pY_j$ for some i.

□

2 Construction of two kinds of proximity relations

Suppose A,B are non-empty sets and $\mathbb{K} = \{K_\alpha; \alpha \in A\}, \mathbb{L} = \{L_\beta; \beta \in B\}$ are collections of intervals $K_\alpha, L\beta \in \mathfrak{I}n_0(\mathbb{R})$ for any pair of indices $\langle \alpha, \beta \rangle \in A \times B$. We define $\mathbb{K}p_1\mathbb{L}$ if there exists a pair of intervals of real numbers $\langle K_\alpha, L_\beta \rangle \in \mathbb{K} \times \mathbb{L}$ such that $K_\alpha \varkappa L_\beta$ or $\mathbb{K} \cap \mathbb{L} \neq \emptyset$. (For definition of relation $\varkappa$ see section 4.) Further, denote $\mathbb{C} = \{\langle \mathbb{LA}(K_\alpha), F_n^{(\alpha)}; n \in \mathbb{N}_0 \rangle; \alpha \in A\}$ and $\mathbb{D} = \{\langle \mathbb{LA}(L_\beta), F_n^{(\beta)}; n \in \mathbb{N}_0 \rangle; \beta \in B\}$. Then we define $\mathbb{C}p_C^{(1)}\mathbb{D}$ whenever $\mathbb{K}p_1\mathbb{L}$ holds.

Proposition 2. *Binary relations $p_1, p_C^{(1)}$, where $p_1 \subset (exp\mathfrak{I}n_0(\mathbb{R}) \times \mathfrak{I}n_0(\mathbb{R}))$, $p_1^{(C)} \subset (exp\mathbb{CLA} \times exp\mathbb{CLA})$, satisfy axioms (prox1) – (prox4) of a proximity, thus structures $(\mathfrak{I}n_0(\mathbb{R}), p_1)$, $(\mathbb{CLA}, p_C^{(1)})$ are proximity spaces.*

Proof. We should verify that relations $p_1, p_C^{(1)}$ satisfy (prox1) – (prox4).

Consider the relation p_1, first. Evidently, $\emptyset non p_1 \mathfrak{I}n_0(\mathbb{R})$. Suppose $\mathbb{K}p_1\mathbb{L}$, which means that there exists a pair of intervals $\langle K, L \rangle \in \mathbb{K} \times \mathbb{L}$ with the property $K \cap (a,b) \neq \emptyset \neq L \cap (a,b)$ for some interval $(a,b) \in \mathfrak{I}n_0(\mathbb{R})$ of the length $\omega(a,b) = 1$ or that there exists an interval $J \in \mathbb{K}$ and also $J \in \mathbb{L}$, i.e. $\mathbb{K} \cap \mathbb{L} \neq \emptyset$. Then also $\mathbb{L}p_1\mathbb{K}$, thus the relation p_1 is symmetric, hence the condition (prox2) is satisfied. Further, if $\mathbb{K} \cap \mathbb{L} \neq \emptyset$, then with respect to the definition of the relation p_1 we have $\mathbb{K}p_1\mathbb{L}$, i.e. condition (prox3) is satisfied, as well. Now, consider a collection $\mathbb{V} = \{V_\gamma; \gamma \in C\}, C \neq \emptyset$, satisfying $(\mathbb{K} \cup \mathbb{L})p_1\mathbb{V}$. Let $K \in \mathbb{K} \cup \mathbb{L}, L \in \mathbb{V}$ be intervals such that $K \varkappa L$ or there is an interval J with the property $J \in \mathbb{K} \cup \mathbb{L}$ and $J \in \mathbb{V}$. In the first eventuality $K \cap (a,b) \neq \emptyset \neq L \cap (a,b)$ for some interval $(a,b) \in \mathfrak{I}n_0(\mathbb{R})$ of the length 1. Since $K \in \mathbb{K}$ or $K \in \mathbb{L}$ and $L \in \mathbb{V}$ then either $\mathbb{K}p_1\mathbb{V}$ or $\mathbb{L}p_1\mathbb{V}$. In the second case similarly $J \in \mathbb{K} \cap \mathbb{V}$ or $J \in \mathbb{L} \cap \mathbb{V}$, thus either $\mathbb{K}p_1\mathbb{V}$ or $\mathbb{L}p_1\mathbb{V}$.

Now, on the contrary, suppose either $\mathbb{K}p_1\mathbb{V}$ or $\mathbb{L}p_1\mathbb{V}$. Then either the system $\mathfrak{I}n_0(\mathbb{R})$ contains an interval (a,b) of the length $\omega(a,b) = 1$ and $K \cap (a,b) \neq \emptyset \neq L \cap (a,b)$ for some pair of intervals $\langle K, L \rangle \in \mathbb{K} \times \mathbb{V}$ or $\langle K, L \rangle \in \mathbb{L} \times \mathbb{V}$. Then we have $\langle K, L \rangle \in (\mathbb{K} \times \mathbb{V}) \cup (\mathbb{L} \times \mathbb{V}) = (\mathbb{K} \cup \mathbb{L}) \times \mathbb{V}$, thus $(\mathbb{K} \cup \mathbb{L})p_1\mathbb{V}$ holds. If $\mathbb{K} \cap \mathbb{V}) \neq \emptyset$ or $\mathbb{L} \cap \mathbb{V}) \neq \emptyset$ then

$$(\mathbb{K} \cup \mathbb{L}) \cap \mathbb{V} = (\mathbb{K} \cap \mathbb{V}) \cup (\mathbb{L} \cap \mathbb{V}) \neq \emptyset.$$

Consequently the condition (prox4) is also satisfied, hence $(\mathfrak{I}n_0(\mathbb{R}), p_1)$ is a proximity space.

Now consider the above defined mapping:

$$\Phi : \mathfrak{I}n_0(\mathbb{R}) \to \mathbb{CLA},$$

where $\Phi(J) = \langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}$ for any interval $J \in \mathfrak{I}n_0(\mathbb{R})$. Suppose that $\mathbb{C} = \{\langle \mathbb{LA}(J_\alpha), F_n^{(\alpha)}; n \in \mathbb{N}_0 \rangle; \alpha \in A\}, \mathbb{D} = \{\langle \mathbb{LA}(J_\beta), F_n^{(\beta)}; n \in \mathbb{N}_0 \rangle; \beta \in B\}$ are two arbitrary collections of chains of groups of differential operators, i.e. subsets of the systems $\mathbb{CLA}$. Consider sets $\mathbb{A} = \{J_\alpha; \alpha \in A\}, \mathbb{B} = \{J_\beta; \beta \in B\}$ of intervals from the system $\mathfrak{I}n_0(\mathbb{R})$ such that $\mathbb{A}p_1\mathbb{B}$. Or we can say if $\mathbb{A}p_1\mathbb{B}$ then we put $\mathbb{C}p_C^{(1)}\mathbb{D}$.

We show that prescriptions defines a proximity for the systems $\mathbb{CLA}$. In fact the mapping Φ is bijection, so we have

$$\{J_\alpha;\alpha\in A\}p_1\{J_\beta;\beta\in B\} \text{ iff } \Phi(\{J_\alpha;\alpha\in A\})p_C^{(1)}\Phi(\{J_\beta;\beta\in B\}).$$

Since $\emptyset nonP$ then also $\emptyset nonp\mathbb{CLA}$ hence (prox1) is satisfied by $p_C^{(1)}$. Similarly $p_C^{(1)}$ is also satisfied. Further, according to definition the proximity space $(\Im n_0(\mathbb{R}),p_1)$ (i.e. in consequence of the implication $\{J_\alpha;\alpha\in A\}\cap\{J_\beta;\beta\in B\}\neq\emptyset \implies \{J_\alpha;\alpha\in A\}p_1\{J_\beta;\beta\in B\}$ we have that the relation $p_C^{(1)}$ satisfies the condition (prox3).

Now suppose that $\mathbb{A},\mathbb{B},\mathbb{C}\subset\mathbb{CLA}$ are collections of chains satisfying $(\mathbb{A}\cup\mathbb{B})p_C^{(1)}\mathbb{C}$. Consider sets of intervals $\{J_\alpha;\alpha\in A\}$, $\{J_\beta;\beta\in B\}$, $\{J_\gamma;\gamma\in C\}$ such that $\Phi(\{J_\alpha;\alpha\in A\})=\{\Phi(J_\alpha);\alpha\in A\}=\mathbb{A}$ and similarly $\{\Phi(J_\beta);\beta\in B\}=\mathbb{B}$, $\{\Phi(J_\gamma);\gamma\in C\}=\mathbb{C}$. Then $\Phi^{-1}(\mathbb{A}\cup\mathbb{B})=\Phi^{-1}(\mathbb{A})\cup\Phi^{-1}(\mathbb{B})p_1\Phi^{-1}(\mathbb{C})$, which implies $\Phi^{-1}(\mathbb{A})p_1\Phi^{-1}(\mathbb{C})$ or $\Phi^{-1}(\mathbb{B})p_1\Phi^{-1}(\mathbb{C})$. Since $(\Im n_0(\mathbb{R}),p_1)$ is a proximity space (satisfying the condition (prox4)), we obtain consequences $\mathbb{A}\cup\mathbb{B}p_C^{(1)}\mathbb{C}$, thus $\Phi(\Phi^{-1}(\mathbb{A}\cup\mathbb{B}))p_C^{(1)}\Phi(\Phi^{-1}(\mathbb{C})$, i.e.

$$(\Phi(\Phi^{-1}(\mathbb{A}))\cup\Phi(\Phi^{-1}(\mathbb{B})))p_C^{(1)}\Phi(\Phi^{-1}(\mathbb{C}),$$

hence either $\mathbb{A}p_C^{(1)}\mathbb{C}$ or $\mathbb{B}p_C^{(1)}\mathbb{C}$.

Now suppose $\mathbb{A}p_C^{(1)}\mathbb{C}$ or $\mathbb{B}p_C^{(1)}\mathbb{C}$. Denote $\{J_\alpha;\alpha\in A\}=\Phi^{-1}(A)$, $\{J_\beta;\beta\in B\}=\Phi^{-1}(B)$ and $\{J_\gamma;\gamma\in C\}=\Phi^{-1}(C)$. Then either $\{J_\alpha;\alpha\in A\}p_1\{J_\beta;\beta\in B\}$ or $\{J_\beta;\beta\in B\}p_1\{J_\gamma;\gamma\in C\}$. From the validity (prox4) for the space $(\Im n_0(\mathbb{R}),p_1)$ there follows

$$(\{J_\alpha;\alpha\in A\}\cup\{J_\beta;\beta\in B\})p_1\{J_\gamma;\gamma\in C\},$$

thus

$$\mathbb{A}\cup\mathbb{C}=\Phi(\Phi^{-1}(\mathbb{A}))\cup\Phi(\Phi^{-1}(\mathbb{B}))=\Phi(\{J_\alpha;\alpha\in A\})\cup\Phi(\{J_\beta;\beta\in B\})=$$
$$=\Phi(\{J_\alpha;\alpha\in A\}\cup\{J_\beta;\beta\in B\}),$$

hence $(\mathbb{A}\cup\mathbb{B})p_C^{(1)}\mathbb{C}$. Consequently the relation $p_C^{(1)}$ satisfies also (prox4) and we have that the system $(\mathbb{CLA},p_C^{(1)})$ is a proximity space in considered sense. □

3 Induced hypergroupoids on proximity spaces

Suppose that (P,p) is a proximity space, i.e. P is a non-empty set and the binary relation p on P satisfies condition (prox1)–(prox4). It is to be noted that the mentioned condition are overtaken from the monograph due to Eduard Čech [1] substantially elaborated by Zdeněk Frolík and Miroslav Katětov). Nevertheless since the time of Efremovič-Smirnov-founders of the theory of proximity spaces there have appeared various approaches and axioms of thous structures in question. Then we will define a binary hyperoperation on the system $expP'=expP\setminus\{\emptyset\}$:

I. Recall that a $p-$neighborhood of a non-empty subset X of a proximity space (P,p) is the set $U\subset P$ such that $(P\setminus U)nonpX$. Evidently, $X\subset U$ and all $p-$neighborhoods, (called

also proximal neighborhoods) of a non-empty set X, form a filter of sets which will be denoted by $\mathscr{F}_p(X)$. Recall that a filter $\mathscr{F}$ of set on a set X is non-empty collection of subsets of X such that $A \in \mathscr{F}, B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$ and $A \in \mathscr{F}, A \subset B$ $subsetX$ implies $B \in \mathscr{F}$. If, in addition, $\mathscr{F} \neq expX$, i.e. $\emptyset \in \mathscr{F}$, then $\mathscr{F}$ is called a proper filter on X. Now, for $X, Y \in expP'$ we define

$$X * Y = \mathscr{F}_p(X) \cup \mathscr{F}_p(Y).$$

Then $(expP', *)$ is a commutative hypergroupoid. For sets $X, Y \in expP'$ such that $X \subset Y$ we have $X \cap Y \neq \emptyset$ for any proximity p on the set P according to the condition (prox3).

II. For arbitrary non-empty sets $X, Y \in expP'$ define $p(X) = \{V; XpV\}, p(Y) = \{U; U \in expP', YpU\}$ or $p(X) = \{V; V \in expP', \langle X, V\rangle \in p\}$. Then we define $X \triangle Y = p(X) \cup p(Y)$ or $X \triangledown Y = p(X) \cap p(Y)$, (notice that $X \triangledown Y = p(X) \neq \emptyset$ for any pair $\langle X, Y\rangle \in expP' \times expP'$, since $P \in p(X) \cap p(Y)$). Evidently structures $(expP', \triangle), (expP', \triangledown)$ are commutative hypergroupoids. These definitions are based on the concept of the nearness (generalized and developed mainly by Horst Herrlich). We have constructed proximities as p_C and further p_1, p_2, thus we obtain hyperoperations as $*p_C, *_1, *_2, \triangle_1, \triangle_2, \triangledown_1, \triangledown_2$, i.e. fourteen various hypergroupoids, algebraic properties of which should be in more detail investigated.

In a proximity space (P, p) is defined the concept of a proximal neighborhood of a subset $X \subset P$ ([1] 25 A. 5., p.442): A proximal neighborhood of a set $X \subset P$ in a proximity space (P, p) is a set $Y \subset P$ such that $Xnonp(P \setminus Y)$, that is, the complement of Y is distant to X in (P, p).

Let p be a proximity for set P and let η be the relation consisting of all pairs $[X, Y]$ such that Y is a p-proximal neighborhood of X, i.e. $[X, Y] \in \eta$ if and only if $X \subset P$ and $Xnonp(P \setminus Y)$. On the other hand clearly $XnonpY$ if and only if $Y \subset P$ and $[X, P \setminus Y] \in \eta$. Thus a proximity is uniquely determined by the proximal neighborhoods. It is to be noted that some authors define a proximity as the relation η.

Remark 1. *Consider proximal neighborhoods [1, 10], [2, p. 853]. Let $\prec$ be a relation for expP such that*

(1) $\emptyset \prec X$ for each $X \subset P$;

(2) $X \prec Y$ implies $X \subset Y$;

(3) If $X \subset X_1 \prec Y_1 \subset Y$ then $X \prec Y$;

(4) If $X \prec Y_i, i = 1, 2$ then $X \prec Y_1 \cap Y_2$;

(5) $X \prec Y$ implies $(P \setminus Y) \prec (P \setminus X)$.

Then there exists a unique proximity p for the set P such that $X \prec Y$ if and only if Y is a p–proximal neighborhood of the set X.

4 Relation of connexion $\varkappa$

We define on the set $\mathfrak{I}n_0(\mathbb{R})$ of intervals of real number a binary relation $\varkappa$ in this way:
For a pair of intervals $J_1, J_2 \in \mathfrak{I}n_0(\mathbb{R})$ we define $J_1 \varkappa J_2$ (or $[J_1, J_2] \in \varkappa$) if there exists an interval

$(a,b) \in Jn_0(\mathbb{R})$ with $|a-b|=1$, i.e. of the length $\omega(a,b)=1$ such that

$$J_1 \cap (a,b) \neq \emptyset \neq J_2 \cap (a,b).$$

The relation $\varkappa$ is called connexion relation or a relation of connexion for the set $\mathfrak{I}n_0(\mathbb{R})$.

We describe basic properties of the relation $\varkappa$ on $\mathfrak{I}n_0(\mathbb{R})$: Similarly as for a proximity relation we denote basic properties of a binary relation ρ on the set P. We say the relation ρ satisfies:

(con 1) if $\emptyset non \rho P$,

(con 2) if ρ is symmetric, i.e. $X\rho Y$ implies $Y\rho X$,

(con 3) if $X \subset P, Y \subset P, X \cap Y \neq \emptyset$ implies $X\rho Y$,

(con 4) if $X_1 \subset P, X_2 \subset P$ then $(X_1 \cup X_2)\rho Y$ if and only if $X_1\rho Y$ or $X_2\rho Y$.

Evidently, conditions (con 1) and (con 2) are satisfied. If $J_1, J_2 \in \mathfrak{I}n_0(\mathbb{R})$ are intervals such that $J_1 \cap J_2 \neq \emptyset$, then for an interval $(a,b) \in \mathfrak{I}n_0(\mathbb{R})$ of the length $\omega(a,b)=1$ such that $(a,b) \cap J_1 \cap J_2 \neq \emptyset$, we obtain $J_1 \varkappa J_2$, thus (con 3) is satisfied, as well. Suppose $J_1, J_2, J \in \mathfrak{I}n_0(\mathbb{R})$ are intervals with the property $(J_1 \cup J_2)\varkappa J$, which means that $(J_1 \cup J_2) \cap (a,b) \neq \emptyset$ and $J \cap (a,b) \neq \emptyset$ for a suitable interval $(a,b) \in \mathfrak{I}n_0(\mathbb{R})$ of the length $\omega(a,b)=1$. Then either $(J_1 \cap (a,b) \neq \emptyset$ and $J \cap (a,b) \neq \emptyset$ or $J_2 \cap (a,b) \neq \emptyset \neq J \cap (a,b)$. Of course all intersections can be also non-empty, simultaneously.

On the other hand, let us suppose that $J_1 \varkappa J$ or $J_2 \varkappa J$. Then for an interval $(a,b) \in \mathfrak{I}n_0(\mathbb{R})$ with the length $\omega(a,b)=1$ either $J_1 \cap (a,b) \neq \emptyset \neq J \cap (a,b)$ or $J_2 \cap (a,b) \neq \emptyset \neq J \cap (a,b)$. Then in both these cases we have $(J_1 \cup J_2) \cap (a,b) = (J_1 \cap (a,b)) \cup (J_2 \cap (a,b)) \neq \emptyset \neq J \cap (a,b)$, thus $(J_1 \cup J_2)\varkappa J$ holds. Consequently also condition (con 4) is satisfied. To be said, that the pair $(P,\varkappa)$ is also called a connexion space. This connexion structure can be transferred onto the system of chains $\mathbb{CLA}$ in the following way. Since a transfer of the connexion structure $(\mathfrak{I}n_0(\mathbb{R}),\varkappa)$ onto system $\mathbb{CLA}$ of chains of groups is very simple: For an arbitrary pair of chains $\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0\rangle$, $\langle \mathbb{LA}(K), F_n; n \in \mathbb{N}_0\rangle$ we define

$$\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0\rangle \varkappa_C \langle \mathbb{LA}(K), F_n; n \in \mathbb{N}_0\rangle$$

if $J \varkappa K$ for $J, K \in \mathfrak{I}n_0(\mathbb{R})$. Since between systems $\mathbb{CLA}$ and $\mathfrak{I}n_0(\mathbb{R})$ exists the one-to-one correspondence, we obtain, evidently, that the binary relation $\varkappa_C$ on $exp\mathbb{CLA}$ is a connexion relation for the system $\mathbb{CLA}$ of chains of groups, i.e. $(\mathbb{CLA}, \varkappa_C)$ is a connexion space in the above sense.

5 Binary hyperoperations defined using filters of neighborhoods

We recall the concept of a set-filter according to [1, 10], 12.B.2., p.207:

Let A be a set. A non-empty collection $\mathscr{F}$ of subsets of A is called a filter on A if

(i) $X \in \mathscr{F}, Y \in \mathscr{F}$ implies $X \cap Y \in \mathscr{F}$,

(ii) $X \in \mathscr{F}, X \subset Y \subset A$ implies $Y \in \mathscr{F}$.

If, in addition, $\mathscr{F} \neq expA$, i.e. $\emptyset \notin \mathscr{F}$, then $\mathscr{F}$ is called a proper filter on A.

As an example can serve: If $\emptyset \neq X \subset A$, then the collection of all Y satisfying $X \subset Y \subset A$ is a proper filter on A. Now, we denote by $\mathscr{O}p_C(\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle)$ a $\varkappa_C-$ neighborhood of the chain $\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle$, i.e. a neighborhood of the mentioned chain in the connexion space $(\mathbb{CLA}, \varkappa_C)$.

From the above example there follows immediately:

Proposition 3. *For any chain $\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}$ the collection $Fl(\mathscr{O}p_C(\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle))$ of all its neighborhoods in the connexion space $(\mathbb{CLA}, \varkappa_C)$ is a proper filter on the system $\mathbb{CLA}$.*

Now we will define a binary hyperoperation on the system $\mathbb{CLA}$:

Let $\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle, \langle \mathbb{LA}(K), G_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}$ be arbitrary chains. We define

$$\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle * \varkappa_C \langle \mathbb{LA}(K), G_n; n \in \mathbb{N}_0 \rangle = \\ = Fl(\mathscr{O}p_C(\langle \mathbb{LA}(J), F_n; n \in \mathbb{N}_0 \rangle)) \cup Fl(\mathscr{O}p_C(\langle \mathbb{LA}(K), G_n; n \in \mathbb{N}_0 \rangle)).$$

Then $*\varkappa_C$ is a binary commutative hyperoperation on the system $\mathbb{CLA}$, thus $(\mathbb{CLA}, *\varkappa_C)$ is a commutative hypergroupoid.

As simple application of the relation of connexion on $\mathfrak{I}n_0(\mathbb{R})$ we will define a binary hyperoperation on that system of intervals:

Recall, first, that for any interval $J \in \mathfrak{I}n_0(\mathbb{R})$ we can denote

$$\varkappa(J) = \{K; K \in \mathfrak{I}n_0(\mathbb{R}), J \varkappa K\}.$$

Evidently $J \in \varkappa(J) \neq \emptyset$ for each interval $J \in \mathfrak{I}n_0(\mathbb{R})$. Now suppose $J_1, J_2 \in \mathfrak{I}n_0(\mathbb{R})$ are arbitrary intervals. Put

$$J_1 *_\varkappa J_2 = \varkappa(J_1) \cup \varkappa(J_2).$$

Then $J_1 *_\varkappa J_2 = J_2 *_\varkappa J_1 \neq \emptyset$ thus we have that $(J \in \mathfrak{I}n_0(\mathbb{R}), *_\varkappa)$ is an extensive commutative hypergroupoid. Using a connexion relation $\varkappa$ we will construct a proximity relation for sets $\mathfrak{I}n_0(\mathbb{R})$ and $\mathbb{CLA}$. We denote – as above – $\mathbb{K} = \{K_\alpha; \alpha \in A\}, \mathbb{L} = \{L_\beta; \beta \in B\}, K_\alpha, L_\beta \in \mathfrak{I}n_0(\mathbb{R})$ with $\emptyset \neq A \subset \mathbb{R}, \emptyset \neq B \subset \mathbb{R}, \langle \alpha, \beta \rangle \in A \times B$. Further we put $\mathbb{K} p_2 \mathbb{L}$ if for every pair $\langle K_\alpha, L_\beta \rangle \in \mathbb{K} \times \mathbb{L}$ there holds $K_\alpha \varkappa L_\beta$ or $\mathbb{K} \cap \mathbb{L} \neq \emptyset$. Similarly as in (2), denoting $\mathbb{C} = \{\langle \mathbb{LA}(K_\alpha), F_n^{(\alpha)}; n \in \mathbb{N}_0 \rangle; \alpha \in A\}$ and $\mathbb{D} = \{\langle \mathbb{LA}(L_\beta), G_n^{(\beta)}; n \in \mathbb{N}_0 \rangle; \beta \in B\}$ we define $\mathbb{C} p_C^{(2)} \mathbb{D}$ whenever $\mathbb{K} p_2 \mathbb{L}$.

Proposition 4. *Binary relations $p_2 \subset exp\mathfrak{I}n_0(\mathbb{R}) \times exp\mathfrak{I}n_0(\mathbb{R})$, $p_C^{(2)} \subset exp\mathbb{CLA} \times exp\mathbb{CLA}$ satisfy conditions (prox1)–(prox4), thus $(\mathfrak{I}n_0(\mathbb{R}), p_2)$, $(\mathbb{CLA}, p_C^{(2)})$ are proximity spaces.*

Proof. Accordimg to the definition of the relation p_2 we have $\emptyset p_2 \mathfrak{I}n_0(\mathbb{R})$ i.e. the relation p_2 satisfies condition (prox1).

Suppose that $\mathbb{K} = \{K_\alpha; \alpha \in A\}, \mathbb{L} = \{L_\beta; \beta \in B\}$ are sets of intervals from $\mathfrak{I}n_0(\mathbb{R})$ having the property $\mathbb{K} p_2 \mathbb{L}$. If $\mathbb{K} \cap \mathbb{L} \neq \emptyset$ then evidently $\mathbb{L} p_2 \mathbb{K}$. Suppose that for each pair $\langle K_\alpha, L_\beta \rangle \in \mathbb{K} \times \mathbb{L}$

holds $K_\alpha \varkappa L_\beta$, i.e. there is a collection of intervals $J_{\alpha,\beta} \in \mathfrak{I}n_0(\mathbb{R})$ with the property $K_\alpha \cap J_{\alpha,\beta} \neq \emptyset \neq J_{\alpha,\beta} \cap K_\beta$ for any indices α, β, which means $K_\beta \varkappa K_\alpha$, thus $\mathbb{K} p_2 \mathbb{L}$. Hence the condition (prox2) is also satisfied. The relation p_2 clearly satisfies the condition (prox3), as well.

Finally, suppose that $\mathbb{V} = \{V_\gamma; \gamma \in C\} \subset \mathfrak{I}n_0(\mathbb{R})$ for a non-empty set C and $(\mathbb{K} \cup \mathbb{L}) p_2 \mathbb{V}$ holds. Then either for any pair $\langle K_\alpha, K_\beta \rangle \in (\mathbb{K} \cup \mathbb{L}) \times \mathbb{V}$ we have $K_\alpha \varkappa K_\beta$ or

$$\emptyset \neq (\mathbb{K} \cup \mathbb{L}) \cap \mathbb{V} = (\mathbb{K} \cap \mathbb{V}) \cup (\mathbb{L} \cap \mathbb{V}).$$

This means that either for some intervals $J_{\alpha,\beta} \in \mathfrak{I}n_0(\mathbb{R})$ there holds $K_\alpha \cap J_{\alpha,\beta} \neq \emptyset \neq K_\beta \cap J_{\alpha,\beta}$ or $\mathbb{K} \cap \mathbb{V} \neq \emptyset$ or $\mathbb{L} \cap \mathbb{V} \neq \emptyset$. these possibilities imply that either $\mathbb{K} p_2 \mathbb{V}$ or $\mathbb{L} p_2 \mathbb{V}$.

Now suppose, on the contrary, that either $\mathbb{K} p_2 \mathbb{V}$ or $\mathbb{L} p_2 \mathbb{V}$. Then one of the following relationships is possible:

(i) $\mathbb{K} \cap \mathbb{V} \neq \emptyset$,

(ii) $\mathbb{L} \cap \mathbb{V} \neq \emptyset$,

(iii) for any pair of intervals $\langle K_\alpha, V_\gamma \rangle \in \mathbb{K} \times \mathbb{V}$ there exists a collection of intervals $J_{\alpha,\gamma} \in \mathfrak{I}n_0(\mathbb{R})$ such that $K_\alpha \cap J_{\alpha,\gamma} \neq \emptyset \neq V_\gamma \cap J_{\alpha,\gamma}$,

(iv) for any pair of intervals $\langle L_\beta, V_\gamma \rangle \in \mathbb{L} \times \mathbb{V}$ there exists a collection of intervals $J_{\beta,\gamma} \in \mathfrak{I}n_0(\mathbb{R})$ with the property $L_\beta \cap J_{\beta,\gamma} \neq \emptyset \neq V_\gamma \cap J_{\beta,\gamma}$.

In case (i) or in case (ii) we have

$$(\mathbb{K} \cup \mathbb{L}) \cap \mathbb{V} = (\mathbb{K} \cap \mathbb{V}) \cup (\mathbb{L} \cap \mathbb{V}) \neq \emptyset$$

thus $(\mathbb{K} \cup \mathbb{L}) p_2 \mathbb{V}$ holds.

In case (iii) $K_\alpha \in \mathbb{K} \cup \mathbb{L}$, thus $(\mathbb{K} \cup \mathbb{L}) \varkappa \mathbb{V}$ which implies the validity of the relationship $(\mathbb{K} \cup \mathbb{L}) p_2 \mathbb{V}$. Similarly in case (iv) we have $L_\beta \in \mathbb{L} \cup \mathbb{K}$, thus $(\mathbb{K} \cup \mathbb{L}) \varkappa \mathbb{V}$ again, hence $(\mathbb{K} \cup \mathbb{L}) p_2 \mathbb{V}$. Consequently, also condition (prox4) is satisfied. Therefore the relation p_2 is a proximity for the set of intervals $\mathfrak{I}n_0(\mathbb{R})$. Then considering the mapping $\Phi : \mathfrak{I}n_0(\mathbb{R}) \to \mathbb{CLA}$ defined by the rule $\Phi(J) = \langle \mathbb{LA}(J); F_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}$ for each $J \in \mathfrak{I}n_0(\mathbb{R})$ – as in the proof of previous proposition – and applying Lemma – similarly as above – we obtain that $(\mathbb{CLA}, p_C^{(2)})$ (for $\{\langle \mathbb{LA}(J_\alpha), F_n^{(\alpha)}; n \in \mathbb{N}_0 \rangle; \alpha \in A\} p_C^{(2)} \{\langle \mathbb{LA}(J_\beta), F_n^{(\beta)}; n \in \mathbb{N}_0 \rangle; \beta \in B\}, A \neq \emptyset \neq B$) is a proximity space. □

6 Presheaves of continuous mappings

For creation of structures of artificial neurons can be used classical topological systems called presheaves. These systems are defined for abstract sets over quasi-ordered sets as their basis, but the mentioned systems are also modified for various topological objects.

A presheaf of sets over a quasi-ordered set $(A, \leq)$ is a pair $\varphi = \langle \{P_a \mid a \in A\}, \{f_{ab} \mid a \leq b\} \rangle$ such that $\{P_a\}$ is a family of sets, each f_{ab} is a mapping of P_a into P_b and the following two conditions are fulfilled:

(a) f_{aa} is the identity mapping of $\{P_a\}$.

(b) if $a \leq b$ and $b \leq c$, then $f_{ac} = f_{bc} \circ f_{ac}$.

If B is ordered subset of $(A, \leq)$ then the presheaf $\langle \{P_a \mid a \in B\}, \{f_{ab} \mid a \leq b, a \in B, b \in B\} \rangle$ will be called the restriction of φ to B and will be denoted by φ_B. The mappings f_{ab} are called connecting mappings of φ and the quasi/ordered set $(A, \leq)$ is called the base of φ.

Let P and Q be closure spaces. For each $X \subset P$ let $C_X = C(X, Q)$ and for $X \supset Y$ let f_{XY} be the mapping of C_X into C_Y which assigns to each $g \in C_Y$ the domain-restriction $g \mid Y$ to Y. Clearly $\langle \{C_X\}, \{f_{xy}\} \rangle$ is presheaf over $\langle expP, \supset \rangle$. This presheaf will be called the prasheaf of continuous mappings of P into Q and the mappings f_{xy} are usually called restriction mappings.

CONCLUSION

Using proximity relations – overtaken from general topology ([1]) – there are constructed systems of artificial differential neurons. The obtained systems are provided by an algebraic hyperstructure [7, 11, 13, 14, 15, 16, 17, 18, 19] which creates hypergroupoids of mentioned artificial neurons and their preshaves.

Proximity spaces belong from general topology point of view into a subcategory of category of uniform spaces. Thus there are not objects of more important interest. Nevertheless in this paper seems to be very interesting and applicable structures to investigate system of artificial neurons.

Binary hyperoperations are defined using neighborhood filters of investigated objects. Consequently the new approach to the theory of artificial neurons is based on concepts of general topology.

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IT COMPONENTS OF THE STUDY OF AIR FORCE PROFESSIONALS IN THE CZECH ARMED FORCES

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Abstract: *Currently, the continuous five-year Master's study program Military Technology – Mechanical (MT–M), which is implemented at the Faculty of Military Technology of the University of Defense (FMT UoD), includes, among other six specialization, two very attractive study specializations, namely Military Pilot (MP) and Air Traffic Controller (ATC). The authors of the contribution from the Department of Aviation and the Department of Informatics and Cyber Operations at the FMT UoD in Brno deal mainly with the IT components of the teaching of MP and ATC study specializations. Two IT subjects, which are included in the 1st semester of their studies, are discussed. The importance of building an algorithmic thinking is emphasized.*

Keywords: air traffic controllers, information technologies, military pilots, study program.

INTRODUCTION

For the benefit of the Ministry of Defense (MoD) and the Czech Armed Forces (CAF), the study program Military Technology – Mechanical (MT–M) [3] is purpose-built. It is a continuous five-year master's study program that includes the following 8 specializations:

- Combat and special vehicles
- Aviation technology – kites and engines
- Airport and technical security
- Air traffic controller
- Military pilot
- Weapons and ammunition
- Air defense weapon systems
- Engineering design.

The Ministry of Defense of the Czech Republic determines the reference numbers for the study of individual specializations. The Military Pilot (MP) and Air Traffic Controller (ATC) specializations, which are guaranteed by the Department of Aviation FMT UoD, are dealt with from an IT point of view in the following text.

1 STUDY AND TRAINING OF MILITARY PILOTS AND AIR TRAFFIC CONTROLLERS WITH MAXIMUM USE OF INFORMATION TECHNOLOGY

The development of the contemporary civilization is significantly determined by the use of information technologies on wider and wider scale. It is very difficult to imagine the functioning of a company without powerful computing resources. Military pilots and air traffic controllers

must be prepared for this fact at the beginning of their studies. The authors believe that the key components of IT knowledge and competences, which are decisive for the given two study specializations, are the areas mentioned below.

1.1 A requirement for deep knowledge of pilots and air traffic controllers in the field of algorithmization and programming

Pilots in a combat situation or in training can perform autopilot programming on their aircraft, allowing them to precisely follow a flight path. This is particularly important when conducting combat operations where specific flight maneuvers need to be followed. Pilots can also perform programming of target evaluation systems. In this way, they can better identify and track potential threats and targets and plan attacks on enemy forces.

Military pilots can also get involved in flight simulator programming that allows them to practice combat missions and tactics without the need for actual flight. Programming these simulators can include creating different scenarios and situations that can help military pilots better prepare for combat actions. Additionally, pilots can program data analysis tools that allow pilots to plan their missions more effectively. These tools can include programs for terrain analysis, overcoming enemy air defenses, tracking movement, and evaluating data from reconnaissance drones or satellites [4,5].

In addition, pilots can use software systems to manage fuel systems. In this way, they can optimize fuel consumption and increase the range of their aircraft, which is especially important during long combat missions. However, the important task is to monitor and map the enemy's forces and then plan and execute combat missions with quality [6].

In conclusion, it can be said that programming is a key skill for military pilots, which helps them to fulfill the given combat tasks. Programming makes it possible to adapt their aircraft and equipment to their needs, increasing their effectiveness and safety in carrying out missions.

1.2 Development and use of flight simulators at the Air Force Department

The development of flight simulators at the Air Force department takes place by creating realistic virtual environments in which students practice piloting aircraft and navigation. This development focuses on creating as authentic simulations of the aviation environment as possible, including aircraft, airports, weather conditions and other factors affecting flying [7].

As part of the development of flight simulators, research and development of new technologies is carried out at the University of Defense with the aim of improving the realism of the simulation. This includes, for example, developing improved physics models of aircraft, improving the visual processing of environments or creating interactive elements for training purposes.

The flight simulators at the University of Defense also enable the training of pilots of real aircraft and the conduct of scientific research in the field of the aviation industry and aviation. Pilot license applicants must use flight simulators as part of their training to gain the necessary skills and experience before they get into real planes [8].

The development of flight simulators at the University of Defense is therefore an important element of education and training in the field of aviation. With advanced technology and

authentic simulations, students and professional pilots can be better prepared for different situations and they can learn to operate the aircraft efficiently and safely.

1.3 Use of information systems at the Air Force Department and at Air Bases

Information systems (IS) are tools of the great importance in aviation and are used in various areas. This is primarily flight management, where IS is a key to the air traffic management. Aircraft control and navigation systems use IT to track flights, control flight paths, communicate between the aircraft and the control center, and ensure flight safety.

Aviation communications and navigation, including GPS and other satellite systems, are very important to pilots and air traffic controllers. IS also enable fast and reliable communication between pilots and the control station, including the use of radio links and data transmissions.

IS plays an important role in aircraft maintenance and repair. Systems for monitoring the condition of engines and other important parts of the aircraft using sensors and monitoring devices help to predict the need for the maintenance or repair. IS also enable the central management of maintenance information, parts removal, repair planning and inventory tracking.

1.4 The use of digital maps for planning flight routes with regard to air traffic safety

Digital maps are an essential tool in aviation, providing important information for pilots and flight navigation. The creation and use of these maps has become an integral part of modern aviation.

Digital mapping begins with the collection of geographic data using technologies such as global positioning systems (GPS), aerial photography from aircraft and drones, laser scanning, and ground mapping. This data is then processed and digitally recorded, which leads to the creation of detailed map documents. Pilots use digital maps to determine their position, plan a flight plan, and track flight paths during a training or combat flight [4]. Digital maps contain information about airspace, navigation points, airports, navigation aids and other important data for the flight of the aircraft itself.

Another use of digital maps is the analysis and planning of flight paths. Digital maps allow pilots and air traffic controllers to evaluate various factors such as weather conditions, airspace restrictions, topography, obstacles, and most importantly air traffic, when planning optimal flight paths for aircraft. Furthermore, digital maps are also used for the safety of the flights themselves. They show areas with restricted or prohibited air traffic, such as military training areas, restricted access zones or danger areas. Pilots can use this information in digital maps for safety flight planning and prevent dangerous situations [9].

Digital maps are used in most aircraft these days and are usually integrated into flight instruments (avionics). These devices allow pilots to view digital maps on onboard displays and interactively switch between different display modes and data layers. Overall, it can be said that the creation and use of digital maps in aviation is a key element for safe and effective air navigation. They provide pilots with important information about space, weather conditions, flight paths and restrictions to help them better plan and execute individual flights [6,10].

1.5 Basic rules for the use of air correspondence during the flight

Air correspondence is governed by established rules based on aviation regulations and regulations. These rules and procedures are used in communications between individual aircraft and ground control stations. All information in air mail must be accurate and unambiguous to avoid misunderstandings. For example, when prohibiting entry into a dangerous area, air correspondence cannot say "not allowed" but "prohibited" and must be clearly and distinctly stated. All correspondence should be as current as possible to minimize the possibility of outdated information. In air correspondence, it is important to follow security regulations and not to provide sensitive information to persons who are not authorized to do so. These aviation correspondence policies help minimize errors and misunderstandings in communications between military entities and ensure the safety and efficiency of air traffic [5,6].

2 THE ATTRACTIVENESS OF THE MILITARY PILOT STUDY PROGRAM AT THE UNIVERSITY OF DEFENCE

From the students' point of view, the VP specialization is clearly the most attractive specialization of the MT-M study program. The admission procedure is based on mathematics, English language and proof of physical fitness at a relatively high level. Selected applicants with the presumption to become a pilot of a military aircraft then go for a detailed medical examination with many tests, which is carried out at the Institute of Aviation Health in Prague [7,8].

In the 1st year of the MT-M study program in the academic year 2023-24, at the beginning of the studies, there was the following interest in studying the MP and ATC specializations. A total of 20 students expressed interest in studying the MP specialization, and another 7 students were considering this study. A total of 8 students were interested in studying the ATC specialization, and another student was thinking about this study. The previous education of the given students is documented in the table **Tab. 1**.

Secondary school type	**No of MP**	**No of ATC**
Multi-year (6 or 8) grammar school	8	3
Four-year grammar school	9	1
Technically oriented secondary school	7	3
Military secondary School	2	1
Other (non-technically oriented) secondary school	1	1

Tab. 1. Secondary school type of new students of the FMT UoD interested in MP and ATC specializations in the academic year 2023-24
Source: own

Previous work experience and previous studies at another university are not exceptional among the students interested in the MP and ATC specializations.

3 THE CURRENT SITUATION IN THE FIELD OF INFORMATION TECHNOLOGY AT THE UNIVERSITY OF DEFENSE

In the winter semester of the 1st year of study, the MT-M study program has the subject Information Technology in the Armed Forces (ITAF) in the scope of 56 teaching hours, which is guaranteed by the Department of Informatics and Cyber Operations FMT UoD, and the subject MATLAB basics in the scope of 28 teaching hours, which is guaranteed by the Department of Military Robotics FMT UoD. The structure of the ITAF subject is shown in table **Tab. 2**, the basic mandatory literature is [1] and [2]. The structure of the MATLAB basics course is shown in table **Tab. 3**. The both subjects are ended with a credit.

1.	Lectures Exercises	2 2	Introduction into the subject, basic terms
2.	Lectures	4	Cyber security
3.	Lectures	2	Information systems of the CAF
4.	Lectures	2	Viruses in IT - malware
5.	Lectures	2	Protection of classified information
6.	Lectures	4	Command and control systems
7.	Lectures	2	Introduction to algorithmization
8.	Lectures Exercises	2 2	Data types and data structures
9.	Lectures Exercises	2 2	Basic commands of a higher programming language
10.	Lab. exercises	6	Basic programming techniques
11.	Lectures	4	Introduction to computer networks, history, models of network architecture
12.	Lectures	6	Link and network layer protocols
13.	Lectures	4	Routing, application layer protocols
14.	Lectures Lab. exercises	2 2	Basic configuration of the network with respect to security
15.	Lab. exercises	4	Design and realization of simple networks

Tab. 2. Structure of the subject ITAF in the academic year 2023-2024.
Source: own

The ITAF subject ends with a credit, which consists of two parts. Firstly, the students have to pass a written examination from the first part of the subject (topics 1 to 10), and secondly, they have to pass an electronic test from the second part of the subject (topics 11 to 15). This electronic test is prepared on the Moodle base.

1.	Lectures	2	Introduction to the study of the subject and the MATLAB environment
2.	Exercises	2	Elementary work with numbers
3.	Lectures	2	Vectors and matrices
4.	Exercises	2	Working with vectors and matrices
5.	Lectures	2	Other data types

6.	Exercises	2	Working with other data types
7.	Lectures	2	2-D and 3-D graphics
8.	Lectures	2	2-D graphics
9.	Lectures	2	Application programming - scripts and functions
10.	Exercises	2	Practical creation of scripts and functions
11.	Lectures Exercises	2 2	Other selected job options
12.	Exercises	2	Independent student work, problem solving
13.	Exercises	2	Independent student work, evaluation of works

Tab. 3. Structure of the subject MATLAB basics in the academic year 2023-2024
Source: [3]

Especially part focused on algorithmization and programming plays a key role in required style of thinking a technically educated professional. The ITAF course contains 16 lessons of the basics of programming. The JavaScript programming language is used and it seems to be a good solution for the programming part of the course. An assignment for students in programming (topic 10 in the Table 2) can be as follows.

Example No 1:
Generate random whole numbers from the interval <-3;7> into a vector named peter that has 13 components. Find out and print the indexes of folders that do not contain either of the two numbers entered by the user. Print the data on the web page according to the pattern below.

```
Numbers stored in vector peter:
??
??
…
??

Tested numbers: ?? and ??

Indexes found:
??, ??, …, ??
```

The solution of the problem in Example No 1 can be divided in two files (File 1, File 2) because we want the JavaScript code to be separated. The student can run the peter.html file in any web browser available at his/her notebook.

File 1: `peter.html`

```
<!DOCTYPE html>
<html>
 <head>
  <title>Example No 1</title>
 </head>
 <body>
  <script src="peter.js"></script>
 </body>
</html>
```

File 2: `peter.js`

```
var peter=[],i,test_num1,test_num2,first=false;
document.write("Numbers stored in vector peter:<br>");
for(i=0;i<13;i++){
 peter[i]=Math.round(Math.random()*10)-3;
 document.write(peter[i]+"<br>");
}
test_num1=Number(prompt("Enter 1. tested number",-1));
test_num2=Number(prompt("Enter 2. tested number",2));

document.write("<br>Tested numbers: "+test_num1+" and "+test_num2)
document.write("<br><br>Indexes found:<br>");
for(i=0;i<13;i++){
 if((peter[i]!==test_num1)&&(peter[i]!==test_num2)){
  if(!first){
   document.write(i);
   first=true;
  }else{
   document.write(", "+i);
  }
 }
}
```

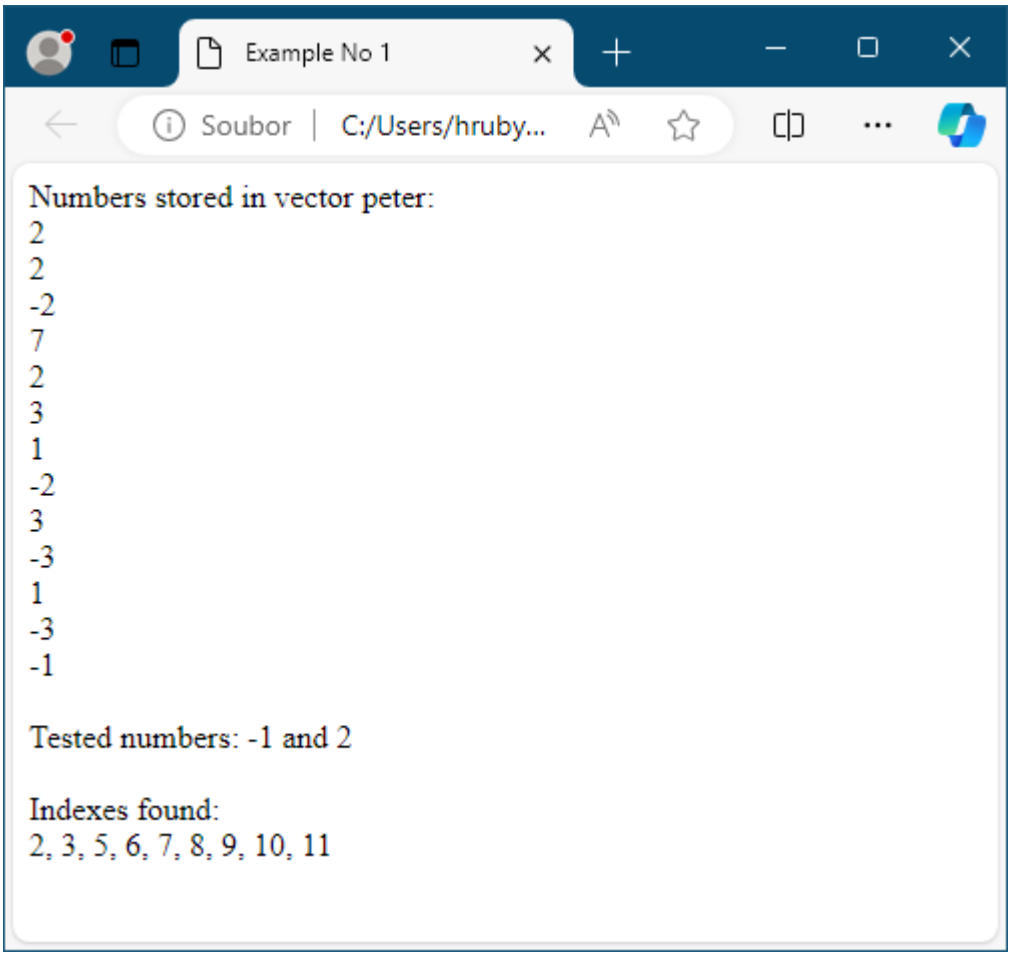

Fig 1. Outcome on the screen of the Example No 1 which was run in MS Edge web browser. Source: Own

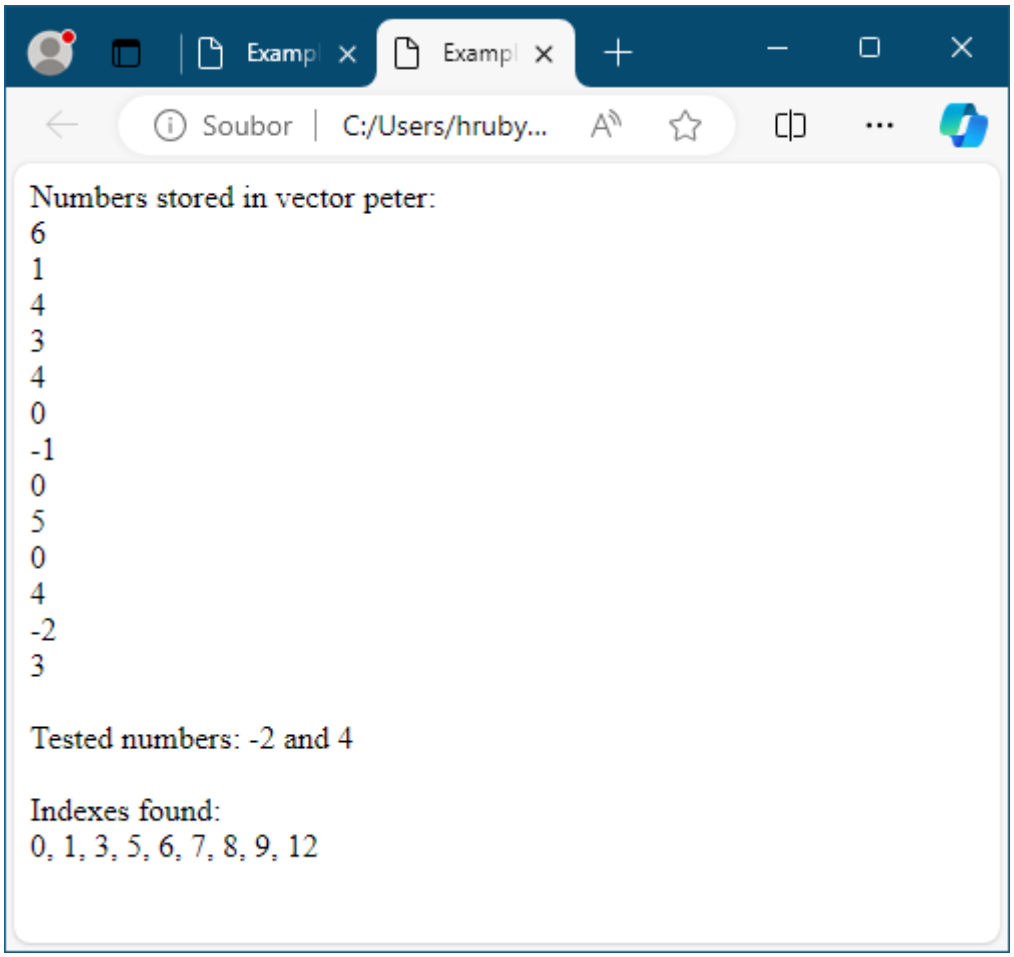

Fig 2. Outcome on the screen of the Example No 1 which was run in MS Edge web browser. Source: Own

4 THE MAIN ADVANTAGES OF INFORMATION TECHNOLOGIES IN TEACHING MILITARY PILOTS AND AIR TRAFFIC CONTOLLERS

Information technology to support the accredited study of the Military Pilot and Air Traffic Controller program is the key to ensuring the safety and efficiency of the Air Force. This support includes a wide range of technologies and systems that assist pilots and air traffic controllers in learning and training.

One of the main areas where IT support plays an important role is flight simulation. Flight simulators and virtual reality allow pilots to train in a variety of situations and perform exercises that would be costly or too dangerous in a real aircraft. Information systems ensure the proper functioning of these simulators and provide a realistic environment in which pilots and air traffic controllers can learn and improve their skills.

Another important area is data management and analysis. IT systems collect and store data from various aircraft, sensors and systems. This data is then analyzed and used to assess individual pilot performance, diagnose and repair the aircraft, and optimize operational processes.

The use of artificial intelligence (AI) in this area can automate data analysis and provide pilots and engineers with valuable information for quick and efficient decision-making. IT support also includes management of communication systems and networks. The Air Force relies on a variety of communications technologies to enable pilots, air traffic controllers, and component commanders to transmit critical information and instructions when needed. The safe and reliable operation of these systems is essential for the successful execution of operational missions.

IT support also includes cyber security. The Air Force is often the target of cyber-attacks that can compromise its operational capabilities. Therefore, it is important to have IT systems and protocols in place that protect sensitive data and prevent unauthorized access. The application of AI in the IT support of the study of military pilots and air traffic controllers can include, for example, machine learning, which helps to improve flight simulators and ensures that exercises are better adapted to the individual needs of pilots.

Another option is to use AI to automatically review and analyze data from aircraft and sensors, enabling faster problem detection and production shutdowns. AI can also be used to detect cyber threats and prevent attacks [9,10,11].

Overall, IT support of military pilot and air traffic controller studies and AI application brings many benefits such as increased safety, efficiency and quality of training and enables the Air Force to maintain training quality and combat capabilities in the ever-evolving dynamic environment of the Air Force.

CONCLUSION

There is great interest in studying the Military Pilot and Air Traffic Controller specifications at the Defense University, Faculty of Special Technologies. The selection of applicants is very strict, especially for the Military Pilot specialization. Emphasis is placed on knowledge of mathematics, the English language, as well as physical and mental fitness, which is checked at the Institute of Aviation Health in Prague. It is also important to note that both specifications, i.e. Military Pilot and Air Traffic Controller have moved to the new bachelor's study program from the current academic year.

Modern military operations cannot be done without the use of IT technologies and support. Whether they are Military pilot performing air missions or Air traffic controller, IT support plays a key role in the efficiency and success of their operations.

Military pilots use a wide variety of IT systems and equipment to help them both plan and execute air missions. One of the most important tools is the so-called “mission planning software”, which allows them to plan the flight path and other flight parameters. These systems take into account various factors such as weather conditions, terrain and potential threats.

Pilots also use advanced navigation systems that allow them to accurately track the position of the aircraft, as well as aerial targets or objects of interest on the ground. Thanks to these systems, pilots have a real-time overview of their position and the situation around them. This is especially important when performing complex maneuvers or navigating in unfamiliar terrain.

Flight simulation is another important part of IT support, which allows pilots to carry out training for various tasks and thus increase their combat readiness. Aircraft simulators are

equipped with advanced graphics and physics models that enable realistic and accurate flight simulations. These simulations include not only flight maneuvers and aircraft reactions, but also specific operations such as in-flight refueling or the use of airborne weapon systems.

IT support plays an even more important role in air traffic management. Air traffic controllers use sophisticated systems to control and track aircraft. These controllers are responsible for coordinating flights and ensuring the safety of the airspace. IT systems allow them to monitor the position of the planes, evaluate their speed or height, and give the necessary warnings and instructions to the pilots.

Another important part of Air Force IT support is the processing and analysis of big data that is generated during flight operations. This data can provide important information about traffic, security risks or possible improvements. Analytical tools and algorithms allow air traffic controllers to effectively evaluate this data and subsequently improve the safety and efficiency of air operations.

Military pilots and Air traffic controllers face a future in which IT support will play an increasingly important role. With the development of artificial intelligence and other technologies, it will be possible to automate processes even more, increasing the safety and efficiency of military aviation operations.

The use of the acquired IT competences in the ITAF and MATLAB basics subjects in the winter semester of the 1st year of study is realized both during the subsequent study of both specializations and in future military and possibly subsequent civilian practice.

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CALCULATION THE EIGENVALUES OF A 3x3 MATRIX IN MS EXCEL.

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Abstract: *One of the important topics in undergraduate mathematics is the calculation of eigenvalues of matrices. This article describes the principle of operation and the use, as a teaching aid, of a model created in MS Excel, in which the numbers forming a 3x3 matrix are written or copied and the output is three calculated eigenvalues (including the situation where two of them are a complex union). This model has proved to be a useful teaching tool, especially for setting exercises. The core of the model is a chain of calculations, at the end of which are the calculated roots of the characteristic equation, in this case cubic, which are the eigenvalues of the given matrix. Once this part had been mastered, there was nothing to prevent the creation of another model which, instead of the matrix and the subsequent calculation based on the determinant calculation, had as input the coefficients of the general cubic equation and as output its roots (this model was created as a by-product, but it too has its rich applications in mathematical, economic and statistical calculations).*

Keywords: Eigenvalues, characteristic equation, solution of cubic equation, complex numbers in MS Excel.

INTRODUCTION

Calculating the eigenvalues of a 3×3 matrix is an important skill taught in university mathematics courses. Knowledge of this problem is the basis for other economic applications such as Saaty's method in the area of decision making on the selection of a suitable option ([10], p. 70). However, these applications usually deal with higher-order matrices, for which the WolframAlpha tool [11] is a suitable tool for finding eigenvalues. [1], [2], [4], [5], [6] But here it builds on the foundations acquired in mathematics. And it is for these mathematical lessons that a model has been created in MS Excel to generate different types of 3x3 matrix assignments for which we immediately see the results in the form of eigenvalues of the matrices, see Figure 1. Of course, the situation where the eigenvalues are in the complex domain is also envisaged.

The purpose of this article is to describe how this model works. The model has proven itself in the simple and fast creation of a large number of variants of sample problems for teaching purposes as well as for exams and credit tests. The main advantage of this model is its ease of use, since it is much easier to copy matrices into MS Excel cells, or to write or generate those using random numbers, than it is to insert data into e.g. WolframAlpha. Most importantly, it is much easier to copy and save the results from the model in MS Excel, with the added benefit of working even when the Internet connection is down. This is the answer to the question why create your own model when you can use the WolframAlpha tool. The ease of use of the model is offset by the relatively large amount of work that had to be sacrificed in order to create such a model, as the limits of MS Excel were reached. The calculations involved are mainly those related to the subtraction of complex numbers. However, it was the search for ways to implement the calculations involved in solving the general cubic equation that we found

interesting, both from a mathematical point of view and from the point of view of working with MS Excel, and this is what our article is about.

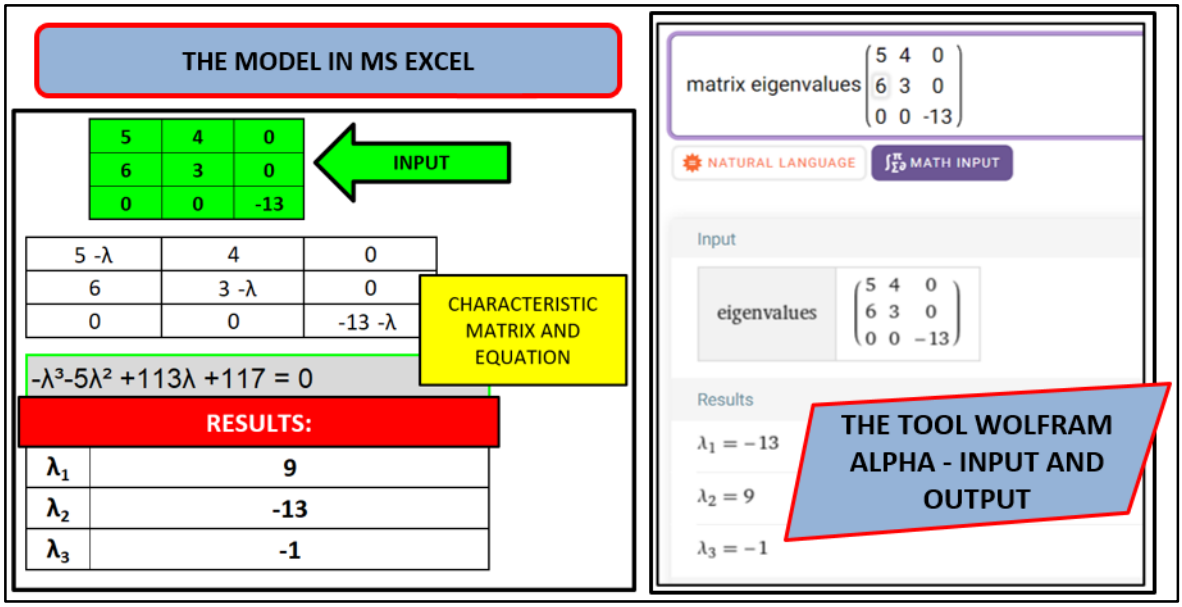

Figure 1. Demonstration of the input and output of the model in MS Excel and WolframAlpha
Source: own according to [11]

A by-product is another model in MS Excel whose input is the four coefficients of a general cubic equation and whose output is the roots of this equation. In order to find the eigenvalues of the third-degree matrices, it is necessary to solve the third-degree characteristic equation. [9], Thus, both models have their central part in common. In other words, the model for computing the eigenvalues has been slightly modified to become a model for solving cubic equations. The advantage and strength of MS Excel lies in the possibility of modularly combining individual calculations, modifying them or copying the entire models created into other models in MS Excel. A specific case is the copying of a model for solving a cubic equation into a statistical model, where a real need arose for solving a cubic equation that serves as an auxiliary calculation for determining the third parameter in the log-normal distribution in the three-parameter version, ([3], p. 59). Another example is taken from the model of cost function shapes, where the total cost function as a function of output is a third-degree polynomial, and here too it is necessary to solve the cubic equation.

Even in the actual classroom, the model has been successful in allowing students to see how their own numbers change as the numbers in the matrix change. Important rules such as the sum of the eigenvalues of a matrix is equal to the sum of the numbers on the main diagonal, the product of the eigenvalues is equal to the determinant of the matrix, the eigenvalues are zero when the rows or columns are linearly combined, etc. are well and clearly explained. This is an illustrative didactic tool for diversifying and improving the quality of teaching.

1 TRANSFORMATION OF THE INPUT IN THE CHARACTERISTICS FORMULA

The input to the model is a table (a group of nine cells) with three columns and three rows into which the specified values (matrix elements) are inserted. The search for the eigenvalues of this matrix is based on formula 1, where we must obtain a value of zero for the determinant.

$$det|A - \lambda \cdot E| = 0 \qquad (1)$$

As an example for the situation of the following step, formula 2 is given for the given matrix A for a better illustration.

$$A = \begin{pmatrix} 10 & 2 & 0 \\ 3 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}; \quad det \begin{vmatrix} 10-\lambda & 2 & 0 \\ 3 & 1-\lambda & -1 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0 \tag{2}$$

Now the Sarrus rule ([7], p. 20), [8] is applied to calculate the determinant of the third-order matrix, the first and second rows are written under the third row, the values on the diagonals are multiplied, etc. The result of the determinant is a third-order polynomial equal to zero on the right hand side, this equation is called the characteristic equation. The result of the determinant is a third order polynomial with zero on the right hand side, this equation is called the characteristic equation. As the matrix is of order three, it is a cubic equation. In general, it is an equation written in formula 3.

$$\alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta = 0 \tag{3}$$

where the coefficients α to δ are real numbers (we are looking at calculations with real matrix elements). We use these rules in the model, but since it is a model, the result is not four numbers, but calculations of values in four cells, which contain all the calculations described above, containing the values of the nine input cells. The calculation of the coefficients, consisting of the successive multiplication of the addition and subtraction of the cells, is done by several auxiliary calculations. The exception is the coefficient α, which is always equal to -1. The values of the coefficients are associated with the corresponding powers of the variable λ using the text function "CONCATENATE" and also with the symbol "+" (if the coefficient is positive) and "=0" on the right hand side, the output of this function is the automatic writing of the corresponding characteristic equation, this principle is illustrated in Figure 2.

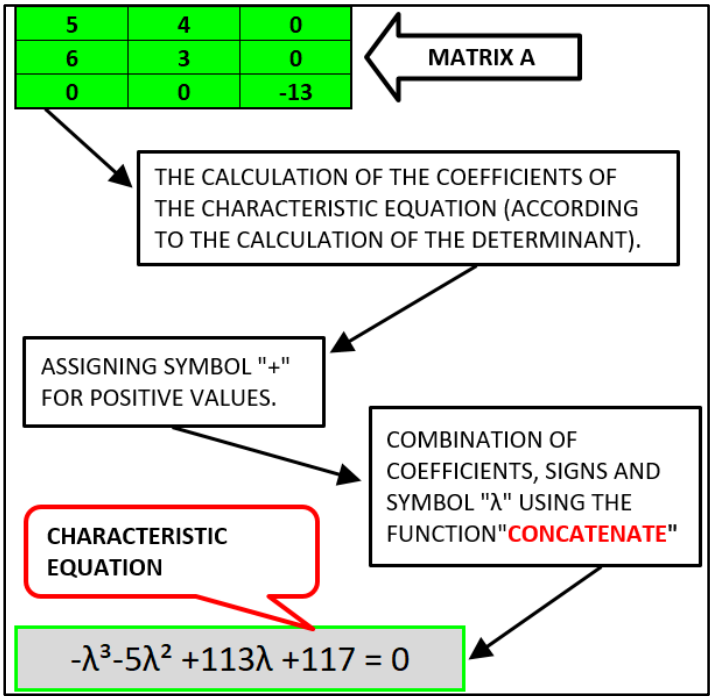

Figure 2. Automatic generation of the characteristic equation by transformation of input values.
Source: own

The equation given by formula 3 is divided by α, in this case by -1, in the case of our model this is the division of the values in the cells. This gives the cubic equation described by formula 4, where the eigenvalues we are looking for are obtained as solutions of this equation.

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{4}$$

2 ALGORITHMISATION OF THE SOLUTION OF A CUBIC EQUATION IN MS EXCEL

For the algorithmisation of the calculations in the MS Excel environment, we considered that the most appropriate way of solving the problem was to use T. Harriot's substitution [8], [9]. The first substitution is described in formula 5.

$$\lambda = t - \frac{a}{3} \tag{5}$$

Then change formula 4 to formula 6, using the auxiliary values of p and q given by formula 7.

$$t^3 + p \cdot t + q = 0 \tag{6}$$

$$p = b - \frac{a^2}{3};\ \ q = c + \frac{2a^3 - 9ab}{27} \tag{7}$$

Then the further substitution described by formula 8 is performed, after which formula 6 is modified to the form given by formula 9.

$$t = y - \frac{p}{3} \tag{8}$$

$$y^6 + q \cdot y^3 - \frac{p^3}{27} = 0 \tag{9}$$

We are already able to solve the equation in this form (the unknown is y, p and q are numbers, see formula 7).

For the sake of clarity, we will make the substitution given by formula 10, so that formula 9 becomes formula 11.

$$y^3 = z,\ respectively\ \ y^6 = z^2,\ respectively\ \ y = \sqrt[3]{z} \tag{10}$$

$$z^2 + q \cdot z - \frac{p^3}{27} = 0 \tag{11}$$

On the basis of the above formulas the calculations performed in MS Excel cells work, i.e. from the original values of a, b, c the values of p, q are calculated and using them the values of the coefficients of the quadratic equation given by formula 11 are calculated (the quadratic term is 1, the linear term is q, etc.).

The next step is to set up the calculation for the solution of this quadratic equation (formula 11), taking into account that the solution of this quadratic equation can be two complex associated numbers - the discriminant to be evaluated can be negative.

Use the "IF" function to branch the calculation in two directions:

(a) discriminant ≥ 0: the calculation is performed, with the output being two cells with values (there may be a situation where these values are the same, i.e. discriminant = 0).
b) discriminant < 0: the value of the discriminant is multiplied by -1, a positive number is produced, this can be the square root, but the result of the square root is entered as the imaginary part of the complex number, denoted by the symbol 'i', and the rest of the calculation must be done using calculations with complex numbers. The second root of the quadratic equation is then calculated from the first using the IMCONJUGATE function, which assigns a complex number to the complex number. From this stage, the calculations in the model can no longer be set up as classical calculations with cells and functions, but must be set up to calculate with complex numbers, as discussed in the next subsection.

2.1 Complex number calculations in MS Excel

MS Excel can calculate with complex numbers, but only with functions in the "Design" category (or "Engineering" or "Project", depending on the type of MS Excel). Contrary to classical calculations where you simply enter, for example, cell plus cell, you have to use the "IMSUM" function to add complex numbers. The calculation operations with cells are therefore performed using functions that allow calculations with complex numbers. A list of these functions used in the model described is given in Table 1.

Numerical operations	Function name
Adding complex numbers	IMSUM
Subtracting complex numbers	IMSUB
Multiplication of complex numbers	IMPRODUCT
Division of complex numbers	IMDIV
Combining the real and imaginary components of a complex number.	COMPLEX
Output is the real component of the complex number.	IMREAL
The output is the imaginary component of the complex number.	IMAGINARY
Absolute value of a complex number	IMABS
Calculates the angle in radians of a complex geometric shape.	IMARGUMENT
Assigns a complex number to a complex number	IMCONJUGATE

Table 1. MS Excel functions for complex number operations
Source: Author's description of selected MS Excel functions

Depending on the values entered, there may be a situation where the calculations are performed with real numbers, but to deal with the possibility that the values are complex, the calculation must be performed using the functions listed in Table 1.

2.2 Substituting backwards to get the result

After solving the quadratic equation given by formula 11, the results are entered in two cells corresponding to the values of z_1 and z_2, from which we must calculate the third root, see formula 10, and thus obtain the values of the unknown y. As mentioned, z_1 and z_2 can be complex numbers, so the calculation for exponentiating complex numbers by a third must be used, according to formula 12.

$$z^n = |z|^n \cdot (\cos n\varphi + i \cdot \sin n\varphi), \qquad where\ n = \frac{1}{3} \tag{12}$$

Through a chain of calculations and auxiliary calculations using the functions listed in Table 1, we arrive at an output that results in six cells. That is, the value of z_1 gives three search values for the variable y (see formula 10) and the value of z_2 gives three more. However, y is an auxiliary value based on the substitution, see formula 8, and from each value of y_1 to y_6 the calculation of values t_1 to t_6 is carried out, and from these the calculation given by the substitution of formula 5 is then carried out, and thus we obtain the results in the form of values λ_1 to λ_6 (all the calculations are still carried out using the functions given in Table 1). The flow of this calculation phase in the model is illustrated in Fig. 3 and, for better clarity, the principle of operation of the calculations is schematically described in Fig. 4.

λ = t - a/3	
λ1	9,00000000000003+1,02140518265514E-14i
λ2	-0,999999999999997-9,76996261670138E-15i
λ3	-13
λ4	9,00000000000003-1,02140518265514E-14i
λ5	-13
λ6	-1+9,76996261670138E-15i

Figure 3. Viewing some of the auxiliary calculations running in the model
Source: own

We are almost there, but there are still some important steps to be taken to solve the cubic equation or to find the values of the three eigenvalues. Two important conclusions can be drawn from Figure 3. First, the solutions that originally came from the auxiliary value of z_1, i.e. λ_1, λ_2, λ_3, are identical to the solutions that came from the auxiliary value of z_2, i.e. λ_4, λ_5, λ_6. However, we cannot proceed on the assumption that the main result - the eigenvalues of the matrix - will be the first triplet of values from Figure 3 and ignore the second triplet. For there may be a situation where z_1 or z_2 takes values of zero. If z_1 is zero, the result is the second triplet originally calculated from z_2 (i.e. λ_4, λ_5, λ_6) and conversely, if z_2 is zero, the result is the first triplet originally calculated from z_1 (i.e. λ_1, λ_2, λ_3). The linking of these calculated values to the cells intended for the model output is therefore done using the "IF" function, depending on the values of z_1 and z_2, but in combination with another possibility that may occur. This is a situation where z_1 and z_2 are simultaneously zero. This situation occurs when the auxiliary values of p and q (see formula 7) are zero. Consequently, formula 6 degenerates to formula 13 as a consequence of the zero values of p and q.

$$t^3 + 0 \cdot t + 0 = 0 \tag{13}$$

Therefore, “t” is also zero. We insert this into the substitution described in formula 5 and obtain the result, which is formula 14.

$$\lambda = 0 - \frac{a}{3} \tag{14}$$

This situation occurs when the cubic equation has a triple root, i.e. all three eigenvalues of the matrix are calculated using the formula 14.

It should be added that the implementation of the calculation from the values of z_1 and z_2 to the values of λ_1 to λ_6 could be simplified in the sense that instead of six calculations there would be only three and the input to these calculations would be the value of z_1 if it is not zero, if it is zero the input would be z_2, with a branch to the next mentioned condition where both z_1 and z_2 are zero and the solution has a specific form see formula 14. However, we have deliberately not gone down this route so as not to lose the full anatomy of the calculations arising from formula 9, which describes a sixth degree equation with up to six solutions, some of which are double roots. In fact, another by-product of this model is a demonstration of the steps involved in solving a general cubic equation using our method, which may be pedagogically useful.

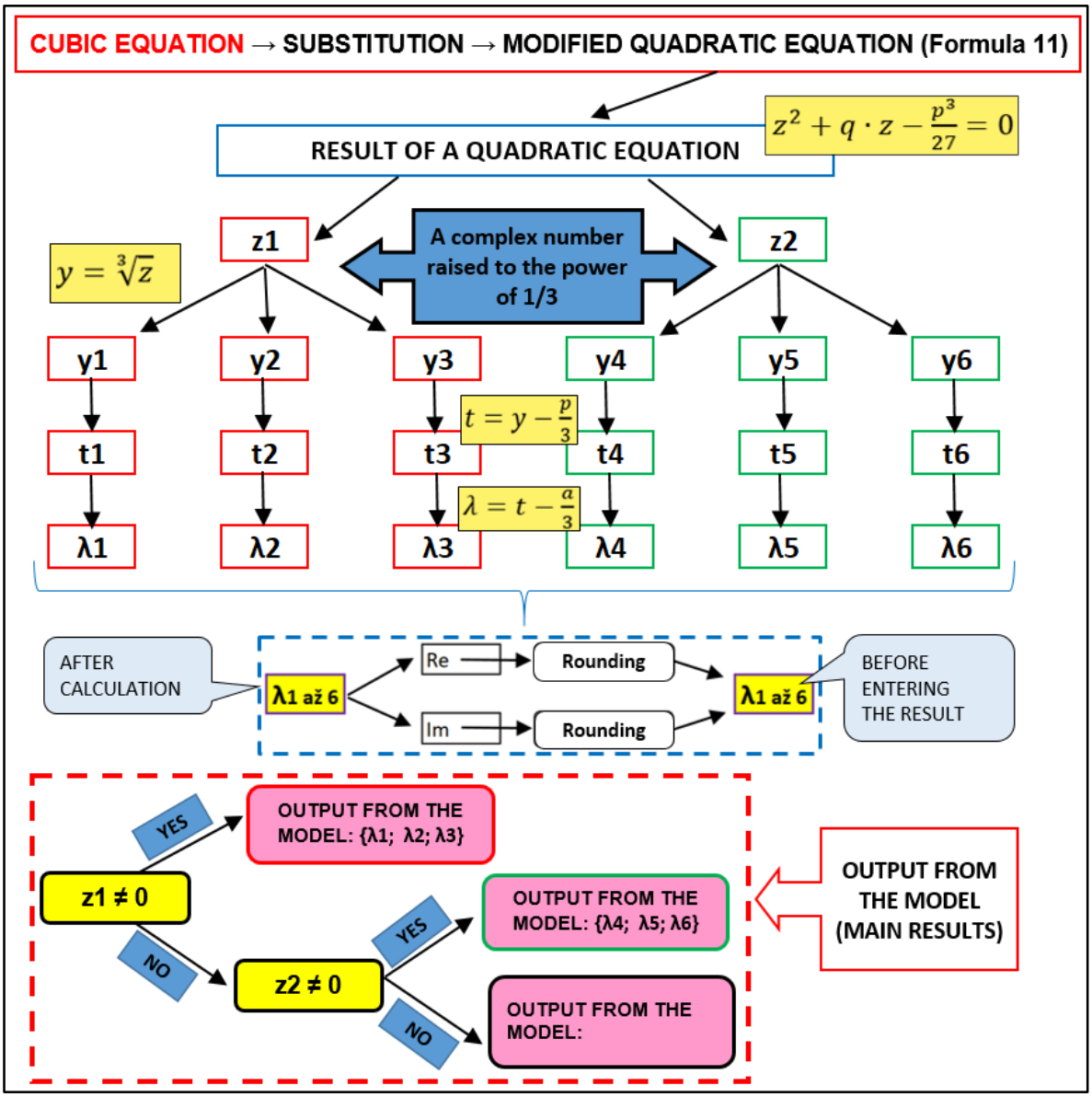

Figure 4. Model calculation algorithmisation
Source: own

It should also be noted that although this model was originally designed to compute the eigenvalues of a 3x3 matrix, it was assumed from the outset that it could easily be cloned, with slight modification, to solve general cubic equations. Therefore, there was no need to deal with the question of the difference in computational procedures between the third-degree characteristic equation and the general cubic equation.

At this stage we have described the main principles of the calculations and the operation of the model, nothing less, but we still need to describe one element that is included in the model. Let's go back to Figure 3, from which a second important conclusion follows, namely that it is necessary to reduce the unnecessarily high number of decimal places in the results, such as the value: -0.999999999999997 - 9.7699626E-15i, which is confusing because the real numbers of the results look complex due to the very small values of the imaginary components (this is a consequence of the calculations). Normally we would solve this by using the "ROUND" function, but this doesn't work for complex numbers. This problem is solved by an auxiliary calculation that forms an intermediate step between the calculation of the values λ_1 to λ_6 and the insertion of these values into the cells marked as model outputs, depending on the values of z_1 and z_2. The auxiliary calculation consists of splitting the complex number into its real and imaginary components, applying a rounding function, and then synthesising both components into a complex number. This completes the chain of set computations by outputting three cells intended for the output of the model, representing the eigenvalues of a 3x3 matrix, which operates in the case of eigenvalues from the complex number domain.

CONCLUSION

The paper described a method that was used to create a model in MS Excel capable of finding three times three of its three eigenvalues for a completely arbitrary matrix (the output is three numbers that can be identical in some cases, see the description of the calculation procedure above). It was further explained that the model created was used to create another model capable of solving general cubic equations, as both solve much the same problem. The individual auxiliary calculations included in the model can also serve as an illustration and educational aid for solving general cubic equations.

The reader has had the opportunity to learn that MS Excel can calculate with complex numbers and how these calculations are performed. This can be put to good use when creating exercises or teaching models to enrich the teaching of complex numbers in secondary school mathematics.

The model described here is another alternative to the tool WolframAlpha, which has the advantage that it cannot only solve this problem, but also (among other things) find eigenvalues of higher order matrices. For teaching purposes, however, third-order matrices are sufficient, and the strength of this model lies in the ease of copying input and output values and generally working with it, given the user-friendly and comfortable MS Excel environment. In practice, we have found this model very useful for teaching and especially for creating problem examples. The fact that the model can be used in offline mode is also an advantage.

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A QUANTUM MECHANICAL INTERPRETATION OF THE LORENTZ TRANSFORMATION

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This essay is dedicated to 120 Years Anniversary of Relativity Theory and 100 Years Anniversary of Quantum Mechanics

Abstract: *In this paper we attempt to connect three worlds – special relativity, quantum mechanics and simple mathematics to show, that the Lorentz transformation has a certain quantum mechanical interpretation that follows implicitly from the famous Einstein postulates. To do this, we use mostly elementary linear algebra, some minimum from the theory of function spaces and other widely known mathematical knowledge. In short, we extend the Lorentz transformation to a Hermitian operator on the Hilbert space of wave packets and identify the corresponding observable as the relativistic Doppler shift factor. Then the Lorentz transformation itself may be considered as a retract of its certain quantum counterpart.*

Keywords: relativity, quantum, eigenvalues, momentum, wave packets, red/blue shift, retract.

INTRODUCTION

In 1905, almost 120 years ago, Albert Einstein published his four papers that contributed significantly to the foundations of modern physics. Einstein's contribution to quantum theory is most closely associated with the paper [8], for which Einstein received the Nobel Prize in 1921 for the explanation of the photoelectric effect, and not with the paper [9], in which he established the foundations of special relativity. The latter paper seems to have nothing in common with quantum theory. Nevertheless, we will try to show that Einstein's ideas, expressed by his two famous postulates, and which allow to derive the (no less famous) Lorentz transformation independently on Maxwell-Lorentz theory of electrodynamics, but as a consequence of a deeper understanding of the topology of space and time, have something – or even a lot – in common with ideas of quantum theory. To do this, we need to look at these two apparently distant areas through the prism of eigenvalue theory of matrices and linear operators. Speaking without claim to mathematical rigorosity but inspired by the terminology of topology and category theory, we will show that the physical situation in which the matrix of the Lorentz transformation acts on vectors associated with the coordinate system, is a *retract* of another, more general situation in which a linear operator – a certain extension of the Lorentz transformation – acts on a space of wave functions.

Although it is difficult to pinpoint the exact date of the birth of modern quantum theory, the upcoming year of 2025 has been designated by the United Nations as the International Year of Quantum Science and Science and Technology, marking 100 years since the initial development of quantum mechanics by the group of physicists around Max Born. That group also included Werner Heisenberg, Wolfgang Pauli, Pascual Jordan, and others. Max Born first used

the term *quantum mechanics* in his joint work [3] with Pascual Jordan. We therefore dedicate this paper to the two anniversaries of these fascinating fields of science. However, we cannot fail to mention mention that in 2025 it will also be 75 years since the publication of the joint work of Samuel Eilenberg and Saunders Mac Lane [6], in which they proposed a general theory of mathematical structures called *category theory*. As we will show, the categorial construction of the *retraction* may serve as a bridge between the Lorentz transformation and its quantum counterpart, operating on the space of wave packets.

Regarding the terminology used, we refer the reader to [10] and [11] regarding the terms associated with special relativity and [2], [5] or [13] for the foundations of quantum theory. We especially recommend [2] for those readers who are more interested in the mathematical background of quantum, as well as the monograph [4], devoted to the advanced theory of linear operators and functional analysis, including also some foundations of general topology, and which focuses on applications in quantum physics. Mathematics we use in this paper is almost elementary – it includes some linear algebra, in particular eigenvalue theory, and some awareness of function spaces, differential equations and the theory of mathematical structures may also be useful. We should also note that the principle of special relativity as well as quantum theory have some interest in computer science – in distributed systems and quantum computing – but these topics are far beyond the scope of this paper. For the convenience of the reader, we repeat here Einstein's postulates in a brief formulation that is sufficient for our purposes. The reader can find them in textbooks [10], [11] including some more detailed explanations and discussions around or directly in Einstein's works [7] or [9]:

1. *The laws of physics take the same form in all inertial frames of reference.*
2. *The speed of light in free space has the same value c in all inertial frames of reference.*

For the sake of our intention – derivation of the Lorentz transformation in a special form in the next section – under the first postulate we consider in particular the properties also called as *symmetry* and *reciprocity*. However, these rather geometric properties alone do not provide sufficient information for the derived relations to be in accordance with reality and have to be supplemented by other assumptions of a physical nature, including spatial homogeneity, isotropy, very low gravitation and, possibly, some other conditions.

Finally, it should be noted that although the authors are interested in physics, they are not professional physicists and so do not claim to cover the entire subject or to have a complete and absolutely correct view of the topic. They see their work as a mathematically guided, interdisciplinary essay which may also serve as a starting point for further research, rather than a fully rigorous, monothematic work.

Throughout this paper, we will use the following settings and denotations. Consider two inertial frames of reference, say S and S', equipped with the Cartesian coordinate systems (x,y,z,t) and (x',y',z',t'), respectively. The frame S' is moving moves along the x-axis of the frame S with velocity v with respect to S, and vice versa, the frame S is moving along the x'-axis of the frame S' with velocity $-v$ with respect to S'. The corresponding axes of the two coordinate systems are parallel and, for simplicity, we assume no motion along the axes y, z (or y', z') . Further we assume that for $t = 0$ all other coordinates in both reference frames are equal to zero.

Then, from the point of view of special relativity, the quadruples (x,y,z,t) and (x',y',z',t') refer to the same point in spacetime, if they are connected by the Lorentz transformation given by the equations

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - \frac{v}{c^2}x), \tag{1}$$

which is inverse form is

$$x = \gamma(x' + vt'), \quad y = y', \quad z = z', \quad t = \gamma(t' + \frac{v}{c^2}x'), \tag{2}$$

where the coefficient $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is called the Lorentz factor.

Since the coordinates $y = y'$, $z = z'$ do not change in time, we will for simplicity assume that the pairs (x,t) and (x',t') refer to the same point in spacetime, if are connected by the corresponding equations in (1) and (2) above.

Einstein's second postulate and the mathematical intuition suggest that the Lorentz transformation has something in common with matrices and their eigenvectors, as we will see in the next section. Of course, this is not a completely new idea in the physics literature, but we will need to refer to the derivation in the next step.

1 DERIVATION OF THE LORENTZ TRANSFORMATION

Suppose that the frame S' is moving along the x-axis of the frame S with the velocity v with respect to S, and vice versa, the frame S is moving along the x'-axis of the frame S' with the velocity $-v$ with respect to S'. We will attempt to find a linear transformation defined by a matrix L, such that

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = L \begin{pmatrix} x \\ ct \end{pmatrix},$$

which is compatible with Einstein's second postulate. Since, according to this postulate, the speed of light is always equal to $\pm c$ (depending on the orientation of the motion of light with respect to the x and x' coordinates), it follows

$$L \begin{pmatrix} \pm ct \\ ct \end{pmatrix} = \begin{pmatrix} \pm ct' \\ ct' \end{pmatrix},$$

which gives

$$L \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{t'}{t} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{3}$$

Obviously, the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the eigenvectors of L, corresponding to their eigenvalues $\zeta_{\pm}$.

Considering the light trajectory of length $x = ct$, with respect to the frame S compared to the distance $x' = ct'$ in S', observed contracted by the (so far) unknown factor α, we get the equation

$$ct = vt + \alpha ct'.$$

Analogously, after swapping the roles of S and S', the first postulate yields

$$ct' = -vt + \alpha ct.$$

This gives

$$(c-v)t = \alpha ct'$$
$$(c+v)t' = \alpha ct,$$

and, subsequently,

$$c^2 - v^2 = c^2\alpha^2,$$

which implies

$$\alpha = \sqrt{1 - \frac{v^2}{c^2}}.$$

Hence,

$$\zeta_+ = \frac{t'}{t} = \frac{1 - \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(1-\beta),$$

where we denoted $\gamma = \frac{1}{\alpha}$ and $\beta = \frac{v}{c}$.

Similarly, we get

$$\zeta_- = \frac{t'}{t} = \frac{1 + \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(1+\beta)$$

for the case that the light (or, equivalently, the frame S' relative to S) moves in the opposite direction.

Let

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Due to the fact that the columns of Q are the eigenvectors of L, we have

$$L \cdot Q = Q \cdot \begin{pmatrix} \zeta_+ & 0 \\ 0 & \zeta_- \end{pmatrix},$$

so

$$L = Q \cdot \begin{pmatrix} \zeta_+ & 0 \\ 0 & \zeta_- \end{pmatrix} Q^{-1} = Q \cdot \begin{pmatrix} \zeta_+ & 0 \\ 0 & \zeta_- \end{pmatrix} Q^T = \gamma \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1-\beta & 0 \\ 0 & 1+\beta \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\gamma}{2} \begin{pmatrix} 1-\beta & 1+\beta \\ 1-\beta & -1-\beta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}. \quad (4)$$

Then

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix},$$

which, as a consequence, gives the usual equations that form the Lorentz transformation (1) or its inverse (2).

Now, let us show how to find the ratio $\frac{t'}{t}$ in the equation (3) in an alternative and shorter way. From geometric relations it follows that the distance ct' is viewed as $ct - vt$ from the reference frame S and ct is observed as $ct' + vt'$ from S'. Because of linearity, symmetry and reciprocity, ensured by Einstein's first postulate, the ratio of the distance seen from the other frame to its real length should be the same for both frames.

Hence

$$\frac{(c-v)t}{ct'} = \frac{(c+v)t'}{ct},$$

so

$$\frac{c-v}{c+v} = \left(\frac{t'}{t}\right)^2,$$

which yields

$$\zeta_+ = \frac{t'}{t} = \sqrt{\frac{c-v}{c+v}} = \gamma(1-\beta), \tag{5}$$

and by changing the orientation of the velocity vector of light c, we also get

$$\zeta_- = \frac{t'}{t} = \sqrt{\frac{c+v}{c-v}} = \gamma(1+\beta), \tag{6}$$

Note that ζ_+ and ζ_- are widely known as relativistic Doppler shift factors, used in astronomy and astrophysics. It holds $\zeta_+ \cdot \zeta_- = 1$, where $\zeta_+ \leq 1$ (redshift factor), $\zeta_- \geq 1$ (blueshift factor) and $\gamma = \frac{1}{2}(\zeta_+ + \zeta_-)$.

For the purpose of our next considerations, let us denote

$$h_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad h_- = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{7}$$

2 QUANTUM INTERPRETATION AND ITS CONSEQUENCES

The matrix L is symmetric and therefore Hermitian. Let us first restrict our considerations to one-dimensional geometric model of the relative motion of the frames S and S'. This means that the velocity vector of the relative motion of the frames, as well as the direction of propagation of the light wave, is parallel to both axes x, x' of the frames. Then L represents a certain Hermitian linear operator with eigenvalues ζ_+ and ζ_- and the corresponding orthonormal base of eigenvectors, formed by the columns of the matrix Q.

Attempting a quantum mechanical interpretation of L, it seems clear that the two eigenvectors of L represent the states associated with the two possible directions of propagation of the light wave. For example, let Ψ_+ and Ψ_- be two plane waves moving parallel to axes x, x' with

velocities c and $-c$, respectively. Then we can imagine that L is at the same time a Hermitian linear operator, acting on a two-dimensional space of all wave functions which are linear combinations of Ψ_+ and Ψ_-.

Then there are several natural questions – whether there exists a quantum mechanical interpretation of L, how it can be defined mathematically and whether there exists an observable with a real physical meaning, represented by this extended definition of L. Note that even in the one-dimensional case, the quantum mechanical extension of L cannot be completely represented in a matrix form since there are still infinitely many linearly independent wave functions propagating along the axes x, x'. Being $\mathscr{L}$ such an extension of L, if it really exists, it has to operate on an infinitely dimensional Hilbert space of certain wave functions.

In the following, we must restrict our considerations to wave functions propagating in one direction only, forward or backward, and to all superpositions of such wave functions. Unfortunately, the decomposition of a general wave function into two components, forward and backward propagating, is a difficult problem, which does not seem to be completely solved yet. There are some partly positive as well as negative results. For example, as it is shown in [16], in inhomogeneous media there exist waves which cannot be expressed as a linear superposition of two waves propagating in opposite directions.

An example of splittable wave functions in the sense mentioned above there are the *wave packets*, defined here as superpositions of plane wave functions. By superpositions we mean linear combinations in a generalized sense, consisting of potentially infinite series of components or even uncountably many components, combined by integral transforms. The set $\mathscr{H}$ of wave packets, representing the quantum states in a given model, should be complete with respect to the norm-induced topology, where the norm is given by the inner product properly defined on $\mathscr{H}$. For practical reasons, the wave functions studied in quantum mechanics use complex numbers, which makes the theory simpler. In other words, we assume that $\mathscr{H}$ is a certain Hilbert space over the field $\mathbb{C}$.

Let $\Psi \in \mathscr{H}$ be a wave packet. Then Ψ can be split into two, say forward propagating $\Psi_+ \in \mathscr{H}$ and backward propagating $\Psi_- \in \mathscr{H}$ wave functions in a unique way, such that $\Psi = \Psi_+ + \Psi_-$, where each of the wave packets Ψ_+, Ψ_- is a superposition of forward propagating or backward propagating plane waves, respectively. Let us denote by $\mathscr{H}^+$, $\mathscr{H}^-$ the subspaces of $\mathscr{H}$ of forward propagating and backward propagating packets, respectively. We put

$$\mathscr{L}(\Psi) = \zeta_+ \cdot \Psi_+ + \zeta_- \cdot \Psi_- \ ,$$

where ζ_+, ζ_- are from (5) and (6), which gives

$$\mathscr{L}(\Psi_+) = \zeta_+ \cdot \Psi_+ \ , \ \mathscr{L}(\Psi_-) = \zeta_- \cdot \Psi_- \ .$$

Let $\mathscr{P} : \mathscr{H} \to \mathscr{H}$ be a linear mapping – that is, in the terminology commonly used in quantum mechanics, a linear operator. We may assume $\mathscr{P}$ be Hermitian (since quantum mechanics uses Hermitian operators to represent observable quantities), although we do not need this assumption for the next step. Suppose that $\Psi = \Psi_+ + \Psi_-$, Ψ_+, $\Psi_- \in \mathscr{H}$ are elements of the subspace of $\mathscr{H}$ of the eigenvectors of the operator $\mathscr{P}$ (we also admit that Ψ_+ or Ψ_- can be indentically equal to zero function), which correspond to the same eigenvalue $p \in \mathbb{C}$ (if $\mathscr{P}$ is Hermitian, it is easy to show that $p \in \mathbb{R}$). Then, because of linearity, we have

$$\mathscr{L}\circ\mathscr{P}(\Psi)=\mathscr{L}(p\cdot\Psi)=p\cdot\mathscr{L}(\Psi)=\zeta_+p\cdot\Psi_+ +\zeta_-p\cdot\Psi_-=$$
$$=\zeta_+\mathscr{P}(\Psi_+)+\zeta_-\mathscr{P}(\Psi_-)=\mathscr{P}(\zeta_+\cdot\Psi_+ +\zeta_-\cdot\Psi_-)=\mathscr{P}\circ\mathscr{L}(\Psi) \quad (8)$$

Although many authors implicitly assume that the Hermitian operators encountered in quantum mechanics have complete sets of eigenstates, it should be noted that this is not always true – as can be shown by counterexamples, for example in [2]. Assuming that there is a complete set of eigenstates (basis of eigenvectors) of $\mathscr{P}$ in the Hilbert space $\mathscr{H}$ of the considered wave packets, from (8) it follows that the operators $\mathscr{L}$, $\mathscr{P}$ commute in $\mathscr{H}$ and so there exists a complete set of eigenstates shared by both operators (which is a well-known fact, see [2], [5] or [14]). Note that to ensure that $\mathscr{L}$ is really a Hermitian operator, a modification of the inner product on $\mathscr{H}$, making the subspaces $\mathscr{H}^+$, $\mathscr{H}^-$ orthogonal, is required. Further, it follows that for the eigenstates Ψ_+ and Ψ_- of $\mathscr{P}$ we have

$$\mathscr{P}\circ\mathscr{L}(\Psi_+)=\mathscr{P}(\zeta_+\cdot\Psi_+)=p\,\zeta_+\cdot\Psi_+$$

and, similarly

$$\mathscr{P}\circ\mathscr{L}(\Psi_-)=\mathscr{P}(\zeta_-\cdot\Psi_-)=p\,\zeta_-\cdot\Psi_- \ .$$

Now, suppose that $\mathscr{P}=-i\hbar\frac{\partial}{\partial x}$ is the usual momentum operator, defined in one dimension. Then the forward propagating packet Ψ_+ and the backward propagating packet Ψ_-, respectively, are the eigenstates of $\mathscr{P}\circ\mathscr{L}$ with the red-shifted momentum $p\,\zeta_+$ and the blue-shifted momentum $p\,\zeta_-$, respectively, as the corresponding eigenvalues of $\mathscr{P}\circ\mathscr{L}$.

There are many attempts in the literature to introduce various relativistic modifications of the usual momentum operator, known from non-relativistic quantum mechanics, based on miscellaneous theoretical concepts. For more details, the reader is referred, for example, to [13]. The composite operator $\mathscr{P}\circ\mathscr{L}$ is another momentum-like operator respecting the redshift and blueshift factors, with the expectation value $\gamma p=\frac{1}{2}(\zeta_++\zeta_-)p$ (under the assumption that both directions of propagation of the used wave functions are equally probable), which is consistent with the relativistic transverse Doppler effect. This suggests that the observable represented by $\mathscr{L}$ should be a physical quantity closely related to the relativistic Doppler shift factor. Note that the quantum or quantized redshift phenomenon has recently been studied (in a different, broader sense) in astronomy and astrophysics, for example in [12] or [15].

For the general model in three dimensions, the situation would be much more complicated and the methods and constructions used would be rather technical – so they go far beyond our intentions and possibilities of this simple paper. In short, since we can still choose the coordinate systems in the frames S and S' in the way as before, the main difference is that the wave packets can now consist of plane waves propagating from different directions, not necessarily parallel to the axes x,x' and to each other. Thus the redshift and blueshift factors analogous to ζ_+ and ζ_- are applicable only to those wave packets whose plane components propagate in the same direction. In such a case, an orthogonal projection of the relative velocity of the frames S and S' to that direction should be used. If all directions of propagation are equally probable, the authors believe that it can be proved in a rather technical way, that the expectation value associated with $\mathscr{P}\circ\mathscr{L}$, where the momentum operator $\mathscr{P}=-i\hbar\nabla$ now has its general form, would still be γp, but without any simple or nice relation to the redshift and blueshift factors.

3 CONCLUSION

Whatever the operators $\mathscr{L}$ or $\mathscr{L} \circ \mathscr{P} = \mathscr{P} \circ \mathscr{L}$ have some practical utility in physics or not, the existence of $\mathscr{L}$ itself is guaranteed by a special interpretation of the Lorentz transformation, and so it follows implicitly from Einstein's postulates (and, of course, from the postulates of quantum mechanics) as we used them in Section 1.

Furthermore, let $\pi : \mathscr{H} \to \mathbb{R}^2$, $\mu : \mathbb{R}^2 \to \mathscr{H}$ be two linear mappings such that π maps any vector of a selected but fixed orthonormal base in $\mathscr{H}^+$ to h_+ and any vector of a selected but fixed orthonormal base in $\mathscr{H}^-$ to h_-, and $\mu(h_+)$, $\mu(h_-)$ are elements of the corresponding selected orthonormal bases in $\mathscr{H}^+$, $\mathscr{H}^-$, respectively, where h_+, h_- are the eigenvectors of L defined in (7).

Then $\pi \circ \mu = id_{\mathbb{R}^2}$, so, in the sense of the of category theory [1], the morphism (linear mapping) $\pi : \mathscr{H} \to \mathbb{R}^2$ is called a *retraction*. Using the usual topological terminology as a source of inspiration, we can can further say that the object $\mathbb{R}^2$ is a *retract* of the object $\mathscr{H}$ such that $\pi \circ \mathscr{L} \circ \mu$ is the linear mapping represented by the matrix L. In other words, as we mentioned rather vaguely in the introductory section, this means that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\mathscr{L}} & \mathscr{H} \\ \uparrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R}^2 & \xrightarrow{L} & \mathbb{R}^2 \end{array}$$

In this way, some of the essential ideas of these three worlds – special relativity, quantum mechanics, and mathematics – can meet at even an elementary level, linked together by a simple category theory construction with origins in linear algebra – and this is exactly what we want to show in this short essay.

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BASIC ALGEBRAIC NOTIONS AND THEIR COUNTERPARTS IN HYPERCOMPOSITIONAL ALGEBRA

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Abstract: *In this contribution we give an overview of some basic concepts of hypercompositional structure theory. The contribution is intended to show parallels between the usual concepts of classical algebra and the hypercompositional concepts. We intend to introduce the reader to the topic of hypercompositional structure theory so that one could easily see how natural the generalizations of classical concepts are.*

Keywords: hyperfield, hypergroup, hyperring, H_v–group, H_v–ring, Krasner hyperring, semihypergroup.

INTRODUCTION

Whenever introductory courses of mathematics are included in university curricula, basic algebraic notions are present. This is so because students need to grasp the idea of basic algebraic structures as soon as possible because they need to take the advantage of the level of abstraction they offer. Therefore, both *algebraic structures with one* and *with two operations* are studied. In order to do so, students need to be exposed to notions such as *neutral element*, *inverse element*, *operations* and their arity, properties of binary operations and, consequently, to concepts that combine these notions into higher concepts such as *semigroup*, *group*, *ring*, etc. with a varying level of detail depending on needs of the future graduates.
However, the algebraic reasoning at university courses, no matter how specialized, happens almost exclusively within the area of *single-valued operations*. Only a few algebraic courses dare step out of the realm of single-valued operations and discuss their hypercompositional counterparts.
In this contribution we will outline the basic concepts of what some call *algebraic hyperstructures* while others prefer to use the term *hypercompositional algebra*. We will be interested in showing the relation between the usual, well known algebraic concepts and their "odd" multivalued counterparts. We hope that readers find this contribution interesting enough to include some of the concepts we are going to present in their standard algebraic courses.

1 BACKGROUND AND INTRODUCTORY REMARKS

Standard algebra uses the name *algebraic operation*. In order to define this we need a nonempty set and its Cartesian power n.

Definition 1 *By an n-ary algebraic operation on a non-empty set G we mean a mapping $f: G^n \to G$. The number $n \in \mathbb{N}$ is called* arity *of the operation f. A pair (G, f) is called a* groupoid *or a* magma.[1]

Standard algebraic courses focus on *binary* algebraic operations, i.e. on a special case where $n = 2$. The reason is obvious – everybody has become familiar with two special binary operations during their elementary school education. For this reason, symbols "$+$" and "$\cdot$" are usually used instead of the general notation f and we talk about the *additive*, or *multiplicative* symbolics, respectively. Cases of $n = 1$ (i.e. *unary* operations) and $n = 3$ (i.e. *ternary* operations) are studied occassionally. In these, as well as in the general case of n-ary operations, the general symbolics $f(a_1, \ldots, a_n)$ is used instead of specific symbols such as $a + b$ or $a \cdot c$.

The background idea of algebraic operations is the fact that we look for its result *within the same set*. In other words, the sum of integers is an integer again, or the product of real numbers is a real number again. No matter how natural this seems, a different perspective is natural as well (albeit not to such an extent and certainly not for elementary school children). For instance, if we say that a result of an "operation" applied on two points in plane is the line segment between them, we are outside the usual definition of algebraic operation because the result *does not fall into the same set*. Which leads to the following definition:

Definition 2 *By an n-ary algebraic hyperoperation on a non-empty set H we mean a mapping $f: H^n \to \mathscr{P}(H)$, where $\mathscr{P}(H)$ is the power set of H. The number $n \in \mathbb{N}$ is called* arity *of the hyperoperation f. A pair (H, f) is called a* hypergroupoid*. For a given n-tuple $(a_1, \ldots, a_n)$ of elements of H, the set $f(a_1, \ldots, a_n)$ is called a* hyperproduct *(or* hypercomposition*) of elements $a_1, \ldots, a_n$.*

Given this perspective and the above example of a line segment defined by its endpoints, H is a set of all points in plane, i.e. pairs of real numbers, and $\mathscr{P}(H)$ is the set of all subsets of $\mathbb{R}^2$. Naturally, in such a definition the result can be an empty set. This is sometimes permitted. However, the default option is "not permitted". If empty sets are permitted as results of hyperoperations, the term *partial hypergroupoid* is used. Notice that partial hypergroupoids are in fact counterparts to partial binary operations. In this respect we write $\mathscr{P}^*(H) = \mathscr{P}(H) \setminus \emptyset$.

The term *hyperoperation* is used as a natural counterpart to the term *operation*. Sometimes, *composition* is used instead of *operation* (and *hypercomposition* instead of *hyperoperation*). This is justified by the idea that new elements are obtained from others by *composition* under *laws of synthesis*. *Composition* and *synthesis* are somewhat more natural ideas than *operation* and *operating* because e.g. number 6 is composed of 2 and 3 (under product) or of 2 and 4 (under addition) whereas $2, 3, 4$ do not operate anywhere or on anything. The following definition, presented for the binary case $n = 2$, is used by Massouros brothers in [8], a paper which we strongly recommend for further study.[2]

Definition 3 [[8], p. 3] *Let E be a nonvoid set. A mapping from $E \times E$ into E is called a* composition on E *and a mapping from $E \times E$ into the powerset $\mathscr{P}(E)$ is called* hypercomposition *on E. A set with a composition or a hypercomposition is called a* magma*.*

As a result, *algebraic hyperstructures* or *hypercompositional structures* are phrases used to name this extension of classical algebraic structures. Using the adjective *algebraic* in *algebraic hyperstructures* is essential because one may encounter *topological hyperstructures*[3] with

[1] English prefers *magma* while Czech textbooks prefer *groupoid*.

[2] To a large extent, this paper is in fact based on [8].

[3] See e.g. [3] for the first ideas, [1] for an overview of the theory or [13], p. 77 for a brief historical overview of this approach.

sometimes overlapping naming (e.g. *hypergroup*), which, however, have a different motivation. Further on, we will stay within the framework of the usual algebraic classes, i.e. we will focus on algebraically defined hyperstructures (or, *hypercompositional structures*). *In this we will focus on the binary case only.*

2 PROPERTIES OF BINARY HYPEROPERATIONS

Given the nature of 2, it is obvious that the standard properties such as *commutativity*, *associativity* and *distributivity* will be generalized by means of equalities between sets. Recall that binary *commutativity* means that, for an arbitrary pair of input elements, the result of the operation is the same regardless of their order. Binary *associativity* extends this requirement to an arbitrary triplet of elements in the sense that the binary operation is performed twice and priorities do not matter. Finally, binary *distributivity* links two binary operations. No matter how trivial the formal definitions are, it is useful to recall them here. We are going to use the standard additive and multiplicative symbolics.

Definition 4 *A binary operation "$\cdot$" on G is called:*

1. Commutative *if, for an arbitrary pair of elements $a,b \in G$, there is*

$$a \cdot b = b \cdot a$$

2. Associative *if, for an arbitrary triplet of elements $a,b,c \in G$, there is*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

We say that the binary operation "$\cdot$" distributes *over the binary operation "$+$" (both of these are operations defined on G) if, for an arbitrary triplet of elements $a,b,c \in G$, there is*

$$\begin{aligned} a \cdot (b+c) &= a \cdot b + a \cdot c \\ (a+b) \cdot c &= a \cdot c + b \cdot c \end{aligned}$$

Naturally, when deciding whether (an) operation(s) on a set is (are) commutative, associative or distributive, we eventually examine equality of *elements of G*.

In the context of Definition 2, commutativity and associativity are discussed as follows:

Definition 5 *A binary hyperoperation (hypercomposition) "$*$" on H is called:*

1. Commutative *if, for an arbitrary pair of elements $a,b \in H$, there is*

$$a * b = b * a$$

2. Associative *if, for an arbitrary triplet of elements $a,b,c \in H$, there is*

$$a * (b * c) = (a * b) * c$$

Even though the wording of Definitions 4 and 5 is the same at first sight, there is a profound difference when performing proofs of theorems because we are comparing sets, i.e. subsets of

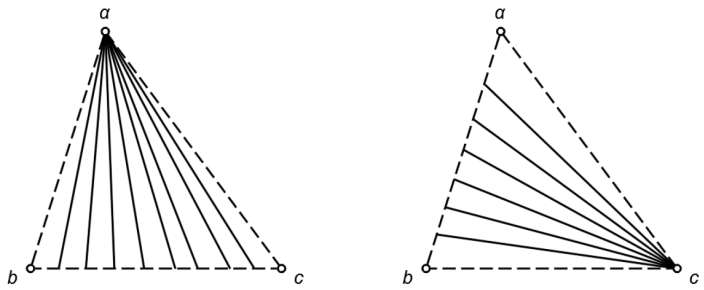

Figure 1: Associativity in hypercompositional structures, taken from [8], Example 10.

$\mathscr{P}(H)$. To be more precise, associativity of "$*$" means that, for an arbitrary triplet of elements $a,b,c \in H$, there is

$$a*(b*c) = \bigcup_{x \in b*c} a*x = \bigcup_{y \in a*b} y*c = (a*b)*c.$$

This can be best demonstrated on the already used example of a line segment with endpoints a,b. *Commutativity* of hyperoperation "$*$" means that both $a*b$ and $b*a$ generates the same line segment. As far as *associativity* is concerned, if we take three points (noncolinear ones in order to have a better idea), then $a*(b*c)$ means that we first draw a line segment between b and c and then connect each point on that line segment to point a. On the other hand, $(a*b)*c$ means that we connect each point on the line segment between a and b with point c. In other words, associativity of the hyperoperation means that both $a*(b*c)$ and $(a*b)*c$ generates the same triangle between noncolinear points a,b,c.

The issue of generalizing *distributivity* in the context of algebraic hyperstructures is rather a complex one. Naturally, one can rewrite the standard formulas used in the context of operations and get e.g

$$\begin{aligned} a \odot (b \oplus c) &= a \odot b \oplus a \odot c \\ (a \oplus b) \odot c &= a \odot c \oplus b \odot c \end{aligned}$$

for an arbitrary triplet of elements $a,b,c \in H$, where "$\oplus$" and "$\odot$" are hyperoperations (hypercompositions) on H. However, this would be a pure formalism ignoring any kind of meaning and usefulness of the defined concept. Notice that, first of all, there is no obvious requirement for *two* hyperoperations as it might be equally useful to regard one hyperoperation and one operation. (Which one, by the way?)

Therefore, we will first of all focus on hyperstructures (hypercompositional structures) with one hyperoperation (hypercomposition) only. In order to do this we have to discuss the issue of some special elements.

3 SPECIAL ELEMENTS OF (HYPER)OPERATIONS

Prior to defining concepts such as *semigroup*, *group*, *ring*, *field*, etc. one needs to discuss elements which have some special properties related to the operations in question. In classical algebra this includes *neutral elements* (or *identities*), *inverse elements* and *absorbing elements*.

Binary neutral elements (or identities) do not change the value of the other element in the pair when the operation is applied on it. On the other hand, absorbing elements always change

the value of the other element in the pair to itself. Finally, elements inverse to a given elements are such that the result of the binary operation applied on the pair (`element`, `its_inverse`) is exactly the neutral element. For classical algebraical operations we know that there is at most one neutral element (or identity), at most one inverse and at most one absorbing element.

Definition 6 *In a magma $(G,\cdot)$ (i.e. in a groupoid) we say that $e \in G$ is*

1. *An* identity *(or a* neutral element*) of G if, for all $a \in G$, there is $a \cdot e = e \cdot a = a$.*
2. *A* zero *(or an* absorbing element*) of G if, for all $a \in G$, there is $a \cdot e = e \cdot a = e$.*

If a magma (or groupoid) $(G,\cdot)$ has an identity e, then we say that $a^{-1} \in G$ is the inverse of $a \in G$ if there is $a^{-1} \cdot a = a \cdot a^{-1} = e$.

In the context of hyperstructures (hypercompositional structures), identities or inverses or zeros need not be unique. This is once again caused by the fact that the result of a hyperoperation is a set instead of a unique element. Thus we get the following definition:

Definition 7 *In a hypergroupoid $(H,*)$ we say that $e \in H$ is*

1. *An* identity *(or a* neutral element*) of H if, for all $a \in H$, there is $a \in a*e \cap e*a$.*
2. *A* scalar identity *of H if, for all $a \in H$, there is $a*e = e*a = \{a\}$.*
3. *A* zero *(or an* absorbing element*) of H if, for all $a \in H$, there is $a*e = e*a = \{e\}$.*

If in a hypergroupoid $(H,*)$ there is such an identity e, that for a pair of elements $a,a' \in H$ there is $e \in a'*a \cap a*a'$, we say that a' is an inverse *of a.*

Prenowitz and later Jantosciak discussed hypercompositional structures from the point of view of various geometries. The following example is included in [5]. Another paper discussing this approach is [4]. Notice that the geometrical approach to hypercompositional structures is rather old because Prenowitz considered it already as early as 1940s. For a certain overview of these results and details on this approach see [12].

Example 1 *[5] Let H be a set of points of a circle united with the center of the circle e – see Fig. 2. Define the hypercomposition "$\cdot$" on H in the following way:*

$$a \cdot b = \begin{cases} a & \text{for } a = e \text{ or } b = e \\ a & \text{for } b = a \\ \{a, e, a^{-1}\} & \text{for } b = a^{-1} \\ \text{the arc between } a \text{ and } b & \text{for } a \neq b, b \neq a^{-1}, a \neq e, b \neq e \end{cases}$$

In this example, the center of the circle is the scalar identity.

Notice that unlike identities, the special case of a *scalar identity* is always unique – if it exists. This is important once we start discussing the issue of groups and their counterparts. Before proceeding to algebraic structures with one operation, though, let us mention another special type of elements – *idempotent* one.

Definition 8 *In a magma $(G,\cdot)$ (i.e. in a groupoid) an element $a \in G$ is called* idempotent *if there is $a \cdot a = a$. In a magma $(H,*)$ (i.e. in a hypergroupoid) an element $a \in H$ is called* idempotent *if there is $a \in a*a$.*

Example 2 *Suppose $(\mathbb{Z},*)$, where $a*b = \{x \in \mathbb{Z} \mid a+b \leq x\}$. It is easy to see that every negative integer is an identity of $(\mathbb{Z},*)$. If we denote $i(a)$ the set of inverses of an arbitrary element $a \in \mathbb{Z}$, then there is $i(a) = \{x \in \mathbb{Z} \mid x \leq -a\}$. Also, $(\mathbb{Z},*)$ has no scalar identities or absorbing elements, and all negative integers are idempotent elements of $(\mathbb{Z},*)$.*

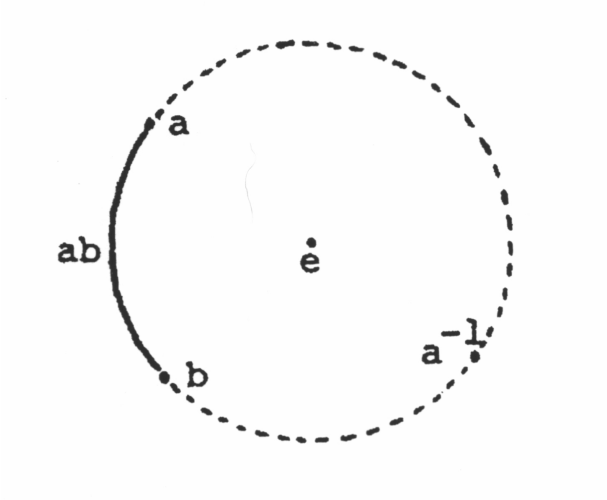

Figure 2: Hypercomposition defined in Example 1

4 ALGEBRAIC (HYPER) STRUCTURES WITH ONE (HYPER)OPERATION

In classical algebra we work with concepts such as *semigroup* or *group*. In the theory of hypercompositional structures, semigroups have a one-to-one counterpart.

Definition 9 *By a* semigroup *we mean an associative groupoid, i.e. a pair* $(G,\cdot)$*, where "*$\cdot$*" is an associative operation on G. By a* semihypergroup *we mean an associative hypergroupoid, i.e. a pair* $(H,*)$*, where "*$*$*" is an associative operation on H. By a* commutative semi(hyper)group *we mean such a semi(hyper)group that the (hyper)operation is commutative.*

However, defining counterparts to groups is not so straightforward because, as Definition 7 suggests, in the context of hyperoperations there can exist multiple identities. Therefore, we have to distinguish between several different concepts. First of all, however, recall the usual definition of a *group*.

Definition 10 *By a* group *we mean a semigroup* $(G,\cdot)$ *with an identity such that every element of G has an inverse.*

However, an alternative definition works equally well. In order to give it, we first have to introduce the notion of *reproduction* (*reproductive law*).

Definition 11 ([8], p. 8) *The law of synthesis "*$\perp$*" in H is called* reproductive *if, for all* $c \in H$*, there is* $x \perp H = H \perp x = H$*.*

Example 3 *Suppose a three-element set* $H = \{1,2,3\}$*. The following laws of synthesis are reproductive. However,* $(H,\perp)$ *on the left is a groupoid while* $(H,\perp)$ *on the right is a hypergroupoid.*

$\perp$	1	2	3
1	1	2	3
2	2	1	3
3	3	2	1

$\perp$	1	2	3
1	{1,3}	{3}	{2}
2	{2}	{1,2,3}	{2}
3	{2}	{1}	{1,2,3}

Definition 12 *Let H be a set and "$\perp$" an associative and reproductive law of synthesis on H.*

1. *If, for all $a,b \in H$, there is "$a \perp b \in H$", we call $(H,\perp)$ a* group.
2. *If, for all $a,b \in H$, there is "$a \perp b \in \mathscr{P}^*(H)$", we call $(H,\perp)$ a* hypergroup.

Remark 1 ([8], proof of Theorem 2) *Suppose that $(H,\perp)$ is a group of Definition 12. Then, for all $x \in H$ there is $x \in x \perp H$. Therefore (since H is a group) there exist some $e \in H$ such that $x \perp e = x$. Now assume an arbitrary $y \in H$. Since "$\perp$" is reproductive, there is $y \in H \perp x$. In other words, there exists $z \in H$ such that $y = z \perp x$. Thus*

$$y \perp e = (z \perp x) \perp e = z \perp (x \perp e) = z \perp x = y.$$

In a completely analogous manner we can prove that there exists $e' \in H$ such that $e' \perp y = y$ for all $y \in H$. Thus we have $e = e' \perp e = e'$, which means that there exists $e \in H$ such that

$$e \perp a = a = a \perp e$$

for all $a \in H$. Thus, in a group of Definition 12 there exists an element e with the property that is the usual defining property of the neutral element.

Now suppose an arbitrary $a \in H$. Since H is a group, there must be $e \in a \perp H$, where e is the above neutral element. In other words, there exists an element $a' \in H$ such that $e = a \perp a'$. Yet, by the reproductive law, there is not only $e \in a \perp H$ but also $e \in H \perp a$, which means that there exists also an element $a'' \in H$ such that $e = a'' \perp a$. However

$$a' = e \perp a' = (a'' \perp a) \perp a' = a'' \perp (a \perp a') = a'' \perp e = a'',$$

which means that a' and a'' coincide. If we denote both as a^{-1}, we can see that a^{-1} is the (usual group) inverse of a.

Now, in order to define a somewhat "more natural" counterpart of a group, we need one more auxiliary notion.

Definition 13 *A hypergroup is called* regular *if it has at least one identity and each element has at least one inverse. A regular hypergroup is called* reversible *if for any triple $x,y,z \in H$ there is*

1. *if $y \in a * x$, then there exists an inverse a' of a such that $x \in a' * y$ and*
2. *if $y \in x * a$, then there exists an inverse a'' of a such that $x \in y * a''$.*

Finally, we can define a hyperstructure counterpart to a group which has the usual property of unique inverses.

Definition 14 *A hypergroup H is called* canonical *if*

1. *it is commutative,*
2. *it has a scalar identity,*
3. *every element has a unique inverse,*
4. *it is reversible.*

Non-commutative canonical hypergroups are called *quasi-canonical hypergroups* (or *polygroups*).

Example 4 *Consider the set* $(\mathbb{N}, \leq)$ *of all natural numbers together with the usual ordering of numbers by size. For all* $x, y \in \mathbb{N}$ *define*

$$x * y = \begin{cases} \max\{x, y\} & \text{for } x \neq y \\ \{z \in \mathbb{N} \mid z \leq x\} & \text{for } x = y \end{cases}$$

Then $(\mathbb{N}, *)$ *is a canonical hypergroup with the smallest element (*1 *or* 0 *depending on convention) as a scalar identity. Moreover, for all* $x \in \mathbb{N}$ *there is* $x = x^{-1}$.

5 ALGEBRAIC (HYPER) STRUCTURES WITH TWO (HYPER)OPERATIONS

As far as classical algebraic structures with two operations are concerned, it is the (*commutative*) *ring*, generalizing the properties of integers, that is the key concept. Since division of integers need not produce an integer, a more specialized concept is needed – a *field*. On the other hand, if one wants to generalize properties of natural numbers, a more general concept – a *semiring* – is used.

Definition 15 *By a* ring *we mean a non-void set R endowed with two operations "*$+$*" and "*$\cdot$*" such that*

1. $(R, +)$ *is a commutative group,*
2. $(R, \cdot)$ *is a semigroup,*
3. *the operation "*$\cdot$*" distributes over the operation "*$+$*".*

By a commutative ring *we mean such a ring that* $(R, \cdot)$ *is a commutative semigroup. By a* ring with identity *we mean such a ring that* $(R, \cdot)$ *has the identity. By a* field *we mean such a ring that* $(R \setminus \{0\}, \cdot)$*, where* 0 *is the identity of* $(R, +)$*, is a commutative group.*

Semirings can be defined in various ways depending on how we treat the requirement for the identity of the multiplicative operation, or for the existence of an absorbing element, i.e. how much we want to stress the link to natural numbers. In this respect we may distinguish between *semirings with identity* and *semirings*.

Definition 16 *By a* semiring *we mean a non-void set R endowed with two operations "*$+$*" and "*$\cdot$*" such that*

1. $(R, +)$ *is a commutative semigroup,*
2. $(R, \cdot)$ *is a semigroup,*
3. *the operation "*$\cdot$*" distributes over the operation "*$+$*".*

By a commutative semiring *we mean such a ring that* $(R, \cdot)$ *is a commutative semigroup. By a* semiring with identity *we mean such a ring that* $(R, \cdot)$ *has the identity.*

Now, if one wants to look for hypercompositional counterparts of various types of (semi)-rings, one has to sort out whether both or just one (and in that case which one) operation will be replace by a hyperoperation. The most widely used approach replaces the additive operation only. Thus we get the following definition:

Definition 17 *By a (*Krasner*)* hyperring *we mean a non-void set R endowed with a hyperoperation "$\oplus$" and an operation "$\cdot$" such that:*

1. *$(R,\oplus)$ is a canonical group with scalar identity 0,*
2. *$(R,\cdot)$ is a semigroup with absorbing element 0,*
3. *the operation "$\cdot$" distributes over the operation "$\oplus$".*

If moreover $(R\setminus\{0\},\cdot)$ is a group, we call $(R,\oplus,\cdot)$ a hyperfield.

Hyperfields and later *hyperrings* were introduced by Krasner [7] as early as 1957. If readers are not familiar with French, we recommend [6], where Krasner explains his motivation for introducing these notions. Notice in [6] he introduced a construction of hyperfields (hyperrings) which after some time of initial interest resulted in a question whether there exist hyperrings which can be constructed in a different way. Since the answer is – luckily for the hyperstructure theory – a positive one, studying hyperrings, hyperfields and similar hypercompositional structures has become a relevant direction of research. We recommend a survey paper by Nakasis [11] for some more background on this topic or a paper by Massouros [9] for an example of a hyperfield obtained by an approach different from Krasner's one.

Krasner hyperrings can be generalized in several ways. If both operations turn into hyperoperations, we talk about *superrings* [10] (or *strong hyperrings in the general sense*). If one starts calling Krasner hyperrings *additive ones*, one can study *multiplicative hyperrings* [14, 15] where insted of $(R,\oplus,\cdot)$ we deal with $(R,+,\odot)$ with an additive *operation* and a multiplicative *hyperoperation*.

Finally, one needs to consider the issue of distributivity. The usual idea of equality can be generalized (if such a generalization is needed) in two different ways – either as *non-void intersection* or as *inclusion*. The non-void intersection is motivated by the study of *weak hyperstructures*, introduced by Vougiouklis in [17], where equality required in commutativity and / or associativity is replaced by non-void intersection. Thus e.g. instead of $a*b=b*a$ we require that $a*b\cap b*a\neq\emptyset$ only. In this, *weak distributivity* is used to define *H_v-rings*.[4]

Definition 18 *By an H_v–ring we mean a non-void set R endowed with two hyperoperations "$\oplus$" and "$\odot$" sych that*

1. *$(R,\oplus)$ is an H_v–group, i.e.*
   (a) *there is $a\oplus(b\oplus c)\cap(a\oplus b)\oplus c\neq\emptyset$ for all $a,b,c\in R$,*
   (b) *the hyperoperation "$\oplus$" is reproductive*
2. *$(R,\odot)$ is an H_v–semigroup, i.e. there is*
   (a) *there is $a\odot(b\odot c)\cap(a\odot b)\odot c\neq\emptyset$ for all $a,b,c\in R$,*
3. *the hyperoperation "$\odot$" is weakly distributive over "$\oplus$", i.e. for all $x,y,z\in R$ there is*

$$\begin{aligned} x\odot(y\oplus z) &\cap (x\odot y)\oplus(x\odot z)\neq\emptyset \\ (x\oplus y)\odot z &\cap (x\odot z)\oplus(y\odot z)\neq\emptyset, \end{aligned} \tag{1}$$

[4] Weak hyperstructures are called *H_v-structures*.

By *inclusion distributivity* we mean that the requested equality in distributive laws is replaced by inclusion. In other words, relations

$$\begin{aligned} a\odot(b\oplus c) &= a\odot b\oplus a\odot c \\ (a\oplus b)\odot c &= a\odot c\oplus b\odot c \end{aligned}$$

required to be valid for all $a,b,c\in H$, where "$\oplus$" and "$\odot$" are hyperoperations (hypercompositions), are replaced by

$$\begin{aligned} a\odot(b\oplus c) &\subseteq a\odot b\oplus a\odot c \\ (a\oplus b)\odot c &\subseteq a\odot c\oplus b\odot c \end{aligned}$$

In this respect we make distinction between *hyperrings in the general sense* (two hyperoperations), *additive* and *multiplicative* hyperrings, i.e. inclusively distributive hypercompositional structures, on one hand and their *good* (sometimes called also *strong*) counterparts with the usual "equality-type" distributivity on the other hand. A nice overview of all these types is given in Vougiouklis [16]. For a deeper insight with exercises and results on various types of hyperrings see book [2].

CONCLUSION

This contribution was meant as an introduction to the topic of hypercompositional algebra. We beleive that it can be used by all those who want to extend their knowledge of algebra beyond the standard university courses.

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THE SERIES OF THE RECIPROCALS OF THE ODD FIBONACCI NUMBERS MODIFIED BY PLUS ONE AND MINUS ONE

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Abstract: *This paper is inspired by intriguing YouTube videos that explore a particular type of Fibonacci series. The first video is by Michael Penn, a mathematics professor at Randolph College in Virginia, USA, who contributes to the popularization of mathematics through his website alongside his university teaching and research. The second video is by Kamaal Mirza, a mathematics teycher at the University of Punjab, Pakistan. In this paper, we first examine the series of the reciprocals of the odd Fibonacci numbers modified by adding one, which is the focus of both videos. Using the golden ratio, we derive the sum of this series. Next, we analyze the series of the reciprocals of the odd Fibonacci numbers modified by subtracting one, and similarly derive its sum using the golden ratio. These derived formulas serve as examples of simple sums for series involving the reciprocals of Fibonacci numbers.*

Keywords: Fibonacci numbers, golden ratio, partial fractions, telescoping series, sum of the series, computer algebra system Maple.

INTRODUCTION

This paper, inspired by YouTube videos [1] and [2], has been written to popularize some series of the reciprocals of the Fibonnaci numbers and add to them other similar series, which also have a relatively simple sum expressed by the square root or golden ratio. With this article, the author loosely follows his previous articles concerning the sum of special convergent numerical series, such as articles [3] and [4].

The Fibonacci numbers appeared for the first time in ancient Indian mathematics before the beginning of the era. In Europe, they were first described by Italian mathematician Leonard Pisani (Leonardo of Pisa), also known as Fibonacci (about 1175−1250), in the year 1202 in his book Liber Abaci. The name *Fibonacci sequence* was first used by the 19th-century number theorist douard Lucas (1842−1891). The Fibonacci numbers includes the rule of golden proportions. In essence, this is an observation that the ratio of any two sequential Fibonacci numbers approximates to the value of 1.618033 ...

The golden ratio and Fibonacci numbers have attracted attention from mathematicians, artists, architects, sculptors and musicians for centuries. Golden Ratio was associated with Ancient Greek art and architecture, and named after Greek sculptor Phidias (4th and 5th Century BC). The golden ratio appears in the works of many of the greatest European mathematicians, such as e.g. Abraham de Moivre (1667−1754), Nicolaus I Bernoulli (1687−1759), Leonhard Euler (1707−1783), and Jacques Philippe Marie Binet (1786−1856). In 1910, inventor Mark Barr (1871−1950) gave the golden ratio the name of the Greek letter *phi* in lower case (φ).

The Fibonacci numbers, or also the Fibonacci sequence, are one of the most famous and most widely numbers in modern mathematics, that have interesting and amazing properties.

Although very simple and known, the Fibonacci numbers are still very popular and offer many new topics for further research work. For example, the following articles have been published in recent years: [5], [6], [7], [8], [9], and [10]. The Fibonacci numbers are a central topic, for example, of the monographies [11], [12], [13], [14], [15], [16], [17], For example, the following articles have recently dealt with the issue of the golden ratio: [18], [19], [20], [21], and [22]. The following monographs deal with the golden ratio, its history and applications: [23], [24], and [25]. It can be stated that the golden ratio and the Fibonacci numbers have numerous applications which range from the infinite series theory, the description of plant growth and the crystallographic structure of certain solids to the development of computer algorithms for searching data bases.

1 BASIC PROPERTIES OF FIBONACCI NUMBERS AND THE GOLDEN RATIO

The *Fibonacci numbers* F_k are for an arbitrary integer $k \geq 2$ defined by recursive formula

$$F_k = F_{k-1} + F_{k-2} \tag{1}$$

with two initial values $F_0 = 0$ and $F_1 = 1$.
By using formula (1) we get the well-known *Fibonacci sequence*

$$\{F_k\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots\}.$$

The ratio of two consecutive Fibonacci numbers converges and approach the *golden ratio*

$$\varphi = \lim_{k\to\infty} \frac{F_{k+1}}{F_k}.$$

This limit has the value

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033988.$$

The Fibonacci numbers have a closed-form expression which is known as *Binet's formula*

$$F_k = \frac{\varphi^k - \psi^k}{\sqrt{5}}, \quad k \geq 0, \tag{2}$$

where ψ is the conjugate of φ, i.e.

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618033988,$$

so for an arbitrary integer $k \geq 0$ we have

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

or

$$F_k = \frac{1}{2^k\sqrt{5}}\left[\left(1+\sqrt{5}\right)^k - \left(1-\sqrt{5}\right)^k\right].$$

The process of deriving formula (2) using generating functions can be found, for example, in the article [26]. Since $\frac{1-\sqrt{5}}{2} = 1 - \frac{1+\sqrt{5}}{2}$, we have that

$$\psi = 1 - \varphi. \tag{3}$$

Because φ and ψ are the solution of polynomial equations $x^2 - x - 1 = 0$, we get by repeated using of the equality

$$\varphi^2 = \varphi + 1 \tag{4}$$

the following higher powers of number φ:

$$\varphi^3 = \varphi^2 + \varphi = 2\varphi + 1,$$

$$\varphi^4 = 2\varphi^2 + \varphi = 3\varphi + 2,$$

$$\varphi^5 = 3\varphi^2 + 2\varphi = 5\varphi + 3,$$

$$\varphi^6 = 5\varphi^2 + 3\varphi = 8\varphi + 5, \ldots$$

By using mathematical induction we can prove that for $k \geq 0$ it holds

$$\varphi^k = F_k\varphi + F_{k-1}.$$

There are plenty of other formulas for the Fibonacci numbers and the golden ratio, but here we present only those that we continue to use in the paper. Finally, we derive two more formulas for the Fibonacci number, which we will continue to use in the paper. From formula (4) we receive by dividing both sides of the equation by constant φ an equality

$$\varphi - 1 - \frac{1}{\varphi} = 0, \qquad \text{whence} \qquad \varphi - 1 = \frac{1}{\varphi}, \qquad \text{i.e.} \qquad 1 - \varphi = -\frac{1}{\varphi},$$

so, by formula (3), we get

$$\psi = -\frac{1}{\varphi}.$$

Since we can Binet's formula for the kth term in the Fibonacci sequence write in the form

$$F_k = \frac{1}{\sqrt{5}}\left[\varphi^k - (-1)^k\frac{1}{\varphi^k}\right], \quad k \geq 0.$$

Whence we get the expression of the odd terms of Fibonacci sequence in the form

$$F_{2k+1} = \frac{1}{\sqrt{5}}\left[\varphi^{2k+1} - (-1)^{2k+1}\frac{1}{\varphi^{2k+1}}\right],$$

i.e.

$$F_{2k+1} = \frac{1}{\sqrt{5}}\left[\varphi^{2k+1} + \frac{1}{\varphi^{2k+1}}\right],$$

so

$$F_{2k+1} = \frac{\varphi^{4k+2} + 1}{\sqrt{5}\varphi^{2k+1}}.$$

Therefore we have the following formulas

$$F_{2k+1} + 1 = \frac{\varphi^{4k+2} + 1}{\sqrt{5}\varphi^{2k+1}} + 1 = \frac{\varphi^{4k+2} + \sqrt{5}\varphi^{2k+1} + 1}{\sqrt{5}\varphi^{2k+1}}, \tag{5}$$

and

$$F_{2k+1} - 1 = \frac{\varphi^{4k+2} + 1}{\sqrt{5}\varphi^{2k+1}} - 1 = \frac{\varphi^{4k+2} - \sqrt{5}\varphi^{2k+1} + 1}{\sqrt{5}\varphi^{2k+1}}, \tag{6}$$

which we continue to use.

2 THE SERIES OF THE RECIPROCALS OF THE ODD FIBONACCI NUMBERS MODIFIED BY PLUS ONE

Let us first recall that odd terms of Fibonacci sequence form this subsequence:

$$\{F_{2k+1}\} = \{1, 2, 5, 13, 34, 89, 233, 610, \ldots\}.$$

Because it represents a subsequence Let us consider the series

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}+1} = \frac{1}{F_1+1} + \frac{1}{F_3+1} + \frac{1}{F_5+1} + \frac{1}{F_7+1} + \cdots = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{14} + \cdots . \qquad (7)$$

Because the terms of this series represent a subsequence of the sequence of the reciprocal Fibonacci numbers and because the ratio of its successive terms tends to the reciprocal of the golden ratio, which is less than one, then the ratio test shows that the series 7 converges (see e.g. [27]). Denote $S(1)$ the sum of the series (7) and determine this sum. According to formula (5) we can rewrite the series (7) to the form

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}+1} = \sum_{k=0}^{\infty} \frac{\sqrt{5}\varphi^{2k+1}}{\varphi^{4k+2}+\sqrt{5}\varphi^{2k+1}+1} = \sqrt{5}\sum_{k=0}^{\infty} \frac{\varphi^{2k+1}}{\varphi^{4k+2}+\sqrt{5}\varphi^{2k+1}+1}.$$

By substitution $u(k) = \varphi^{2k+1}$, briefly $u = \varphi^{2k+1}$, we can write this series in the form

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}+1} = \sqrt{5}\sum_{k=0}^{\infty} \frac{u}{u^2+\sqrt{5}u+1}.$$

Now, we use the method of partial fraction decomposition. Since the roots of the polynomial equation $u^2+\sqrt{5}u+1=0$ are

$$u_{1,2} = \frac{-\sqrt{5}\pm\sqrt{5-4}}{2} = \frac{-\sqrt{5}\pm 1}{2} = \begin{cases} -\dfrac{1+\sqrt{5}}{2} = -\varphi, \\ \dfrac{1-\sqrt{5}}{2} = \psi = -\dfrac{1}{\varphi}, \end{cases}$$

we can write rational fraction $\dfrac{u}{u^2+\sqrt{5}u+1}$ in the form

$$\frac{u}{u^2+\sqrt{5}u+1} = \frac{A}{u+\varphi} + \frac{B}{u+1/\varphi}.$$

By multiplying this equation by the polynomial $u^2+\sqrt{5}u+1$ we get

$$u = A\left(u+\frac{1}{\varphi}\right) + B(u+\varphi).$$

To obtain an equation with an unknown coefficient A, we substitute $u=-\varphi$. Then we receive an equation

$$-\varphi = A\left(\frac{1}{\varphi}-\varphi\right),$$

i.e.

$$\varphi = A\,\frac{\varphi^2-1}{\varphi},$$

whence we get

$$A = \frac{\varphi^2}{\varphi^2 - 1},$$

so by (4) we have

$$A = \frac{\varphi^2}{\varphi} = \varphi.$$

If we substitute $u = -\dfrac{1}{\varphi}$, we get an equation

$$-\frac{1}{\varphi} = B\left(-\frac{1}{\varphi} + \varphi\right),$$

i.e.

$$\frac{1}{\varphi} = B\,\frac{1-\varphi^2}{\varphi},$$

whence we get

$$B = \frac{1}{1-\varphi^2},$$

so by (4) we have

$$B = -\frac{1}{\varphi}.$$

The coefficients

$$A = \varphi \qquad \text{and} \qquad B = -\frac{1}{\varphi}$$

give the partial fraction decomposition in the form

$$\frac{u}{u^2 + \sqrt{5}u + 1} = \frac{\varphi}{u + \varphi} - \frac{1}{\varphi(u + 1/\varphi)}.$$

After substituting $u = \varphi^{2k+1}$ we receive an equation

$$\frac{\varphi^{2k+1}}{\varphi^{4k+2} + \sqrt{5}\varphi^{2k+1} + 1} = \frac{\varphi}{\varphi^{2k+1} + \varphi} - \frac{1}{\varphi(\varphi^{2k+1} + 1/\varphi)},$$

i.e.

$$\frac{\varphi^{2k+1}}{\varphi^{4k+2} + \sqrt{5}\varphi^{2k+1} + 1} = \frac{1}{\varphi^{2k} + 1} - \frac{1}{\varphi^{2k+2} + 1}.$$

Now we can write the series (7) in the form

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1} + 1} = \sqrt{5} \sum_{k=0}^{\infty} \left(\frac{1}{\varphi^{2k} + 1} - \frac{1}{\varphi^{2k+2} + 1} \right).$$

The sum $S(1)$ we determine as a limit of the sequence of partial sums $\{S(1)_n\}$. We get

$$S(1) = \lim_{n\to\infty} S(1)_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{F_{2k+1} + 1} = \sqrt{5} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{\varphi^{2k} + 1} - \frac{1}{\varphi^{2k+2} + 1} \right).$$

By writing out terms of the nth partial sum, we receive

$$S(1)_n = \sqrt{5}\left[\left(\frac{1}{2}-\frac{1}{\varphi^2+1}\right)+\left(\frac{1}{\varphi^2+1}-\frac{1}{\varphi^4+1}\right)+\left(\frac{1}{\varphi^4+1}-\frac{1}{\varphi^6+1}\right)+\cdots\right.$$
$$\left.\cdots+\left(\frac{1}{\varphi^{2n-2}+1}-\frac{1}{\varphi^{2n}+1}\right)+\left(\frac{1}{\varphi^{2n}+1}-\frac{1}{\varphi^{2n+2}+1}\right)\right].$$

Due to the telescopic properties of this nth partial sum, all summands will be canceled, with the exception of the first and last summands. Because $\varphi > 1$, then $\lim\limits_{n\to\infty}\dfrac{1}{\varphi^{2n+2}+1}=0$. So we get

$$S(1)=\sum_{k=0}^{\infty}\frac{1}{F_{2k+1}+1}=\sqrt{5}\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{\varphi^{2n+2}+1}\right)=\frac{\sqrt{5}}{2}=\frac{1+\sqrt{5}}{2}-\frac{1}{2}=\varphi-\frac{1}{2}. \quad (8)$$

3 THE SERIES OF THE RECIPROCALS OF THE ODD FIBONACCI NUMBERS MODIFIED BY MINUS ONE

Let us consider the series

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k+1}-1}=\frac{1}{F_3-1}+\frac{1}{F_5-1}+\frac{1}{F_7-1}+\frac{1}{F_9-1}+\cdots=\frac{1}{1}+\frac{1}{4}+\frac{1}{12}+\frac{1}{33}+\cdots. \quad (9)$$

For the same reason, which was mentioned at the beginning of Section 2, this series converges. Denote $S(-1)$ the sum of the series (9) and determine this sum. According to formula (6) we can rewrite the series (9) to the form

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k+1}-1}=\sum_{k=1}^{\infty}\frac{\sqrt{5}\varphi^{2k+1}}{\varphi^{4k+2}-\sqrt{5}\varphi^{2k+1}+1}=\sqrt{5}\sum_{k=1}^{\infty}\frac{\varphi^{2k+1}}{\varphi^{4k+2}-\sqrt{5}\varphi^{2k+1}+1}.$$

Again using substitution $u(k)=\varphi^{2k+1}$, briefly $u=\varphi^{2k+1}$, we can write this series in the form

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k+1}-1}=\sqrt{5}\sum_{k=1}^{\infty}\frac{u}{u^2-\sqrt{5}u+1}.$$

Since the roots of the polynomial equation $u^2-\sqrt{5}u+1=0$ are

$$u_{1,2}=\frac{\sqrt{5}\pm\sqrt{5-4}}{2}=\frac{\sqrt{5}\pm 1}{2}=\begin{cases}\dfrac{\sqrt{5}-1}{2}=-\psi=\dfrac{1}{\varphi},\\ \dfrac{\sqrt{5}+1}{2}=\varphi,\end{cases}$$

we can use the method of partial fraction decomposition and write rational fraction $\dfrac{u}{u^2-\sqrt{5}u+1}$ in the form

$$\frac{u}{u^2+\sqrt{5}u+1}=\frac{A}{u-1/\varphi}+\frac{B}{u-\varphi}.$$

By multiplying this equation by the polynomial $u^2-\sqrt{5}u+1$ we get

$$u=A(u-\varphi)+B\left(u-\frac{1}{\varphi}\right).$$

If we substitute $u = \dfrac{1}{\varphi}$, we receive an equation with an unknown coefficient A

$$\frac{1}{\varphi} = A\left(\frac{1}{\varphi} - \varphi\right),$$

i.e.

$$\frac{1}{\varphi} = A\,\frac{1-\varphi^2}{\varphi},$$

whence we get

$$A = \frac{1}{1-\varphi^2},$$

so by (4) we have

$$A = -\frac{1}{\varphi}.$$

If we substitute $u = \varphi$, we get an equation

$$\varphi = B\left(\varphi - \frac{1}{\varphi}\right),$$

i.e.

$$\varphi = B\,\frac{\varphi^2-1}{\varphi},$$

whence we get

$$B = \frac{\varphi^2}{\varphi^2-1},$$

so by (4) we have

$$B = \frac{\varphi^2}{\varphi} = \varphi.$$

So we received partial fraction decomposition in the form

$$\frac{u}{u^2-\sqrt{5}u+1} = \frac{-1/\varphi}{u-1/\varphi} + \frac{\varphi}{u-\varphi} = \frac{-1}{\varphi u-1} + \frac{\varphi}{u-\varphi}.$$

After substituting $u = \varphi^{2k+1}$ we receive an equation

$$\frac{\varphi^{2k+1}}{\varphi^{4k+2}-\sqrt{5}\varphi^{2k+1}+1} = \frac{-1}{\varphi^{2k+2}-1} + \frac{\varphi}{\varphi^{2k+1}-\varphi},$$

i.e.

$$\frac{\varphi^{2k+1}}{\varphi^{4k+2}-\sqrt{5}\varphi^{2k+1}+1} = \frac{1}{1-\varphi^{2k+2}} - \frac{1}{1-\varphi^{2k}}.$$

Now we can write the series (9) in the form

$$\sum_{k=1}^{\infty} \frac{1}{F_{2k+1}-1} = \sqrt{5}\sum_{k=1}^{\infty}\left(\frac{1}{\varphi^{2k}-1} - \frac{1}{\varphi^{2k+2}-1}\right).$$

The sum $S(-1)$ we determine as a limit of the sequence of partial sums $\{S(-1)_n\}$. We get

$$S(-1) = \lim_{n\to\infty} S(-1)_n = \lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{F_{2k+1}-1} = \sqrt{5}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{\varphi^{2k}-1} - \frac{1}{\varphi^{2k+2}-1}\right).$$

By writing out terms of the nth partial sum, we receive

$$S(1)_n = \sqrt{5}\left[\left(\frac{1}{\varphi^2-1}-\frac{1}{\varphi^4-1}\right)+\left(\frac{1}{\varphi^4-1}-\frac{1}{\varphi^6-1}\right)+\left(\frac{1}{\varphi^6-1}-\frac{1}{\varphi^8-1}\right)+\cdots\right.$$
$$\left.\cdots+\left(\frac{1}{\varphi^{2n-2}-1}-\frac{1}{\varphi^{2n}-1}\right)+\left(\frac{1}{\varphi^{2n}-1}-\frac{1}{\varphi^{2n+2}-1}\right)\right].$$

Due to the telescopic properties of this nth partial sum, all summands will be canceled, with the exception of the first and last summands. Because $\varphi > 1$, then $\lim\limits_{n\to\infty}\dfrac{1}{\varphi^{2n+2}-1}=0$. So we get

$$S(-1)=\sum_{k=1}^{\infty}\frac{1}{F_{2k+1}-1}=\sqrt{5}\lim_{n\to\infty}\left(\frac{1}{\varphi^2-1}-\frac{1}{\varphi^{2n+2}-1}\right)=\frac{\sqrt{5}}{\varphi^2-1}=\frac{\sqrt{5}}{\varphi}=\frac{2\sqrt{5}}{1+\sqrt{5}}. \quad (10)$$

Note that we can remove the radical from the denominator of the fraction $\dfrac{2\sqrt{5}}{1+\sqrt{5}}$ and receive the sum $S(-1)$ in the form $S(-1)=\dfrac{2\sqrt{5}(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})}=\dfrac{2\sqrt{5}(1-\sqrt{5})}{-4}=\dfrac{10-2\sqrt{5}}{4}=\dfrac{5-\sqrt{5}}{2}$.

4 NUMERICAL VERIFICATION

We solve the problem to determine the values of the sums $S(1)$ and $S(-1)$. We use on the one hand an approximate direct evaluation of the sums

$$S(1)=\sum_{k=0}^{t}\frac{1}{F_{2k+1}+1} \qquad \text{and} \qquad S(-1)=\sum_{k=1}^{t}\frac{1}{F_{2k+1}-1},$$

where the upper index $t = 10, 10^2, 10^3$, using the basic programming language of the computer algebra system Maple 2022, and on the other hand the formulas (8) and (10) for evaluation the sums $S(1)$ and $S(-1)$ respectively. We use the following very simple procedure `sums1m1`.

```
> with(combinat,fibonacci):
> sums1m1:=proc(t)
    local i,k,s1,sm1,s1t,sm1t;
    s1t:=0; sm1t:=0;
    s1:=sqrt(5)/2;
    sm1:=(5-sqrt(5))/2;
    print("t=",t,"s(1)=",evalf[12](s1),"s(-1)=",evalf[12](sm1));
    for k from 0 to t do
        s1t:=s1t+1/(fibonacci(2*k+1)+1);
    end do;
    for k from 1 to t do
        sm1t:=sm1t+1/(fibonacci(2*k+1)-1);
    end do;
    print("st(1)=",evalf[12](s1t),"st(-1)=",evalf[12](sm1t));
    print("diff1=",evalf[12](abs(s1t-s1)),"diff-1=",evalf[12](abs(sm1t-sm1)));
  end proc:
> for t in 10,100,1000 do
    sums1m1(t);
  end do;
```

The approximate values of the sums $S(1)$ and $S(-1)$ rounded to 11 decimals are

$$S(1) = 1.11803398875 \quad \text{and} \quad S(-1) = 1.38196601125.$$

Computation of these two sums and three pairs of the sums $S(1,t)$ and $S(-1,t)$, where $t = 10, 10^2, 10^3$, took about 1 second. The absolute errors, i.e. the differences $|S(1) - S(1,t)|$ and $|S(-1) - S(-1,t)|$, are $5.6 \cdot 10^{-5}$ for $t = 10$ and 0 for $t = 10^2$ and $t = 10^3$.

CONCLUSION

We dealt with the sum of the series of reciprocals of the odd Fibonacci numbers modified by plus one and minus one, i.e. with the series

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}+1} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{F_{2k+1}-1}.$$

We derived that the sum $S(1)$ of the first series is given by the formula

$$S(1) = \varphi - \frac{1}{2} \doteq 1.118034$$

and the sum $S(-1)$ of the second one is given by the formula

$$S(-1) = \frac{\sqrt{5}}{\varphi} \doteq 1.381966,$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. We verified these results by computing the partial sums

$$\sum_{k=0}^{t} \frac{1}{F_{2k+1}+1} \qquad \text{and} \qquad \sum_{k=1}^{t} \frac{1}{F_{2k+1}-1}$$

for $t = 10, 10^2, 10^3$ using the computer algebra system Maple 2022.
The series above so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a simple formula in closed form.

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ON SEQUENCES OF POSITIVE NUMBERS PRESCRIBED BY MEANS

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Abstract: *The aim of this article is to study such monotone sequences (a_n) of positive numbers that for each $n \geq 2$ satisfy the equality $a_n = K(a_{n-1}, a_{n+1})$, where the function $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the mean, i.e. each value of $K(x,y)$ lies between $\min\{x,y\}$ and $\max\{x,y\}$ ($\min\{x,y\} \leq K(x,y) \leq \max\{x,y\}$). Such known sequences are, for example, arithmetic (geometric) sequence, because starting from the second term, each of its terms is equal to the arithmetic (geometric) mean of its neighboring terms.*
Further, this accomplishment generalized and extended some results from the papers [6], [8], where it is referred the properties of the logarithmic sequence (a_n), i.e. such a sequence that for every $n \geq 2$ satisfies $a_n = L(a_{n-1}, a_{n+1})$, where $L(x,y)$ is logarithmic mean of positive numbers x, y defined as follows:

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y. \end{cases}$$

Keywords: means of two-variables, logarithmic mean, power-mean, exponent of convergence of sequences, sequences prescribed by means.

Introduction

The core of this paper is to investigate the relations between convergence exponent of sequences and other characteristics specified for monotone sequences of positive numbers. We focus on such monotone sequences (a_n) of positive numbers that for each $n \geq 2$ satisfy the equality $a_n = K(a_{n-1}, a_{n+1})$, for some mean $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. Such sequences that satisfy this equality for each $n \geq 2$ are called K-sequences. In the present paper some basic estimations, limits of the terms of K-sequences are investigated.
Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. Let us denote $\mathscr{S}$ the system of all nondecreasing sequences of positive real numbers.

Similar area has been studied by many mathematicians. For this paper, we were inspired by [1], [2], [3], [4], [5], [6], [8].

Definitions

In this part, we recall some basic definitions. The following definitions are from papers [3], [5], [6], [8], [10], and [11].

- A function $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a *mean* on $\mathbb{R}^+$ if for all $x, y \in \mathbb{R}^+$ we have

$$\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}.$$

It is obvious that $M(x,x) = x$ for all $x \in \mathbb{R}^+$.

The mean $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called *symmetric* if

$$M(x,y) = M(y,x)$$

for all $x, y \in \mathbb{R}^+$.

The mean $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a *strict* mean on $\mathbb{R}^+$ if for all $x, y \in \mathbb{R}^+$ with $x \neq y$

$$\min\{x,y\} < M(x,y) < \max\{x,y\}.$$

The mean $M : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called *homogeneous* if

$$M(zx,zy) = zM(x,y),$$

for all $x, y, z \in \mathbb{R}^+$.

Classical examples of two-variable strict means on $\mathbb{R}^+$ are:

– The arithmetic, the geometric, and the harmonic mean

$$A(x,y) := \frac{x+y}{2}, \quad G(x,y) := \sqrt{xy}, \quad H(x,y) := \frac{2xy}{x+y}.$$

- The power mean of degree $\alpha \in \mathbb{R}$ of two positive numbers x, y is as follows:

$$M_\alpha(x,y) = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}} \text{ if } \alpha \neq 0 \text{ and } M_0(x,y) = \lim_{\alpha \to 0} M_\alpha(x,y).$$

The case $\alpha = 1$ corresponds to the arithmetic mean, $\alpha = 0$ to the geometric mean, and $\alpha = -1$ to the harmonic mean.

Obviously, the mean M_α is a homogeneous strict mean such that

$$\lim_{y \to \infty} M_\alpha(x,y) = \infty \text{ for } \alpha > 0\,,$$

$$\lim_{x \to 0} M_\alpha(x,y) = 0 \text{ for } \alpha < 0\,,$$

moreover $M_\alpha(x,y)$ is continuous with respect to x and y.

It is well known that $M_0(x,y) = \sqrt{xy}$ and $M_\alpha(x,y)$ is increasing with respect to α for given $x, y \in \mathbb{R}^+$ (see [7]).

- The logarithmic mean (see [5]) of positive numbers x, y is

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y\,. \end{cases}$$

Obviously, the mean L is homogeneous strict mean such that

$$\lim_{y \to \infty} L(x,y) = \infty \text{ and } \lim_{x \to 0} L(x,y) = 0\,,$$

moreover $L(x,y)$ is continuous with respect to x and y.

In [1], [2], there is a generalization of the logarithmic mean and in [12], [3], the following relation between $L(a,b)$ and $M_\alpha(a,b)$ was proven in various ways for arbitrary positive numbers a, b :

$$M_0(a,b) \leq L(a,b) \leq M_{\frac{1}{3}}(a,b),$$

and the equality occurs if and only if $a = b$.

- A sequence (a_n) of positive real numbers is called a logarithmic sequence if

$$a_n = L(a_{n-1}, a_{n+1}) \text{ for every } n \geq 2.$$

- Let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a mean. Then the sequence (a_n) of positive real numbers is called K-sequence if

$$a_n = K(a_{n-1}, a_{n+1}) \text{ for every } n \geq 2.$$

  Thus, the logarithmic sequence is L-sequences.

- In [7], the characteristics *exponent of convergence* of a sequence $(a_n) \in \mathscr{S}_\infty = \{(a_n) \in \mathscr{S} : \lim_{n\to\infty} a_n = \infty\}$ is defined, as the infimum of all positive real numbers α such that the series $\sum_{n=1}^{\infty} a_n^{-\alpha}$ is convergent and, in case no such α exist, we define the exponent of convergence as ∞. Let us denote the exponent of convergence of $(a_n) \in \mathscr{S}_\infty$ by $\lambda((a_n))$. It can be easily seen that $\lambda((a_n))$ can be interpreted as a measure of rate of convergence of (a_n) to infinity. Thus for $(a_n) \in \mathscr{S}_\infty$, we put

$$\lambda((a_n)) = \inf\Big\{\alpha > 0 : \sum_{n=1}^{\infty} \frac{1}{a_n^\alpha} < \infty\Big\}.$$

  If $q > \lambda((a_n))$, then $\sum_{n=1}^{\infty} a_n^{-q} < \infty$ and if $q < \lambda((a_n))$, then $\sum_{n=1}^{\infty} a_n^{-q} = \infty$. In case $q = \lambda((a_n))$, the series $\sum_{n=1}^{\infty} a_n^{-q}$ can be either convergent or divergent.
  From ([7], p. 26, Exercises 113, 114), it follows that the set of all possible values of λ forms the whole interval $[0,\infty]$, i.e. $\{\lambda((a_n)) : (a_n) \in \mathscr{S}_\infty\} = [0,\infty]$ and if $(a_n) \in \mathscr{S}_\infty$, then $\lambda((a_n))$ can be calculated by

$$\lambda((a_n)) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$

- For $0 < q$, we define the set

$$\mathscr{I}_{1/x^q} = \Big\{(a_n) \in \mathscr{S}_\infty : \sum_{n=1}^{\infty} \frac{1}{a_n^q} < \infty\Big\}.$$

  By values of λ, we define the following sets (similarly to [9] and [10]):
  $\mathscr{I}_{<q} = \{(a_n) \in \mathscr{S}_\infty : \lambda((a_n)) < q\}$ for $0 < q \leq \infty$,
  $\mathscr{I}_{\leq q} = \{(a_n) \in \mathscr{S}_\infty : \lambda((a_n)) \leq q\}$ for $0 \leq q \leq \infty$ and
  $\mathscr{I}_0 = \{(a_n) \in \mathscr{S}_\infty : \lambda((a_n)) = 0\}$.
  Obviously, $\mathscr{I}_{\leq 0} = \mathscr{I}_0$ and $\mathscr{I}_{\leq\infty} = \mathscr{S}_\infty$.

  Families $\mathscr{I}_{<q}, \mathscr{I}_{\leq q}$, and $\mathscr{I}_{1/x^q}$ are related to $0 < q < q'$ by following inclusions (similarly to [9], Th.2.1.)

$$\mathscr{I}_0 \subsetneq \mathscr{I}_{<q} \subsetneq \mathscr{I}_{1/x^q} \subsetneq \mathscr{I}_{\leq q} \subsetneq \mathscr{I}_{<q'},$$

  and the difference of successive sets is infinite, so equality does not hold in any of the inclusions.

Overview of known results

In this section, we mention known results related to the topic of this paper and some ones are generalized in the proofs of our theorems.

$(S1)$ Let a_1, a_2 be two positive numbers.

i) If $a_1 = a_2$, then a_1, a_2 are the first two terms of the constant sequence $a_1, a_1, \cdots, a_1, \cdots$ which is logarithmic.

ii) Let $a_1 \neq a_2$. Then there exists a logarithmic sequence (a_k) such that if $a_1 < a_2$, then $a_1 < a_2 < \cdots < a_k < a_{k+1} < \cdots$, and if $a_1 > a_2$, then $a_1 > a_2 > \cdots > a_k > a_{k+1} > \cdots$. ([8], Theorem 2.1.)

$(S2)$ Let (a_k) be a logarithmic sequence of positive numbers. Then we have

i) If $a_1 > a_2$, then series $\sum_{k=1}^{\infty} a_k$ converges and

$$\sum_{k=1}^{\infty} a_k \leq \frac{a_1^2}{a_1 - a_2}.$$

([8], Theorem 2.2.)

ii) If $a_1 < a_2$, then $\lim_{k\to\infty} a_k = \infty$. ([8], Theorem 2.3.)

$(S3)$ Let (a_n) be a increasing logarithmic sequence of positive numbers. Then the following statements hold.

i) The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, moreover

$$a_n > \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \text{ for every } \alpha \geq \frac{1}{3} \text{ and } n \geq 2.$$

([6], Corollary 2.3. and Theorem 2.1.i))

ii) The inequality

$$a_{n+1} - a_n > \left(\sqrt{a_2} - \sqrt{a_1}\right)^2 (n+1)$$

holds for every $n \geq 2$.

([6], Theorem 2.6.)

iii) Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 1.$$

([6], Theorem 2.7. and Corollary 2.8.)

$(S4)$ Let (a_n) be a decreasing logarithmic sequence of positive numbers. Then the following statements hold.

i) Then

$$a_n < \left(\frac{a_2^\alpha - a_1^\alpha}{2}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} \text{ for every } \alpha < 0 \text{ and } n \geq 2.$$

([6], Theorem 2.1.ii))

ii) Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{q^n} = 0 \text{ for every real } q > 0.$$

([6], Theorem 2.7. and Corollary 2.8.)

Results

In this section, we generalize the some results of $(S1)-(S4)$ for a suitable class of means. The following Theorem 1 generalizes the result $(S1)$ and it holds for all known means from part 2.

Theorem 1. *Suppose that $K:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a strict mean such that $K(x,y)$ is continuous with respect to x and y. Let a_1,a_2 be two positive numbers. Then we have*

i) If $a_1=a_2$, then a_1,a_2 are the first two terms of the constant sequence $a_1,a_1,\cdots,a_1,\cdots$, which is a K-sequence.

ii) Let $a_1\neq a_2$. Then there exists a sequence (a_n) which is a K-sequence for a suitable class of means. Precisely, if $a_1<a_2$ and $\lim_{y\to\infty}K(x,y)=\infty$, then (a_n) is an increasing sequence and if $a_1>a_2$ and $\lim_{x\to 0}K(x,y)=0$, then (a_n) is a decreasing sequence.

Proof. The case i) is trivial.
ii) Let $a_1\neq a_2$, say $a_1<a_2$. We shall prove that there exists $y>0$, $y\neq a_1$, such that $a_2=K(a_1,y)$. Let us consider the function $f(y)=K(a_1,y)$ on the interval (a_1,∞). From the statements of this theorem, we have that the function f is continuous on (a_1,∞) and holds

$$\lim_{y\to a_1} f(y)=f(a_1)=a_1 \text{ and } \lim_{y\to\infty} f(y)=\infty. \tag{1}$$

Since function f is continuous, f has the Darboux property on (a_1,∞), so from (1) follows the existence of $a_3\in(a_1,\infty)$ such that $f(a_3)=a_2$, i.e. $K(a_1,a_3)=a_2$. Since K is a strict mean, then $a_1<a_2<a_3$.

It can be shown similarly that if $a_1>a_2$, then there exists $a_3<a_2$ such that $K(a_3,a_1)=a_2$. In this case, let us consider the function $f(x)=K(x,a_1)$ defined on the interval $(0,a_1)$. So we have constructed the term a_3 of the sequence (a_n).

Now is already easy to construct the sequence (a_n) corresponding to ii), by mathematical induction. □

The following Theorem 2 and Theorem 3 generalize some results from $(S2)-(S4)$ with the exception of (S3)-ii) and (S4)-i).

Theorem 2. *Suppose that $K:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a mean. Then we have*

i) If sequence (a_n) of positive real numbers is decreasing K-sequence and

$$M_0(a,b)\leq K(a,b) \text{ for all } a,b\in\mathbb{R}^+,$$

then $\lim_{n\to\infty}a_n=0$, moreover

$$\sum_{n=1}^{\infty}a_n\leq\frac{a_1^2}{a_1-a_2}.$$

This is a refined and generalized (S2)-i).

ii) If sequence (a_n) of positive real numbers is increasing K-sequence and for some $0\leq\alpha$ holds

$$K(a,b)\leq M_\alpha(a,b) \text{ for all } a,b\in\mathbb{R}^+,$$

then $(a_n)\in\mathscr{S}_\infty$ ($\lim_{n\to\infty}a_n=\infty$), moreover $a_n^\alpha>n\left(\frac{a_2^\alpha-a_1^\alpha}{2}\right)$ for every $n\geq 2$ and $(a_n)\in\mathscr{I}_{\leq\alpha}$. Therefore, for every $q>\lambda((a_n))$ real there is

$$(a_n)\in\mathscr{I}_{1/x^q}, \text{ i.e. } \sum_{n=1}^{\infty}\frac{1}{a_n^q}<\infty$$

and for every $0 \le q' < \lambda((a_n))$*, we have*

$$(a_n) \notin \mathscr{I}_{1/x^{q'}},\ \textit{i.e.}\ \sum_{n=1}^{\infty} \frac{1}{a_n^{q'}} = \infty.$$

This is a generalization of (S2)-ii) and (S3)-i) (in case $K = L$*, and* $\alpha \ge \frac{1}{3}$*).*

Proof. i). According to the assumption for every $n \ge 2$ we have

$$\sqrt{a_{n-1}a_{n+1}} \le K(a_{n-1}, a_{n+1}) = a_n \text{ thus } \frac{a_{n+1}}{a_n} \le \frac{a_n}{a_{n-1}} .$$

From this for every $n \in \mathbb{N}$ we get

$$\frac{a_{n+1}}{a_n} \le \frac{a_2}{a_1} = q < 1 .$$

Multiplying these inequalities for $n = 1, 2, \cdots, k$ we obtain

$$a_{k+1} = \frac{a_{k+1}}{a_k} \cdot \frac{a_k}{a_{k-1}} \cdots \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \cdot a_1 \le a_1 \cdot q^k \text{ for all } k = 0, 1, 2, \cdots . \tag{2}$$

Then

$$\sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} a_{k+1} \le \sum_{k=0}^{\infty} a_1 \cdot q^k = a_1 \cdot \frac{1}{1-q} = \frac{a_1^2}{a_1 - a_2} .$$

ii). According to the assumption for every $n \ge 2$ and for $\alpha > 0$ we have

$$K(a_{n-1}, a_{n+1}) = a_n \le \left(\frac{a_{n-1}^{\alpha} + a_{n+1}^{\alpha}}{2} \right)^{\frac{1}{\alpha}} \text{ thus } 2a_n^{\alpha} \le a_{n-1}^{\alpha} + a_{n+1}^{\alpha} .$$

Which is equivalent with

$$a_n^{\alpha} - a_{n-1}^{\alpha} \le a_{n+1}^{\alpha} - a_n^{\alpha} .$$

Therefore for every $n \in \mathbb{N}$

$$a_n^{\alpha} = \sum_{j=1}^{n-1} (a_{j+1}^{\alpha} - a_j^{\alpha}) + a_1^{\alpha} > (n-1)(a_2^{\alpha} - a_1^{\alpha}) > n \frac{a_2^{\alpha} - a_1^{\alpha}}{2} .$$

From this we obtain $a_n > n^{\frac{1}{\alpha}} \cdot c$, where $c = c(a_1, a_2, \alpha) = (\frac{a_2^{\alpha} - a_1^{\alpha}}{2})^{\frac{1}{\alpha}}$ is a positive constant. Thus $(a_n) \in \mathscr{S}_{\infty}$. Then

$$\lambda((a_n)) = \limsup_{n\to\infty} \frac{\log n}{\log a_n} \le \limsup_{n\to\infty} \frac{\log n}{\log c + \frac{1}{\alpha} \log n} = \limsup_{n\to\infty} \frac{1}{\frac{\log c}{\log n} + \frac{1}{\alpha}} = \alpha.$$

In case if $\alpha = 0$, similarly to proof i)., we obtain

$$a_{k+1} = \frac{a_{k+1}}{a_k} \cdot \frac{a_k}{a_{k-1}} \cdots \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} \cdot a_1 \ge a_1 \cdot q^k \text{ for all } k = 0, 1, 2, \cdots ,$$

where $q = \frac{a_2}{a_1} > 1$. Thus $(a_n) \in \mathscr{S}_{\infty}$. Then

$$0 \le \lambda((a_n)) = \limsup_{n\to\infty} \frac{\log n}{\log a_n} \le \limsup_{n\to\infty} \frac{\log n}{\log a_1 + (n-1)\log q} = 0.$$

□

Corollary 1. *Suppose that $K:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a mean. If sequence (a_n) of positive real numbers is decreasing K-sequence and*

$$M_0(a,b)\leq K(a,b) \text{ for all } a,b\in\mathbb{R}^+ ,$$

then the sequence $\left(\frac{1}{a_n}\right)\in\mathscr{S}_\infty$ and $\lambda\left(\left(\frac{1}{a_n}\right)\right)=0$.

Proof. By (2) from Theorem 2, we have

$$\frac{1}{a_n}\geq\frac{1}{a_1}\left(\frac{a_1}{a_2}\right)^{n-1} \text{ for all } n=1,2,\cdots .$$

Since $\frac{a_1}{a_2}>1$ then $\left(\frac{1}{a_n}\right)\in\mathscr{S}_\infty$ and

$$0\leq\lambda\left(\left(\frac{1}{a_n}\right)\right)=\limsup_{n\to\infty}\frac{\log n}{\log\frac{1}{a_n}}\leq\limsup_{n\to\infty}\frac{\log n}{\log\frac{1}{a_1}+(n-1)\log\frac{a_1}{a_2}}=0.$$

□

Theorem 3. *Suppose that $K:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a homogeneous strict mean such that*

$$M_0(x,y)<K(x,y) \text{ for all } x,y\in\mathbb{R}^+ ,\ x\neq y$$

and $K(x,y)$ is continuous with respect to y. Then the following statements hold:

i) If sequence (a_n) of positive real numbers is increasing K-sequence then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1 \text{ and } \lim_{n\to\infty}\frac{a_n}{q^n}=0 \text{ for every real } q>1 ,$$

which is a generalization of (S3)-iii)(in case $K=L$).

ii) If sequence (a_n) of positive real numbers is decreasing K-sequence then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=0 \text{ and } \lim_{n\to\infty}\frac{a_n}{q^n}=0 \text{ for every real } q>0 ,$$

which is a generalization of (S4)-ii)(in case $K=L$).

Proof. According to the assumptions for $n\geq 2$ we get

$$a_n=K(a_{n-1},a_{n+1})=a_{n-1}K\left(1,\frac{a_{n+1}}{a_{n-1}}\right)$$

therefore

$$\frac{a_n}{a_{n-1}}=K\left(1,\frac{a_{n+1}}{a_n}\frac{a_n}{a_{n-1}}\right) . \tag{3}$$

On the other hand from assumption this theorem for every $n\geq 2$ we have

$$\sqrt{a_{n-1}a_{n+1}}<K(a_{n-1},a_{n+1})=a_n , \text{ thus } \frac{a_{n+1}}{a_n}<\frac{a_n}{a_{n-1}} .$$

Since the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is decreasing and $1<\frac{a_{n+1}}{a_n}$ for all $n\geq 2$ in case if (a_n) is increasing then there exists finite $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=x$, where $1\leq x<\frac{a_2}{a_1}$. If the sequence (a_n) is decreasing

then obviously $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 2$ and there exists finite $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$, where $0 \leq x < \frac{a_2}{a_1}$. Then by (3) in case $n \to \infty$ we obtain $x = K(1,x^2)$, which in case $x \in (0,1) \cup (1,\infty)$ it can't be true, because from assumption of this theorem we have $K(1,x^2) > \sqrt{1 \cdot x^2} = x$. Thus, $x = 1$ if (a_n) is increasing and $x = 0$ if sequence (a_n) is decreasing. Further, consider the power series

$$\sum_{n=1}^{\infty} a_n x^n \, .$$

i) In case if (a_n) is increasing sequence then from above we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, which implies that the radius of its convergence is $R = 1$. Thus, for every $0 < x < 1$ the series $\sum_{n=1}^{\infty} a_n x^n$ converges. Consequently $\lim_{n\to\infty} a_n x^n = 0$. Denoting $q = \frac{1}{x}$ we have $q > 1$ arbitrary and $\lim_{n\to\infty} \frac{a_n}{q^n} = 0$.
ii) In case if (a_n) is decreasing sequence then from above we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, which implies that the radius of convergence R of the consedered power series is infinity. Thus for every real $x > 0$ we have $\lim_{n\to\infty} a_n x^n = 0$.

□

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GENERALIZED LAGUERRE DOMAIN IDENTIFICATION OF CONTINUOUS TIME MODELS

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Abstract: *This paper explores the application of Laguerre orthonormal functions, focusing on simple Laguerre functions (SLFs) and generalized Laguerre functions (GLFs), in system modeling, identification, and control theory. GLFs, with an additional free parameter, offer extended capabilities and are relatively novel in control theory research. This article summarizes previous research and presents new examples to illustrate the potential and versatility of Laguerre-based methods in practical control applications, focusing on dynamic system identification and delay estimation.*

Keywords: system modeling, identification, Laguerre functions, control theory

INTRODUCTION

The utilization of Laguerre orthonormal functions in system modeling and identification has a lengthy history dating back to their introduction in [1] and [2]. Over the years, numerous papers have been dedicated to exploring various theoretical approaches in this domain, documenting the evolution and diverse applications of these functions.

Simple Laguerre functions (SLFs) possess several advantageous properties that render them suitable for control applications. Primarily applied in system modeling and identification, SLFs have been extensively utilized, as evidenced by works such as [3, 4, 5, 6, 7]. However, their application in control theory, as demonstrated in [8, 9], remains relatively uncommon. It's noteworthy that SLFs offer potential benefits in this area as well. Generalized Laguerre functions (GLFs), as referenced in [10, 11], extend the capabilities of SLFs by introducing an additional free parameter, potentially enhancing the quality of outcomes obtained through their application.

The utilization of generalized Laguerre functions with a generalization parameter α in system control theory represents a relatively novel area of research. While GLFs have been applied in theoretical mathematics, as evidenced by works such as [12, 13], their potential within the realm of control theory warrants further exploration.

In [14], a method based on GLFs for system identification was introduced, marking a significant step in this direction. This method was subsequently compared with the conventional least squares-based identification with state variable filters (LSSVF) in [15]. The advantage of optimal parameter selection within GLFs and its impact on the quality of identification was highlighted.

Another notable application of the generalization of the simple Laguerre function method is its extension to the dead time estimation problem, as detailed in [16]. This approach builds upon SLF-based dead time estimation methods, as found in [17]. The study demonstrated that through the optimal selection of free parameters, it is feasible to achieve accurate dead time estimation using a reduced number of Laguerre approximation series.

These examples underscore the potential for advancements in control theory through the adoption of GLF-based approaches, presenting avenues for further research and refinement of SLF-based algorithms.

In this article, we present the application of Laguerre functions for the identification of dynamic systems and the approximation of delays. Specifically, we will summarize findings from [15] and [16], which showcase the effectiveness of Laguerre-based methods in these domains. Additionally, we will introduce new examples to further illustrate the potential and versatility of these approaches in practical control applications.

We will utilize the subsequent equation describing the dynamical system incorporating dead time.

$$y(t) = F(q)u(t - T_d) + e(t), \tag{1}$$

1 SYSTEM IDENTIFICATION WITH GENERALIZED LAGUERRE FUNCTIONS

In this section, we will conduct a comparison between two methods for the direct identification of continuous-time dynamical systems. Our focus will be on the direct identification of continuous-time models using sampled discrete-time data of input and output $\{u(t_k), y(t_k)\}$. This problem is of significant engineering interest, as highlighted in references such as [18], [19], and [20].

We will conduct a comparative analysis of two distinct methodologies for continuous-time system identification from sampled data. Firstly, we will provide a brief overview of the method based on Generalized Laguerre Functions (GLF) and compare it with the conventional Least Squares (LS) method employing State Variable Filters (SVF), as discussed in [21] and [20]. The GLF-based identification method was originally proposed in [14, 15]. The continuous-time models 1 (without time delay and noise) in the Laplace domain are expressed as:

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}. \tag{2}$$

The discrete sampled time data of input and output $\{u(t_k), y(t_k)\}$ can be expanded into finite Laguerre series utilizing generalized Laguerre functions $L_n^{(\alpha)}(t, p)$, where α serves as the generalization parameter (with $\alpha = 0$ yielding the simple Laguerre functions) and p represents the time-scale parameter. Specifically,

$$u(t_k) \approx \sum_{n=0}^{N_1} u_n^T L_n^{(\alpha_1)}(t_k, p_1), \quad y(t_k) \approx \sum_{n=0}^{N_2} y_n^T L_n^{(\alpha_2)}(t_k, p_2), \tag{3}$$

$$L_n^{(\alpha)}(t, p) = \sqrt{\frac{2p\Gamma(n+1)}{\Gamma(n+\alpha+1)}} e^{-pt} (2pt)^{\alpha/2} \frac{t^{-\alpha} e^{2pt}}{n!} \frac{d^n}{dt^n}(e^{-2pt} t^{n+\alpha}), \tag{4}$$

The terms u_n^T and y_n^T represent the expansion coefficients for the generalized Laguerre series of the sampled discrete-time data of input and output $\{u(t_k), y(t_k)\}$ over the time interval $[0, T]$. The optimal selection of free parameters α and p has been addressed in [22] and further discussed in [14]. In this context, it was found that the optimal parameter α should be approximated to the nearest even number to facilitate integer transfer function approximation and avoid non-integer issues.

We can approximate the model (1) by the following generalized Laguerre model:

$$\sum_{n=0}^{N_2} y_n^T \Phi(n,\alpha_2,p_2)\frac{P_n^{\alpha_2}(s)}{(s+p_2)^{n+1+\alpha_2/2}} \approx \frac{B(s)}{A(s)}\sum_{n=0}^{N_1} u_n^T \Phi(n,\alpha_1,p_1)\frac{P_n^{\alpha_1}(s)}{(s+p_1)^{n+1+\alpha_1/2}}, \tag{5}$$

$$\mathscr{L}\{L_n^{\alpha}(t,p)\} = \Phi(n,\alpha,p)\frac{P_n^{\alpha}(s)}{(s+p)^{n+1+\alpha/2}}, \tag{6}$$

$$\Phi(n,\alpha,p) = \sqrt{(2p)^{\alpha+1}\Gamma(n+1)\Gamma(n+\alpha+1)}, \quad P_n^{\alpha}(s) = \sum_{m=0}^{n} A_{n,m}^{\alpha} \sum_{i=0}^{n-m} \binom{n-m}{i} s^{n-m-i}p^i, \tag{7}$$

$$A_{n,m}^{\alpha} = \frac{(-1)^m (2p)^m \Gamma(m+\alpha/2+1)}{m!(n-m)!\Gamma(m+\alpha+1)}. \tag{8}$$

After multiplying both sides of equation (5) by the term $(s+p_2)^{N_2+1+\alpha_2/2}(s+p_1)^{N_1+1+\alpha_1/2}$ and carrying out subsequent calculations, we derive the following approximation of the transfer function for the model (1):

$$\tilde{F}(s,N_1,N_2,T) \approx \frac{(s+p_1)^{N_1+1+\alpha_1/2}}{(s+p_2)^{N_2+1+\alpha_2/2}} \times \frac{\sum_{n=0}^{N_2} u_n^T \Phi(n,\alpha_2,p_2) P_n^{\alpha_2}(s)(s+p_2)^{N_2-n}}{\sum_{n=0}^{N_1} u_n^T \Phi(n,\alpha_1,p_1) P_n^{\alpha_1}(s)(s+p_1)^{N_1-n}}, \tag{9}$$

The order of the approximation (9) is given by $N = N_1 + N_2 + 1 + \max(\alpha_1/2, \alpha_2/2)$. The following equation should hold:

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\lim_{T\to\infty} \tilde{F}(s,N_1,N_2,T) = F(s). \tag{10}$$

The count of dominant Hankel singular values of system (9) assists us in determining the order of the original system and in selecting the appropriate order K for the reduced system. The balanced truncation technique involves discarding the portion of the system corresponding to the $N-K$ smallest Hankel singular values in the Singular Value Decomposition (SVD) of the approximated systems, as detailed in papers such as [23]. In our identification procedure, we will utilize the MATLAB implementation "balred" for this approximation.

2 EXAMPLE OF SYSTEM IDENTIFICATION

In the upcoming experiments, we will compare the GLF identification method with an identification approach based on least squares estimation utilizing state variable filters (LSSVF). For the identification process, the input signal was selected as $u(t) = e^{-t}$. The step input responses of the reduced approximated systems with order K (reduced from order N) using both the GLF method and the LS method with SVF (LSSVF) characterized by $L(s) = \frac{1}{(s+\lambda)^n}$ are shown in figures below. The dominant Hankel singular values of the Generalized Laguerre functions approximations of order N are visualized on the logarithmic scale. Additionally, the corresponding relative RMS errors (rRMSE) are provided.

$$rRMSE = \frac{RMS(y(t)-\hat{y}(t))}{RMS(y(t))} * 100\%, \tag{11}$$

where $y(t)$ represents the step input response of the original system and $\hat{y}(t)$ denotes the step input response of the approximated systems. The parameter n in the definition of the state variable filter $L(p)$ corresponds to the order of the original system, while λ is selected to be larger than the estimated bandwidth, as discussed in [19].

The 1st system is the third order dynamical system with two complex conjugate poles and one real pole:

$$F(s) = \frac{1}{5s^3 + 6s^2 + 2s + 1}, \qquad u(t) = e^{-t}. \tag{12}$$

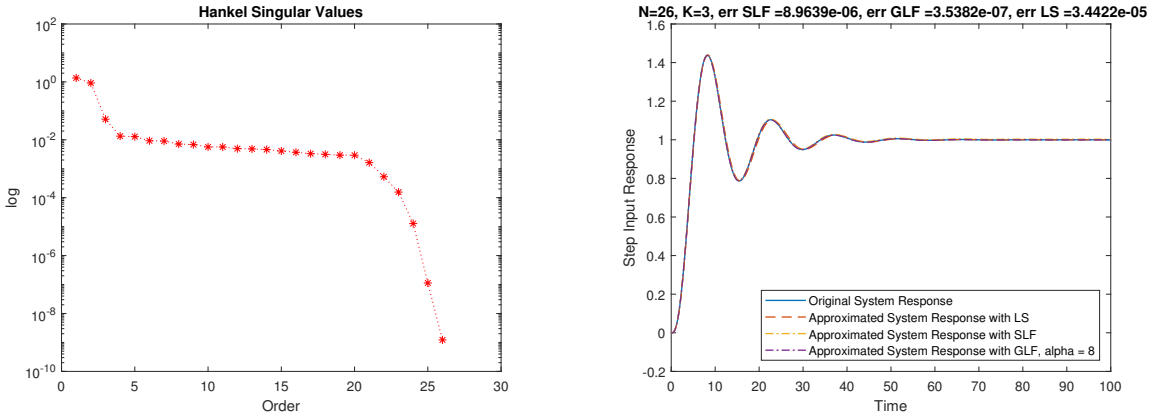

Figure 1: Step input response of the 1st system and Hankel singular values of GLF approximation

The 2nd system is the second order dynamical system with two real poles:

$$F(s) = \frac{5}{s^2 + 7s + 8}, \qquad u(t) = e^{-t}. \tag{13}$$

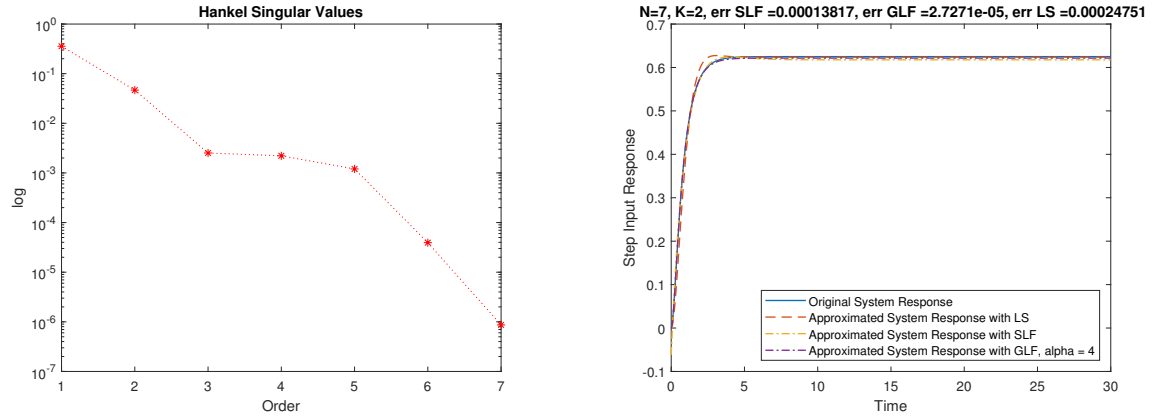

Figure 2: Step input response of the 2nd system and Hankel singular values of GLF approximation

3 DEAD TIME ESTIMATION TECHNIQUES

In the article [17], the estimation of dead time T_d from expansion coefficients $u_n(\alpha, p)$ and $y_n(\alpha, p)$ of input and output signals into the Laguerre functions series was demonstrated. Let's define the vector $\varphi = [\varphi_{k,1}, \varphi_{k,2}, \ldots, \varphi_{k,N+1}]$, where

$$\begin{aligned} \varphi_{k,1} &= u_0(\alpha, p), \\ \varphi_{k,l+1} &= \frac{(-2)^l}{l!(l-1)!} \sum_{m=0}^{k-l} \frac{(k-m-1)!}{(k-m-l)!} u_m(\alpha, p), k \geq l > 0, \\ \varphi_{k,l+1} &= 0, N \geq l > k. \end{aligned} \tag{14}$$

Following these steps, we can generate the following three vectors:

$$Y = [y_0(\alpha,p), y_1(\alpha,p), \ldots, y_N(\alpha,p)]^T,$$
$$\Phi = [\varphi_0, \varphi_1, \ldots, \varphi_N]^T,$$
$$\Theta = [1, pT_d, \ldots, (pT_d)^N]^T e^{-pT_d}.$$

The dead time can be then estimated as

$$\hat{T}_d = p^{-1}(\hat{\Theta}^T(1:N)\hat{\Theta}(1:N))^{-1}\hat{\Theta}^T(1:N)\hat{\Theta}(2:N+1), \hat{\Theta} = \Phi^{-1}Y.$$

In the equation above, MATLAB notation for vector elements is utilized. The selection of the optimal time-scale parameter p and generalization parameter α remains an open question. The determination of the optimal parameters α and p when approximating the given signal $x(t) \in L_2[0,\infty)$ by the truncated series $x_N(t) = \sum_{n=0}^{N} c_n(\alpha,p) L_n^{(\alpha)}(t,p)$ of the GLF was investigated in [22]. The study demonstrated how to select the parameters α and p using only a few specific measurements to minimize the truncation error $\zeta_N(\alpha,p) = \|x(t) - x_N(t)\|_2^2 = \sum_{n=N+1}^{\infty}(c_n(\alpha,p))^2$. According to the findings presented in [22], we can determine the optimal choice of the time-scale parameter as $p = \sqrt{\frac{m_{-1}m_2}{|m_1 m_{-1} - m_0^2|}}$ and the generalization parameter as $\alpha = \frac{m_0}{m_{-1}} 2p$.

The moments m_{-1} to m_2 can be calculated as follows $m_{-1} = (x(t), \frac{1}{t}x(t))$, $m_0 = (x(t), x(t))$, $m_1 = (x(t), tx(t))$, $m_2 = (x'(t), tx'(t))$.

The precise selection of these parameters is particularly crucial when dealing with a low number N of terms in the Laguerre series. In practical computations, we typically compute the initial few terms in the Laguerre expansion and leverage the orthogonal transform's "energy compaction property". This property implies that a significant portion of the total signal energy is concentrated in the first few (generalized) Fourier coefficients of the (generalized) Fourier series for the original data.

In the upcoming section, we will compare the Laguerre estimation method with alternative approaches to dead time estimation, namely delayest and finddelay.

Delayest is a dead time estimation tool available in the MATLAB System Identification Toolbox. It is designed to estimate dead time in a system, given the system input $u(t)$ and output $y(t)$ based on the dataset. Delayest accomplishes this by estimating a low-order ARX model of the dynamical system with a range of delays. Subsequently, it selects the delay corresponding to the best fit. The model takes the form: $y(t) + a_1 y(t-1) + \ldots + a_{n_a} y(t-n_a) = b_1 u(t-T_d) + \ldots + b_{n_b} u(t-n_b-T_d) + e(t)$ Detailed instructions on how to use the Delayest procedure can be found in [24].

The finddelay MATLAB function uses the xcorr function to determine the cross-correlation between each pair of signals at all possible lags specified by the user. The normalized cross-correlation between each pair of signals is then calculated. The estimated delay is given by the negative of the lag for which the normalized cross-correlation has the largest absolute value.

4 EXAMPLES OF DEAD TIME ESTIMATION

For the dead time estimation examples, MATLAB-generated data was employed. In the examples the parameters α_u, α_y, p_u and p_y were computed. The final generalization parameter α and time-scale parameter p were determined as the arithmetic mean of parameters for input and output signals. This parameter selection approach was utilized for the Generalized Laguerre Function (GLF) method in the examples below and it was compared with SLF method with

proposed time-scales $p = 0.8, p = 0.955$ from [17] (SLF08 and SLF0955), as well as with the SLF method using the optimal choice of the time-scale parameter p (SLFopt). Additionally, comparisons were made with the delayest procedure from the MATLAB System Identification Toolbox and with the finddelay MATLAB procedure. In all the MATLAB generated data Gaussian white noise with $SNR_{dB} = 60$ was added to the output. The dead time in all examples was $T_d = 5$. The relative percentage errors $\varepsilon = \left|1 - \frac{\hat{T}_d}{T_d}\right| * 100\%$ were compared for estimated dead time $\hat{T}_d$.

The 1st system $F_1(s)$ is a second order dynamical system with a pair of complex conjugate poles. The 2nd system $F_2(s)$ is a second order dynamical system with two real poles. The data of these systems were generated by MATLAB.

$$F_1(s) = \frac{1}{(4s^2+s+1)}e^{-sT_d}, \quad F_2(s) = \frac{1}{6s^2+8s+1}e^{-sT_d}, \quad u(t) = e^{-t}, N_1 = 10, N_2 = 5. \quad (15)$$

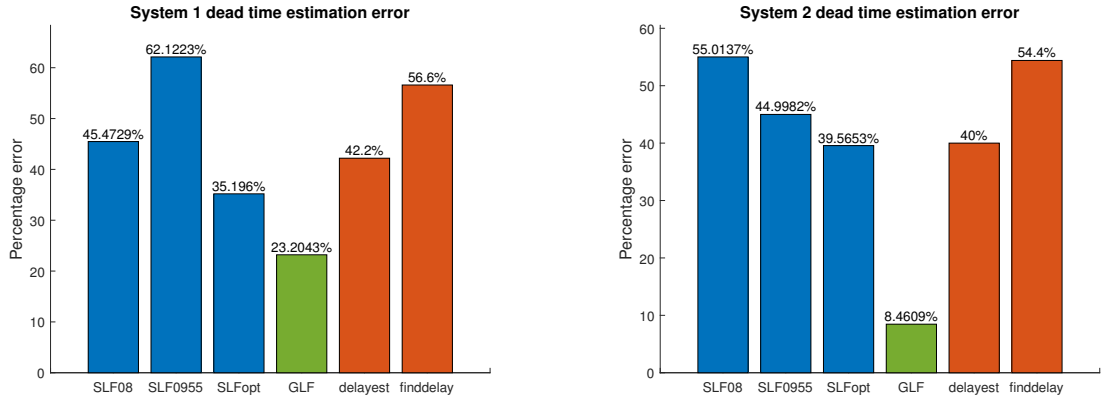

Figure 3: Dead time estimation error of 1st and 2nd system for SLF08, SLF0955, SLFopt, GLF, delayest and finddelay methods

CONCLUSION

The modeled dynamical systems in section 2 were effectively identified using both the generalized Laguerre Function (GLF) and the Least Squares State Variable Filters (LSSVF) methods. Our analysis revealed that employing generalized Laguerre functions within the GLF method, coupled with a proper selection of parameters α and p, yielded superior results compared to the conventional least squares SVF method in the task of system identification.

In the section 4 we have demonstrated the significance of selecting optimal parameters for Laguerre-based dead time estimation to minimize estimation errors. Our comparison involved dead time estimation using both simple Laguerre Functions and generalized Laguerre Functions methodologies, alongside the delayest procedure from the MATLAB System Identification Toolbox and the finddelay procedure. We conducted evaluations on data generated within MATLAB.

Our findings showcased the superior performance of generalized Laguerre-based estimation over the simple Laguerre functions approach, as well as over the delayest and finddelay methods. Particularly, we emphasized the criticality of selecting the optimal free parameters for GLF, especially when dealing with a limited number N of terms in truncated expansions and when working with noisy data characterized by low Signal-to-Noise Ratio (SNR_{dB}).

We found that achieving better results in dead time estimation is feasible by enabling the computation of free parameters α and p independently for input and output signals.

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ON ONE SUFFICIENT CONDITION FOR CONVERGENCE OF SERIES USING IDEALS

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Abstract: *R. Estrada and R. P. Kanwal presented in 1986 a condition ensuring the convergence of series with positive terms, employing the concept of zero asymptotic density of a set. We extend their findings in this study, utilizing the concept of special ideals characterized by quasi-density and the convergence exponent of a sequence. We provide several examples of such ideals where the analogous result applies, along with those where it does not. These results contribute to the broader understanding of series convergence and enrich the theory by exploring the interplay between ideals and density concepts.*

Keywords: convergence of series, ideal, density of sets.

INTRODUCTION

Let us describe the theorem of R. Estrada and R. P. Kanwal (see [2]).

Theorem A. *Let $\sum_{n=1}^{\infty} a_n$ is a series with positive terms. If for any $A \subseteq \mathbb{N}$ such that $d(A) = 0$ we have $\sum_{n\in A} a_n < +\infty$, then $\sum_{n=1}^{\infty} a_n < +\infty$.*

Recall that $d(A)$ is the asymptotic density of set $A \subseteq \mathbb{N}$ (see [4] and Example 3).

We know many ideals, which are created by using some density (see [5], [6]). Now we define the notion of ideal.

Definition 1. *Let $\mathscr{I} \subset 2^{\mathbb{N}}$ is a system of subsets of $\mathbb{N}$. Then $\mathscr{I}$ is an* ideal *on $\mathbb{N}$ if the following holds:*

i) $\emptyset \in \mathscr{I}$,

ii) $A, B \in \mathscr{I}$ *then* $A \cup B \in \mathscr{I}$,

iii) $A \in \mathscr{I}$ *and* $B \subset A$ *then* $B \in \mathscr{I}$.

The ideal $\mathscr{I}$ is called admissible *if $\{a\} \in \mathscr{I}$ for each $a \in \mathbb{N}$.*

In the following we will investigate the ideals with special properties.

Example 2. Let $A \subseteq \mathbb{N}$. Then $\mathscr{I}_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is an admissible ideal on $\mathbb{N}$.

Example 3. An another way to define the admissible ideal is via characteristic function. Let $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$. Define $d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k)$. Denote $\underline{d}(A) = \liminf d_n(A)$ and $\overline{d}(A) = \limsup d_n(A)$. Then $\overline{d}(A)$ is called *upper asymptotic density* and the $\underline{d}(A)$ is called *lower asymptotic density* of $A \subseteq \mathbb{N}$. In case when $\underline{d}(A) = \overline{d}(A) = d(A)$, then we say that A has the asymptotic density. It is clear that $d(A) \in \langle 0,1\rangle$. By using this density we can define the admissible ideal

$$\mathscr{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}.$$

For example, the sets $\mathbb{N}^2 = \{1,4,9,\dots,n^2,\dots\}$, $\mathbb{P} = \{2,3,5,7,\dots\}$ (the set of all prime numbers) have zero density.

Example 4. Let $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$. Put $\delta_n(A) = \frac{1}{S_n}\sum_{k=1}^{n}\frac{\chi_A(k)}{k}$, where $S_n = \sum_{k=1}^{n}\frac{1}{k}$. We define the *lower logarithmic density* of a set $A \subseteq \mathbb{N}$ as $\underline{\delta}(A) = \liminf \delta_n(A)$, and the *upper logarithmic density* of a set $A \subseteq \mathbb{N}$ as $\overline{\delta}(A) = \limsup \delta_n(A)$. In case when $\underline{\delta}(A) = \overline{\delta}(A) = \delta(A)$, then we say that $A \subseteq \mathbb{N}$ has the logarithmic density. It is well known that for each $A \subseteq \mathbb{N}$

$$\underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A). \tag{1}$$

Moreover $S_n = \sum_{k=1}^{\infty}\frac{1}{k} = \log n + \gamma + o\left(\frac{1}{n}\right)$, where γ is the Euler's constant. On the basis of this equality in definition of logarithmic density replace S_n with $\log n$. Then by using this density we can define admissible ideal

$$\mathscr{I}_\delta = \{A \subseteq \mathbb{N} \,:\, \delta(A) = 0\}.$$

From the (1) we have $\mathscr{I}_d \subsetneq \mathscr{I}_\delta$. The inclusion is strict, because the logarithmic density of set $A = \bigcup_{k=2}^{\infty} A_k$, where $A_k = \left\{k^{k^2}+1, k^{k^2}+2, \ldots, k^{k^2+1}\right\}$, $k \in \mathbb{N}$, $k \geq 2$ is equal to 0, but $\overline{d}(A) = 1$ and $\underline{d}(A) = 0$.

The densities in Examples 3 and 4 can be defined via regular matrix (see [6]).

Example 5. Let $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$ and $m \geq 0$. Put $|A(m,n)| = \mathrm{card}(A \cap \{m+1, \ldots, m+n\})$. Define the $\alpha_n(A) = \liminf_{m\to\infty} A(m+1, m+n)$ and $\alpha^n(A) = \limsup_{m\to\infty} A(m+1, m+n)$. It can be shown that there exist $\underline{u}(A) = \lim_{n\to\infty}\frac{\alpha_n}{n}$ and $\overline{u}(A) = \lim_{n\to\infty}\frac{\alpha^n}{n}$ (the lower and upper uniform density of the set $A \subseteq \mathbb{N}$). If $\underline{u}(A) = \overline{u}(A) = u(A)$, then we say that the set A has uniform density. It can be easily seen that for each $A \subseteq \mathbb{N}$

$$\underline{u}(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{u}(A). \tag{2}$$

Then by using this density we can define admissible ideal

$$\mathscr{I}_u = \{A \subseteq \mathbb{N} \,:\, u(A) = 0\}.$$

From the (2) we have $\mathscr{I}_u \subsetneq \mathscr{I}_d$. The inclusion is strict, because the set $B = \bigcup_{k=1}^{\infty} B_k$, where $B_k = \left\{k^3+1, k^3+2, \ldots, k^3+k\right\}$ $(k \in \mathbb{N})$ is in $\mathscr{I}_d$, but it is not in $\mathscr{I}_u$ since $\overline{u}(B) = 1$ and $\underline{u}(B) = 0$ (see [1]).

Recall that the set of all prime numbers is in the ideals $\mathscr{I}_u$, $\mathscr{I}_d$ and $\mathscr{I}_\delta$.

The next ideals are not defined via density.

Example 6. Let $q \in (0,1\rangle$. Put $\mathscr{I}_c^{(q)} = \{A \subseteq \mathbb{N} \,:\, \sum_{a\in A} a^{-q} < +\infty\}$. If $q_1 < q_2$, $q_1, q_2 \in (0,1\rangle$ then $\mathscr{I}_c^{(q_1)} \subsetneq \mathscr{I}_c^{(q_2)}$. Moreover $\mathscr{I}_c^{(q)} \subsetneq \mathscr{I}_d$ for all $q \in (0,1\rangle$. The inclusion is again strict, because the set of all prime numbers is in $\mathscr{I}_d$ but it is not in $\mathscr{I}_c^{(q)}$, since $\sum_{p\in\mathbb{P}}\frac{1}{p} = +\infty$, where $\mathbb{P}$ denotes the set of all prime numbers (see [3]).

Example 7. In this example we introduce a submeasure on $\mathscr{P}(\mathbb{N})$ which will be closely related to the convergence of series of type $\sum_{k=1}^{\infty} a_k^{-t}$, where $\{a_1 < a_2 < \ldots\} \subseteq \mathbb{N}$ and $t > 0, t \in \mathbb{R}$. Put $\lambda(A) = 0$ if $A \subseteq \mathbb{N}$ is finite. If $A = \{a_1 < a_2 < \ldots\}$ is infinite set, then $\lambda(A)$ denotes the exponent of convergence of the sequence $(a_k)_1^{\infty}$. Hence

$$\lambda(A) = \inf\left\{t > 0 : \sum_{k=1}^{\infty} a_k^{-t} < +\infty\right\}.$$

Hence λ is a submeasure on $\mathscr{P}(\mathbb{N})$, which maps $\mathscr{P}(\mathbb{N})$ into $[0,1]$. If $A=\{a_1<a_2<\ldots\}\subseteq\mathbb{N}$ is infinite, then we have

$$\lambda(A)=\limsup_{k\to\infty}\frac{\log k}{\log a_k}.$$

Then we can define ideals $\mathscr{I}_{<q}=\{A\subseteq\mathbb{N}\ :\ \lambda(A)<q\}$, if $0<q\leq 1$ and $\mathscr{I}_{\leq q}=\{A\subseteq\mathbb{N}\ :\ \lambda(A)\leq q\}$, if $0\leq q\leq 1$ (see [12]).

Example 8. Let $p=(p_n)$ be a sequence of positive real numbers that satisfies the following properties:

i) $\lim_{n\to\infty}p_n=+\infty$,

ii) $\limsup_{n\to\infty}\frac{p_n}{n}<+\infty$.

We will call such a sequence premissible. Let $p=(p_n)$ be a premissible sequence. The quasi-density of the set $A\subseteq\mathbb{N}$ is

$$d_p(A)=\lim_{n\to\infty}\frac{A(n)}{p_n}=\lim_{n\to\infty}\frac{|\{k\in A\ :\ k\leq n\}|}{p_n},$$

if this limit exists. We can define the an ideal $\mathscr{I}_p=\{A\subseteq\mathbb{N}\ :\ d_p(A)=0\}$ where the sequence (p_n) has the property $\limsup_{n\to\infty}\frac{p_n}{n}=T\in\mathbb{R}\setminus\{0\}$ (see [7] and [8]).

Moreover the following inclusions hold:

1) $\mathscr{I}_f\subsetneq\mathscr{I}_u\subsetneq\mathscr{I}_d\subsetneq\mathscr{I}_\delta$,

2) $\mathscr{I}_f\subsetneq\mathscr{I}_c^{(q)}\subsetneq\mathscr{I}_d$.

3) $\mathscr{I}_0\subsetneq\mathscr{I}_{<q}\subsetneq\mathscr{I}_c^{(q)}\subsetneq\mathscr{I}_{\leq q}\subsetneq\mathscr{I}_{<q'}\subsetneq\mathscr{I}_c^{(q')}\subsetneq\mathscr{I}_{\leq q'}\subsetneq\mathscr{I}_{<1}\subsetneq\mathscr{I}_c^{(1)}\subsetneq\mathscr{I}_{\leq 1}=2^{\mathbb{N}}$, where $0<q<q'\leq 1$.

We recall the definition of compact submeasure on $\mathbb{N}$.

Definition 9. *The function* $m\colon 2^{\mathbb{N}}\to\langle 0,+\infty)$ *is said to be a* compact submeasure *on* $2^{\mathbb{N}}$ *provided that* m *satisfies the following four conditions:*

1) $A\subseteq B\Rightarrow m(A)\leq m(B)$,

2) $m(A\cup B)\leq m(A)+m(B)$,

3) $\forall a\in\mathbb{N}: m(\{a\})=0$,

4) $\forall\varepsilon>0$ *there exists a decomposition* $\mathbb{N}=A_1\cup A_2\cup\cdots\cup A_s$ *of* $\mathbb{N}$ *such that* $m(A_j)<\varepsilon$, $\forall j=1,2,\ldots,s$.

It is known that the densities d,δ and u satisfy the conditions 1.-4. of definition 9 (see [9]). There exist some functions that are not compact submeasure like Buck's measure and exponent convergence (see [10] and [11]).

1 MAIN RESULTS

There is a natural question that for which of above described ideals the analogy of Theorem A holds. The Theorem A for ideals reads as follows.

Theorem B. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and $\mathscr{I}$ is an ideal on $\mathbb{N}$. If for any $M \in \mathscr{I}$ we have $\sum_{n\in M} a_n < +\infty$, then $\sum_{n=1}^{\infty} a_n < +\infty$.*

On the basis of above theorem if the series with positive terms diverges, then there exists a set $M_0 \in \mathscr{I}$ such that the $\sum_{n\in M_0} a_n$ diverges.

In the following statements we find for which ideals from the first section (Examples 2-8) the Theorem B holds. As first we describe some instances, where the Theorem B is not true.

Theorem 10. *For the ideals $\mathscr{I}_{<q}$ and $\mathscr{I}_{\leq q}$ the statement of Theorem B does not hold.*

Proof. Put $\sum_{n=1}^{\infty} a_n = \sum_{n=1} \frac{1}{n}$ i.e. harmonic series. If $M \in \mathscr{I}_{<q}$, then $\lambda(M) < q$ and $a_n^{-q} \leq a_n^{-\lambda(M)}$. Then $\sum_{n\in M} a_n{}^{\lambda(M)} = \sum_{n\in M} \frac{1}{n^{\lambda(M)}} < +\infty$. But the harmonic series diverges i.e. $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. Under the inclusions 3) we have

$$\mathscr{I}_{\leq q} \subsetneq \mathscr{I}_c^{(q')} \quad \text{where} \quad 0 < q < q' \leq 1.$$

If $M \in \mathscr{I}_{\leq q}$, then $\lambda(M) \leq q < q'$ and $\sum_{n\in M} a_n{}^{-q} = \sum_{n\in M} \frac{1}{n^q} < +\infty$. The series $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ diverges, therefore the Theorem B is not true as in previous case. □

Theorem 11. *For the ideal $\mathscr{I}_u$ the statement of Theorem B holds.*

Proof. It is sufficient to prove, that the $\overline{u}$ (upper uniform density) is a compact submeasure, i.e. we need to prove the properties 1)–4) of Definition 9. The properties 1) and 3) are trivial.

Let $\alpha^s = \limsup_{m\to\infty} A(m+1, m+s)$ and $\beta^s = \limsup_{m\to\infty} B(m+1, m+s)$. Denote $\gamma^s = \limsup_{m\to\infty} (A\cup B)(m+1, m+s)$, where $(A\cup B)(m+1, m+s) = (A\cup B) \cap \{m+1, \ldots, m+s\}$. Obviously $(A\cup B)(m+1, m+s) \leq A(m+1, m+s) + B(m+1, m+s)$. Therefore $\limsup_{m\to\infty}(A\cup B)(m+1, m+s) \leq \limsup_{m\to\infty} A(m+1, m+s) + \limsup_{m\to\infty} B(m+1, m+s)$. Divided by s we have $\frac{\gamma^s}{s} \leq \frac{\alpha^s}{s} + \frac{\beta^s}{s}$. Then if $s \to \infty$

$$\overline{u}(A\cup B) \leq \overline{u}(A) + \overline{u}(B).$$

To prove the 4)[th] property we need to find for all $\varepsilon > 0$ a decomposition of $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_s$ such that $\overline{u}(A_j) < \varepsilon$ for all $j = 1, 2, \ldots, s$. Such decomposition is for example the

$$\mathbb{N} = (d\mathbb{N}) \cup (d\mathbb{N}+1) \cup \cdots \cup (d\mathbb{N}+(d-1)),\ d \in \mathbb{N}.$$

It is clear that $\mathbb{N} = \bigcup_{j=1}^{d} A_j$ and $\overline{u}(A_j) = \frac{1}{d} < \varepsilon$ $(j = 1, 2, \ldots, d)$. □

Below are some theorems about admissible ideals for which the statement of Theorem B holds.

Theorem 12. *For each ideal $\mathscr{I} \supseteq \mathscr{I}_d$ the statement of Theorem B holds.*

Proof. Let $\mathscr{I} \supseteq \mathscr{I}_d$. Under the assumption of theorem for all $M \in \mathscr{I}$ the $\sum_{n\in M} a_n < +\infty$. Then it holds for all $M \in \mathscr{I}_d$ and from Theorem A we have $\sum_{n=1}^{\infty} a_n < +\infty$. □

According to previous theorem for each ideal which contains $\mathscr{I}_d$ the statement of Theorem B holds.

Corollary 13. *For ideal $\mathscr{I}_\delta$ the statement of Theorem B holds.*

Theorem 14. *For ideal $\mathscr{I}_p$ the statement of Theorem B holds, where $p=(p_n)$ is such a sequence that* $\limsup_{n\to+\infty}\frac{p_n}{n}=T\neq 0$.

Proof. Under the results of [9] it is sufficient to prove, that the d_p is a compact submeasure for an admissible sequence $p=(p_n)$, i.e. we need to prove the properties 1)–4) of Definition 9. The properties 1)–3) are trivial.

To prove the 4)$^{\text{th}}$ property we need to find for all $\varepsilon>0$ a decomposition of $\mathbb{N}=A_1\cup A_2\cup\cdots\cup A_s$ such that $d_p(A_j)<\varepsilon$ for all $j=1,2,\ldots,s$. Such decomposition is for example the

$$\mathbb{N}=(d\mathbb{N})\cup(d\mathbb{N}+1)\cup\cdots\cup(d\mathbb{N}+(d-1)),\ d\in\mathbb{N}.$$

It is clear that $\mathbb{N}=\bigcup_{j=1}^{s}A_j$ and $d_p(A_j)=\frac{1}{T}\cdot\frac{1}{d}<\varepsilon$ $(j=1,2,\ldots,s)$, where $\limsup_{n\to+\infty}\frac{p_n}{n}=T<+\infty$, $T\neq 0$, $d_p(A_j)\leq\frac{1}{T}$ and $d_p(\mathbb{N})=\frac{1}{T}$. □

CONCLUSION

In this paper, we have given a sufficient condition for the convergence of series using ideals. The ideals are constructed by means of densities or by means of series convergence. We will give new examples of ideals for which the wording of the given proposition holds or does not hold. Since these are smaller ideals, our results are an enrichment of the theory of convergence of series.

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INCORPORATION OF RUNTIME ASPECTS OF INFORMATION TECHNOLOGY SYSTEMS IN CURRICULUM OF TECHNICAL UNIVERSITIES

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Abstract: *This paper presents our views on the significance of runtime aspects in various IT systems. The significance of runtime aspects is often misunderstood and underestimated, particularly in higher IT education. We will discuss the runtime features of important IT systems and their impact on the performance and security of IT solutions. The article presents ideas for incorporating runtime topics into the curriculum of technical universities.*

Keywords: operating systems, computer networks, performance and tuning, computer security, runtime environments, execution model.

INTRODUCTION – WHAT DO WE MEAN BY RUNTIME?

Runtime, also referred to as execution time or run time, is a crucial phase in the life cycle of a computer program. During this phase, the compiled code is executed by the computer's central processing unit (CPU) or interpreter. It is the period in which the program performs its intended functions and processes data as per the instructions provided in the source code.

The runtime phase of a program involves its interaction with the system's hardware and other software components to execute tasks such as calculations, data manipulation, input/output operations, and more. This phase is critical as it allows for the observation of how a program behaves in a real-world environment and ensures that the desired output is produced.

Optimizing the runtime performance of software is crucial for improving the user experience and making the application more efficient. Profiling tools and performance monitoring techniques are commonly used by developers to identify bottlenecks and enhance the speed and responsiveness of their programs during runtime.

A detailed understanding of runtime is also crucial for eliminating many security-related incidents. It is the runtime phase when all security incidents happen. Detailed discussion is at [1].

1 RUNTIME SYSTEM ASPECTS

Many programming languages have a runtime system, also called a runtime environment or runtime library. The system is essential in providing the required infrastructure and support for program execution.

Here are some important features that a computer runtime system may handle:

- **Memory Management:** The runtime system often takes care of memory allocation and deallocation, managing the program's use of the stack and heap. This includes tasks such as allocating memory for variables and data structures and freeing up memory that is no longer needed.

- **Variable Access:** The runtime system can provide mechanisms for accessing variables, whether stored in registers, on the stack, or in the heap. It ensures proper scoping and visibility of variables during program execution.

- **Parameter Passing:** How parameters are passed between functions or procedures is a runtime concern. This could involve passing parameters via registers, on the stack, or using a combination of both.

- **Interface with Operating System:** The runtime system facilitates communication between the program and the underlying operating system. This includes handling system calls, managing input and output operations, and interacting with external resources.

- **Compiler Assumptions:** The compiler makes certain assumptions about the runtime environment when generating machine code. These assumptions are based on the expected behavior and support provided by the runtime system.

- **Stack and Heap Management:** The runtime system is often responsible for setting up and managing the program's stack and heap. This includes stack frames for function calls, dynamic memory allocation (heap), and releasing resources when they are no longer needed.

- **Garbage Collection:** Some runtime systems incorporate garbage collection, a process that automatically identifies and frees up memory occupied by objects that are no longer in use, helping to manage memory efficiently.

- **Concurrency and Threading:** If a programming language supports concurrency or multithreading, the runtime system may provide features and mechanisms for managing and synchronizing concurrent execution.

- **Dynamic Language Features:** For dynamically-typed languages, the runtime system may handle features like dynamic type checking, reflection, and late binding.

It is essential to have a well-designed runtime system for programs written in a particular language to ensure portability and performance. The runtime system abstracts away many low-level details, providing a consistent environment for programs to run across various platforms and architectures.

2 CONTEXT OF EXECUTION

The context in which execution occurs is indeed critical for understanding how programs operate and what resources they can leverage.

Let's break down some key points:

- **Execution Environment**: Most programs don't run directly on the hardware. They operate within an execution environment provided by an operating system (OS) or a runtime environment. This environment abstracts away hardware details and provides services like memory management, file system access, and process scheduling.

- **Implicit and Explicit Assumptions**: Programs often make assumptions about the resources available during execution. These assumptions can be implicit, such as assuming the availability of certain system calls or hardware features, or explicit, where the program specifically requests resources or services from the execution environment.

- **Multitasking Operating System**: Modern operating systems support multitasking, allowing multiple programs to run simultaneously. This introduces complexities like process scheduling, resource sharing, and inter-process communication, which impact how programs interact with each other and the underlying hardware.

- **Runtime Libraries**: Many programming languages come with runtime libraries that provide additional functionality beyond what's offered by the language itself. These libraries often handle tasks like input/output operations, memory management, and networking, abstracting away low-level details and providing a higher-level interface for developers.

- **Decoupling from Hardware**: The execution environment typically abstracts away direct manipulation of hardware peripherals. Instead, programs interact with abstracted interfaces provided by the OS or runtime environment. This decoupling improves portability and simplifies development by providing a consistent interface regardless of the underlying hardware.

Developers must comprehend the execution context to create reliable and portable software. This enables them to harness the execution environment's capabilities while maintaining compatibility across various platforms and hardware configurations.

3 EXAMPLE – JAVA RUNTIME ENVIRONMENT

Let’s take *Java Runtime Environment*, or *JRE* as an example of a runtime environment that we will use to demonstrate aspects of runtime features.

In summary, the *Java Runtime Environment* provides the necessary components for executing Java applications on various platforms. It abstracts the underlying hardware and operating system details, making Java a versatile and cross-platform programming language.

The three interrelated components for developing and running Java programs are:

- **Java Development Kit (JDK):** The JDK is a software development kit that includes everything needed to develop, debug, and compile Java applications. It encompasses the JRE and additional tools such as the compiler (`javac`),

debugger, and other utilities. In other words, the JDK is a superset of the JRE, adding development and compilation tools to the runtime environment.

- **Java Virtual Machine (JVM):** The JVM is the runtime engine that executes Java bytecode. It is an integral part of both the JRE and the JDK. The JVM interprets or compiles Java bytecode into native machine code, allowing Java applications to run on any device or platform with a compatible JVM. The JVM provides memory management, garbage collection, and other runtime services.

- **Java Runtime Environment (JRE):** The JRE [2] is a subset of the JDK and is designed for running Java applications. It includes the Java Virtual Machine (JVM), class libraries, and other components necessary for the execution of Java programs. The JRE is what end-users need to run Java applications on their machines (see Fig. 1).

In summary, while the JRE is focused on providing the runtime environment for executing Java applications, the JDK includes everything from the JRE plus additional tools for development and compilation. The JVM is an essential component shared by the JRE and the JDK, responsible for executing Java bytecode on the target platform.

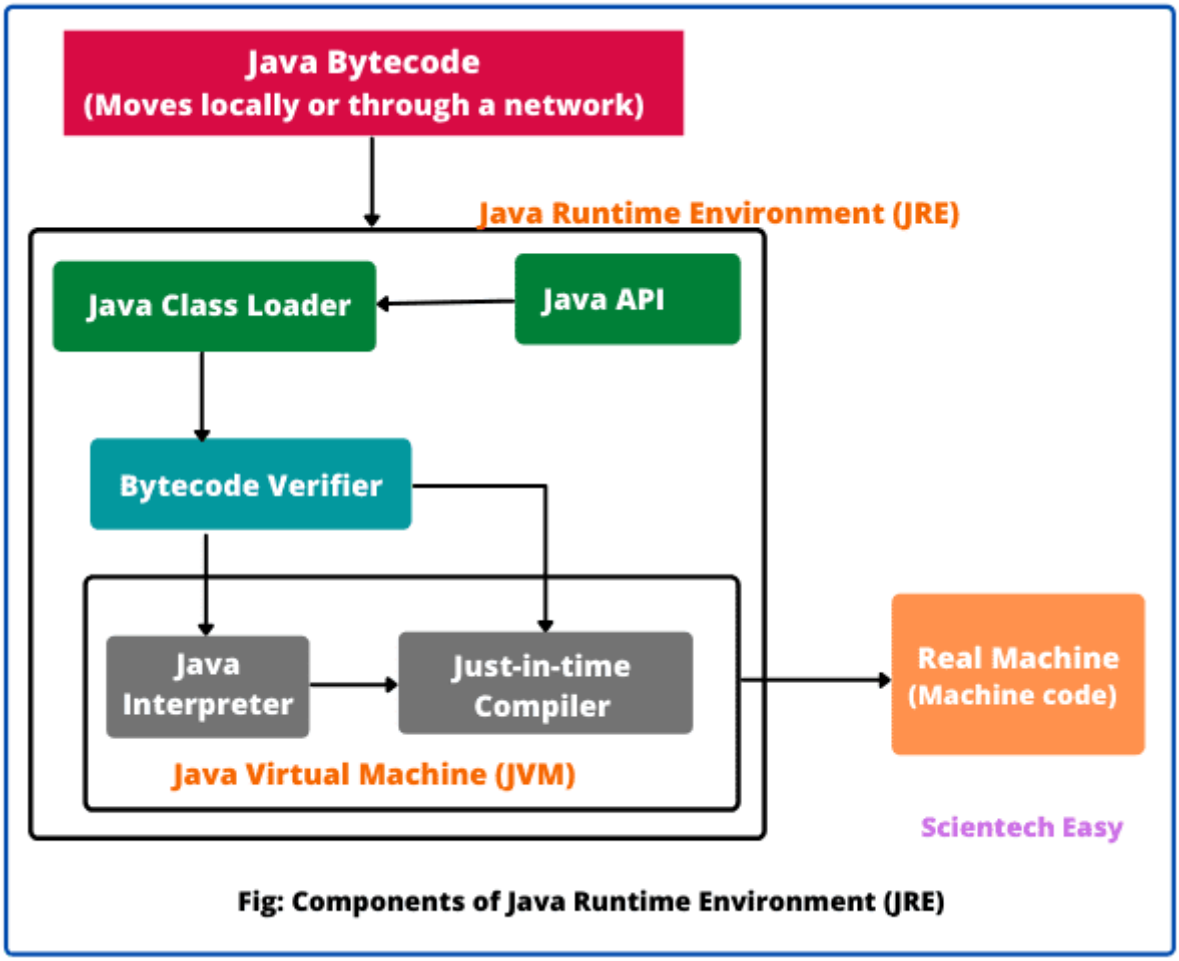

Fig. 1. Components of Java Runtime Environment
Source: **https://www.scientecheasy.com/2021/03/what-is-jre.html/**

The JRE combines Java code created using the JDK with the necessary libraries required to run it on a JVM and then creates an instance of the JVM that executes the resulting program. JVMs are available for multiple operating systems, and programs created with the JRE will run on all of them. In this way, the Java Runtime Environment is what enables a Java program to run in any operating system without modification.

The JDK and JRE interact with one another to create a sustainable runtime environment that enables the seamless execution of Java-based applications in virtually any operating system. The following make up the JRE runtime architecture:

- **ClassLoader**

  The Java ClassLoader dynamically loads all classes necessary to run a Java program. Since Java classes are only loaded into memory when they're required, the JRE uses ClassLoaders to automate this process on demand.

- **Bytecode verifier**

  The bytecode verifier ensures the format and accuracy of Java code before it passes to the interpreter. In the event that code violates system integrity or access rights, the class will be considered corrupted and won't be loaded.

- **Interpreter**

  After the bytecode successfully loads, the Java interpreter creates an instance of the JVM that allows the Java program to be executed natively on the underlying machine.

4. MORE FOCUS ON RUNTIME ASPECTS IN THE CURRICULUM OF INFORMATION TECHNOLOGY STUDY PROGRAMS

We propose to give more focus on the runtime aspects of various IT solutions in these courses delivered at technical universities. We have taken examples from [3]:

4.1 Programming and Algorithms

The objective of this course is to instruct students on how to create algorithms to solve fundamental problems and translate these algorithms into programs using the C programming language. Additionally, the course aims to teach students practical abilities in program testing and debugging. Graduates of the course will have a sound understanding of data types (primitive, pointers, and structures), statements, expressions, and functions in the C language. They will also understand the fundamental principles of recursion and algorithmic complexity.

4.2 Operating Systems and Systems Programming

This course focuses on the design and implementation of the fundamental components that form modern operating systems. These components include threads, processes, context switching, virtual memory, system calls, interrupts, and the interaction between software and hardware using drivers. The students will learn the theoretical concepts of operating system architecture, with an emphasis on kernel architecture. During the course, they will gain practical experience in developing or modifying subsystems (modules) of an operating system kernel.

4.3 Compiler Construction

This is an introductory course on compiler construction aimed at undergraduate students majoring in computer science. The objective of the course is to familiarize students with the fundamental concepts of compilers, enabling them to comprehend the development and execution of programming languages. The core theme of the course is centered around the concept of self-compilation, where students will learn by observing and comprehending the process of compiling a program.

Upon completion of the class, students will have a comprehensive understanding of the distinction between syntax and semantics in programming languages. They will have learned how syntax can be accurately specified and efficiently verified, as well as how to efficiently implement semantics through code generation. Students will also have gained knowledge on how to bootstrap a compiler that compiles itself.

4.4 Web and Database Server Administration

The students will gain familiarity with administering database and web servers and services. They will be equipped with the necessary skills to install, configure, operate, test, and backup intricate database and web service systems.

4.5 Virtualization and Data Centers

The course aims to introduce students to the fundamentals of cloud computing systems. It covers various principles and techniques used in the design and implementation of data center infrastructure, including virtualization, high availability, and software layers. The course provides a comprehensive overview of private, public, and hybrid cloud technologies. Students will learn about current trends in IT infrastructure architecture and how to configure it for both classic and cloud applications. The course also covers the design, validation, and operation of complex infrastructures for modern applications, focusing on scalability and protection against overloads, outages, and data losses.

CONCLUSION

My individual contribution presented in this article is represented by proposing the focus on the runtime aspects of various IT solutions and suggest incorporating it into different university courses. We believe understanding runtime principles improves the performance, security, and efficiency of all IT solutions.

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