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Aims and target group of the conference:

The conference **MITAV 2020** is the seventh annual MITAV conference. It should attract in particular teachers of all types of schools and is devoted to the most recent discoveries in mathematics, informatics, and other sciences as well as to the teaching of these branches at all kinds of schools for any age groups, including e-learning and other applications of information technologies in education. The organizers wish to pay attention especially to the education in the areas that are indispensable and highly demanded in contemporary society. The goal of the conference is to create space for the presentation of results achieved in various branches of science and at the same time provide the possibility for meeting and mutual discussions of teachers from different kinds of schools and orientation. We also welcome presentations by (diploma and doctoral) students and teachers who are just beginning their careers, as their novel views and approaches are often interesting and stimulating for other participants.

Organizers:

Union of Czech Mathematicians and Physicists, Brno branch (JČMF),
in co-operation with
Faculty of Military Technology, University of Defence in Brno,
Faculty of Science, Faculty of Education and Faculty of Economics and Administration,
Masaryk University in Brno,
Faculty of Electrical Engineering and Communication, Brno University of Technology.

Venue:

Club of the University of Defence in Brno, Šumavská 4, Brno, Czech Republic
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Due to measures against the spread of coronavirus and covid-19, the prezence form of the conference was not possible.

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Each MITAV 2020 participant received printed collection of abstracts **MITAV 2020** with ISBN 978-80-7582-307-6. CD supplement of this printed volume contains all the accepted contributions of the conference.

Now, in autumn 2020, this **post-conference CD** was published, containing extended versions of selected MITAV 2020 contributions. The proceedings are published in English and contain extended versions of 7 selected conference papers. Published articles have been chosen from 11 conference papers and every article was once more reviewed.

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DETERMINATION OF INITIAL DATA GENERATING (b, c)-BOUNDED SOLUTIONS OF A TRIANGULAR DIFFERENTIAL SYSTEM OF EQUATIONS

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Abstract: *The paper considers a nonlinear triangular system of discrete equations*

$$u_1(k+1) = q_1(k)\alpha_1(u_1(k)),$$

$$u_2(k+1) = q_2(k)\alpha_2(u_1(k))\alpha_3(u_2(k))$$

where $q_i\colon \{k_0, k_0+1, \dots\} \to (0, \infty)$, $i = 1, 2$ *are given functions,* k_0 *is a natural number and* $\alpha_i\colon [0, \infty) \to [0, \infty)$, $j = 1, 2$ *are increasing continuous positive functions. Initial data for the existence of solutions with coordinates bounded from below and from above by the given functions are determined.*

Keywords: Initial data, triangular system, discrete equation, system of equations, convergent sequence.

INTRODUCTION

Let k_0 be a natural number. Denote by $\mathcal{N}(k_0)$ the set of all natural numbers greater than or equal to k_0, i.e., $\mathcal{N}(k_0) = \{k_0, k_0+1, \dots\}$. In the paper we consider a triangular system of two nonlinear discrete equations

$$u_1(k+1) = q_1(k)\alpha_1(u_1(k)), \tag{1}$$
$$u_2(k+1) = q_2(k)\alpha_2(u_1(k))\alpha_3(u_2(k)) \tag{2}$$

assuming the following: $k \in N(k_0)$, $k_0 \geq 1$ $q_i\colon \mathcal{N}(k_0) \to (0, \infty)$, $i = 1, 2$ and $\alpha_i\colon [0, \infty) \to [0, \infty)$, $j = 1, 2$ are increasing continuous functions.
A function $v = (v_1, v_2)\colon \mathcal{N}(k_0) \to \mathbb{R}^2$ is called a solution to system (1), (2) if this system is satisfied for $u = v$ for all $k \in \mathcal{N}(k_0)$, that is, if

$$v_1(k+1) = q_1(k)\alpha_1(v_1(k)),$$
$$v_2(k+1) = q_2(k)\alpha_2(v_1(k))\alpha_3(v_2(k))$$

for every $k \in \mathcal{N}(k_0)$. Moreover it is easy to see that an initial problem

$$u_1(k_0) = u_1^*, \tag{3}$$
$$u_2(k_0) = u_2^*, \tag{4}$$

where $u_i^* \in [0, \infty)$, $i = 1, 2$ are two fixed initial data, defines a unique solution to system (1), (2) within the meaning as described above.
The problem solved in the paper can be described as follows. Let functions $b_i, c_i\colon \mathcal{N}(k_0) \to \mathbb{R}$, $i = 1, 2$ be given and let

$$0 \leq b_i(k) < c_i(k), \quad \forall k \in \mathcal{N}(k_0), \ \ i = 1, 2. \tag{5}$$

First, based on the known results, sufficient conditions for the existence of a solution
$u = (u_1, u_2)\colon \mathcal{N}(k_0) \to \mathbb{R}^2$, such that

$$b_i(k) < u_i(k) < c_i(k), \quad \forall k \in \mathcal{N}(k_0), \ \ i = 1, 2$$

will be given. Solutions, satisfying inequality (5) are said to be bounded or more exactly, (b, c)-bounded. Next, the main contribution of the paper is the construction of a set of initial data such that every point of this set defines a solution satisfying (5), that is, defines a (b, c)-bounded solution.

1 EXISTENCE OF (b, c)-BOUNDED SOLUTIONS

The following theorem gives sufficient conditions for the existence of at least one (b, c)-bounded solution.

Theorem 1 *Let functions $b_i, c_i\colon \mathcal{N}(k_0) \to \mathbb{R}$, $i = 1, 2, 3$ satisfy inequalities* (5)*. Assume that, for every $k \in \mathcal{N}(k_0)$,*

$$q_1(k)\alpha_1(b_1(k)) < b_1(k+1), \tag{6}$$
$$q_1(k)\alpha_1(c_1(k)) > c_1(k+1) \tag{7}$$

and

$$q_2(k)\alpha_2(c_1(k))\alpha_3(b_2(k)) < b_2(k+1), \tag{8}$$
$$q_2(k)\alpha_2(b_1(k))\alpha_3(c_2(k)) > c_2(k+1). \tag{9}$$

Then, there exists at least one (b, c)-bounded solution

$$u(k) = (u_1(k), u_2(k), \ \ k \in \mathcal{N}(k_0)$$

to system (1)*),* (2)*, that is,*

$$b_1(k) < u_1(k) < c_1(k) \ \ \forall k \in \mathcal{N}(k_0)$$

and

$$b_2(k) < u_2(k) < c_2(k) \ \ \forall k \in \mathcal{N}(k_0).$$

We omit the proof since it can be done simply using [9, Theorem 1] or [10, Theorem 2].

2 INITIAL DATA GENERATING (b, c)-BOUNDED SOLUTIONS

Although Theorem 1 brings sufficient conditions for the existence of at least one (b, c)-bounded solution to system (1), (2), its disadvantage is that it indicates no fixed initial problem (3), (4), generating such a solution. In this part we will construct a set of such initial data. Below we consider each equation of the system (1), (2) separately.

2.1 Initial data generating (b_1, c_1)-bounded solutions to equation (1)

System (1), (2) is a triangular one. Therefore, as equation (1) does not contain the dependent variable u_2, conditions (6), (7), are sufficient for the existence of a (b_1, c_1)-bounded solution to equation (3). Now, let us define sequences $\{u_{1cs}\}_{s=0}^{\infty}$ and $\{u_{1bs}\}_{s=0}^{\infty}$ by the formulas

$$u_{1b0} = b_1(k_0), \quad u_{1c0} = c_1(k_0), \tag{10}$$

$$u_{1b1} = \alpha_1^{-1}\left(\frac{b_1(k_0+1)}{q_1(k_0)}\right), \quad u_{1c1} = \alpha_1^{-1}\left(\frac{c_1(k_0+1)}{q_1(k_0)}\right)$$

where α_1^{-1} denotes the inverse function to function α_1,

$$u_{1b2} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{b_1(k_0+2)}{q_1(k_0+1)}\right)\right),$$

$$u_{1c2} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{c_1(k_0+2)}{q_1(k_0+1)}\right)\right),$$

$$u_{1b3} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\alpha_1^{-1}\left(\frac{b_1(k_0+3)}{q_1(k_0+2)}\right)\right)\right),$$

$$u_{1c3} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\alpha_1^{-1}\left(\frac{c_1(k_0+3)}{q_1(k_0+2)}\right)\right)\right),$$

and, in general,

$$u_{1bs} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots \cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{b_1(k_0+s)}{q_1(k_0+s-1)}\right)\right)\right)\right)\right), \tag{11}$$

$$u_{1cs} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots \cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{c_1(k_0+s)}{q_1(k_0+s-1)}\right)\right)\right)\right)\right) \tag{12}$$

where $s = 1, 2, \ldots$.

Theorem 2 *Let all conditions of Theorem 1 hold. Then, the sequence* $\{u_{1cs}\}_{s=0}^{\infty}$ *defined by* (12) *is a decreasing convergent sequence and the sequence* $\{u_{1bs}\}_{s=0}^{\infty}$ *defined by* (11) *is an increasing convergent sequence, that is,*

$$u_{1cs} > u_{1c,s+1}, \quad s = 0, 1, \ldots, \tag{13}$$

$$u_{1bs} < u_{1b,s+1}, \quad s = 0, 1, \ldots \tag{14}$$

and their limits c_1^*, b_1^*, *where*

$$c_1^* := \lim_{s\to\infty} u_{1cs}, \quad b_1^* := \lim_{s\to\infty} u_{1bs}, \tag{15}$$

exist and satisfy

$$c_1^* \geq b_1^*. \tag{16}$$

Proof. Let us show that the sequence $\{u_{1cs}\}_{s=0}^{\infty}$ is decreasing, i.e., that (13) holds. It means that, by (12), we must to prove that

$$\begin{aligned}
&\alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\right.\\
&\left.\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{c_1(k_0+s)}{q_1(k_0+s-1)}\right)\right)\right)\right)\right)\\
>&\alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\right.\\
&\left.\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-1)}\alpha_1^{-1}\left(\frac{c_1(k_0+s+1)}{q_1(k_0+s)}\right)\right)\right)\right)\right)\right).
\end{aligned} \tag{17}$$

Then, applying the function α_1 to the left-hand and right-hand sides of (17), we get

$$\begin{aligned}
&\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\\
&\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{c_1(k_0+s)}{q_1(k_0+s-1)}\right)\right)\right)\right)\\
>&\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\\
&\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-1)}\alpha_1^{-1}\left(\frac{c_1(k_0+s+1)}{q_1(k_0+s)}\right)\right)\right)\right)\right)
\end{aligned}$$

or, because the value $q_1(k_0)$ is positive,

$$\begin{aligned}
&\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\\
&\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{c_1(k_0+s)}{q_1(k_0+s-1)}\right)\right)\right)\right)\\
>&\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.
\end{aligned}$$

$$\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-1)}\alpha_1^{-1}\left(\frac{c_1(k_0+s+1)}{q_1(k_0+s)}\right)\right)\right)\right).$$

Repeating this procedure in much the same way $(s-2)$-times, we derive

$$\alpha_1^{-1}\left(\frac{c_1(k_0+s)}{q_1(k_0+s-1)}\right) > \alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-1)}\alpha_1^{-1}\left(\frac{c_1(k_0+s+1)}{q_1(k_0+s)}\right)\right). \tag{18}$$

Applying the function α_1 to the left-hand and right-hand sides of (18), we see that inequality

$$\frac{c_1(k_0+s)}{q_1(k_0+s-1)} > \frac{1}{q_1(k_0+s-1)}\alpha_1^{-1}\left(\frac{c_1(k_0+s+1)}{q_1(k_0+s)}\right) \tag{19}$$

must be fulfilled. From (19)

$$c_1(k_0+s) > \alpha_1^{-1}\left(\frac{c_1(k_0+s+1)}{q_1(k_0+s)}\right),$$

or, applying the function α_1 to the left-hand and right-hand sides, we have

$$q_1(k_0+s)\alpha_1\left(c_1(k_0+s)\right) > c_1(k_0+s+1), \tag{20}$$

Inequality (20) is valid because it is a consequence of inequality (7) for $k=k_0+s$. So the sequence $\{u_{1cs}\}_{s=0}^{\infty}$ is decreasing.
In much the same way, using assumption (6), one can prove that the sequence $\{u_{1bs}\}_{s=0}^{\infty}$ is increasing, i.e., that (14) holds.
It is a consequence of (10), (11), (12) and (5) that $u_{1cs} > u_{1bs}$, $s=0,1,\dots$. Therefore, both sequences $\{u_{1bs}\}_{s=0}^{\infty}$. $\{u_{1cs}\}_{s=0}^{\infty}$ are convergent and, for their limits c_1^*, b_1^*, computed by (15), inequality (16) must be valid. □

Lemma 1 *Assume that all conditions of Theorem 1 hold and that the sequences* $\{u_{1bs}\}_{s=0}^{\infty}$, $\{u_{1cs}\}_{s=0}^{\infty}$ *are defined by* (11) *and* (12). *Let, for a fixed* $s\in\{1,2,\dots\}$, $u_{1bs}(k)$ *be the solution of initial problem*

$$u_1(k_0)=u_{1bs},$$

and $u_{1cs}(k)$ *be the solution of initial problem*

$$u_1(k_0)=u_{1cs}. \tag{21}$$

Then

$$u_{1bs}(k_0+s)=b_1(k_0+s) \tag{22}$$

and

$$u_{1cs}(k_0+s)=c_1(k_0+s). \tag{23}$$

Proof. Let $s\in\{1,2,\dots\}$ be fixed. Then, the solution of the initial problem (1), (3) can be computed step by step by the formulas

$$u_1(k_0+1)=q_1(k_0)\alpha_1(u_1(k_0))=q_1(k_0)\alpha_1(u_1^*),$$
$$u_1(k_0+2)=q_1(k_0+1)\alpha_1(u_1(k_0+1))=q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u_1^*)),$$

$$u_1(k_0+3) = q_1(k_0+2)\alpha_1(u_1(k_0+2)) = q_1(k_0+2)\alpha_1(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u_1^*)))$$

and, finally, we derive,

$$\begin{aligned} u_1(k_0+s) = q_1(k_0+s-1)\alpha_1(u_1(k_0+s-1)) = \\ = q_1(k_0+s-1)\alpha_1\left(q_1(k_0+s-2)\alpha_1\left(q_1(k_0+s-3)\ldots\ldots\right.\right. \\ \left.\left.\left(\alpha_1\left(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u_1^*))\right)\right)\right)\right). \quad (24) \end{aligned}$$

Now, in accordance with (21), we set in (24)

$$\begin{aligned} u_1^* = u_{1cs} = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\right. \\ \left.\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{c_1(k_0+s)}{q_1(k_0+s-1)}\right)\right)\right)\right)\right). \end{aligned}$$

This leads to equality $u_1(k_0+s) = c_1(k+s)$ and the formula (23) is proved. The formula (22) can be proved in much the same way using formulas (11) and (24). $\square$.

Lemma 2 *Assume that all conditions of Theorem 1 hold and that the sequences* $\{u_{1bs}\}_{s=0}^{\infty}$, $\{u_{1cs}\}_{s=0}^{\infty}$ *are defined by* (11) *and* (12). *Let a number* $s \in \{1, 2, \ldots\}$ *be fixed. Consider an initial problem*

$$u_1(k_0+s) = u_{1s},$$

where

$$u_{1s} \in (b_1(k_0+s), c_1(k_0+s)). \quad (25)$$

Then,

$$u_{1bs} < u_{1s}^* < u_{1cs}, \quad (26)$$

where

$$\begin{aligned} u_{1s}^* = \alpha_1^{-1}\left(\frac{1}{q_1(k_0)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+1)}\cdots\right.\right. \\ \left.\left.\cdots\left(\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-2)}\alpha_1^{-1}\left(\frac{1}{q_1(k_0+s-1)}\alpha_1^{-1}\left(\frac{u_{1s}}{q_1(k_0+s)}\right)\right)\right)\right)\right)\right). \quad (27) \end{aligned}$$

Moreover, the initial problem

$$u_1(k_0) = u_{1s}^*,$$

defines a solution $u = u_1(k)$ *to equation* (1) *satisfying*

$$u_1(k_0+s) = u_{1s}. \quad (28)$$

Proof. The property (28) follows immediately if, in (24), u_{1s}^* is put instead of u_1^*. The right-hand side of inequality (26) follows from (25) (because $u_{1s} < c_1(k_0+s)$) and from formulas (27), (30) taking into account that α_1^{-1} is an increasing function. Similarly, the left-hand side of inequality (26) can be proved. $\square$

Define interval $I_1 := [b_1^*, c_1^*]$ where, by (15), b_1^* and c_1^* are the limits of sequences $\{u_{1bs}\}_{s=0}^{\infty}$ and $\{u_{1cs}\}_{s=0}^{\infty}$, respectively.

Theorem 3 *Let all conditions of Theorem 1 hold. Consider initial problem* (3) *where* $u_1^* \in I_1$. *Then, this initial problem defines a solution* $u = u_1^*(k)$, $k \in \mathcal{N}(k_0)$ *of equation* (1) *such that the inequality*

$$b_1(k) < u_1^*(k) < c_1(k) \;\; \forall k \in \mathcal{N}(k_0) \tag{29}$$

holds, that is, this solution is (b_1, c_1)*-bounded. Moreover, this solution is expressed by the formula*

$$u_1^*(k) = q_1(k-1)\alpha_1\left(q_1(k-2)\alpha_1\left(q_1(k-3)\ldots\left(\alpha_1\left(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u^*))\right)\right)\right)\right) \tag{30}$$

where $k \in \mathcal{N}(k_0)$.

Proof. In much the same way as in the proof of Lemma 1 we derive

$$\begin{aligned}
u_1^*(k_0+1) &= q_1(k_0)\alpha_1(u_1^*(k_0)) = q_1(k_0)\alpha_1(u_1^*),\\
u_1^*(k_0+2) &= q_1(k_0+1)\alpha_1(u_1^*(k_0+1)) = q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u_1^*)),\\
u_1^*(k_0+3) &= q_1(k_0+2)\alpha_1(u_1^*(k_0+2)) = q_1(k_0+2)\alpha_1(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u_1^*)))\\
&\ldots\\
u_1^*(k_0+s) &= q_1(k_0+s-1)\alpha_1\left(q_1(k_0+s-2)\alpha_1\left(q_1(k_0+s-3)\ldots\right.\right.\\
&\quad\ldots\left.\left.\left(\alpha_1\left(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u^*))\right)\right)\right)\right),\\
&\ldots\,.
\end{aligned}$$

If we replace $k_0 + s$ by k, then

$$u_1^*(k) = q_1(k-1)\alpha_1\left(q_1(k-2)\alpha_1\left(q_1(k-3)\ldots\left(\alpha_1\left(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u^*))\right)\right)\right)\right),$$

where $k \in \mathcal{N}(k_0)$, and formula (30) is proved. Solution $u_1^*(k)$ satisfies inequality (29). This is a consequence of Lemma 1 and Lemma 2. ☐

2.2 Initial data generating (b_2, c_2)-bounded solutions to equation (2)

In this part, assume that all conditions of Theorem 1 hold. Let $u_1^* \in I_1$. Then the initial problem (3) defines, by formula (30), a (b_1, c_1)-bounded solution $u_1^*(k)$, $k \in \mathcal{N}(k_0)$ of equation (1). System (1), (2) is a triangular one. Therefore, as equation (2) contains both the variable u_1 and u_2, conditions (8), (9), are sufficient for the existence of a (b_2, c_2)-bounded solution to equation (2) if u_1 is a (b_1, c_1)-bounded solution. Let us define sequences $\{u_{2cs}\}_{s=0}^{\infty}$ and $\{u_{2bs}\}_{s=0}^{\infty}$ by the formulas

$$u_{2b0} = b_2(k_0), \;\; u_{2c0} = c_2(k_0),$$

$$u_{2b1} = \alpha_3^{-1}\left(\frac{b_2(k_0+1)}{q_2(k_0)\alpha_2(u_1^*(k_0))}\right), \;\; u_{2c1} = \alpha_3^{-1}\left(\frac{c_2(k_0+1)}{q_2(k_0)\alpha_2(u_1^*(k_0))}\right)$$

where α_3^{-1} denotes the inverse function to function α_3,

$$u_{2b2} = \alpha_3^{-1}\left(\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\left(\frac{b_3(k_0+2)}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\right)\right),$$

$$u_{2c2} = \alpha_3^{-1}\left(\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\left(\frac{c_2(k_0+2)}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\right)\right),$$

$$u_{2b3} = \alpha_3^{-1}\left(\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\cdot\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\alpha_3^{-1}\left(\frac{b_2(k_0+2)}{q_2(k_0+2)\alpha_2(u_1^*(k_0+2))}\right)\right)\right),$$

$$u_{2c3} = \alpha_3^{-1}\left(\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\times\right.$$
$$\left.\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\alpha_3^{-1}\left(\frac{c_2(k_0+2)}{q_2(k_0+2)\alpha_2(u_1^*(k_0+2))}\right)\right)\right),$$

and, in general,

$$\begin{aligned} u_{2bs} = \alpha_3^{-1}\Bigg(&\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\Bigg(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots \\ &\cdots\Bigg(\alpha_3^{-1}\Bigg(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))} \\ &\alpha_3^{-1}\Bigg(\frac{b_2(k_0+s)}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg), \end{aligned} \tag{31}$$

$$\begin{aligned} u_{2cs} = \alpha_3^{-1}\Bigg(&\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\Bigg(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots \\ &\cdots\Bigg(\alpha_3^{-1}\Bigg(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))} \\ &\alpha_3^{-1}\Bigg(\frac{c_2(k_0+s)}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\Bigg)\Bigg)\Bigg)\Bigg)\Bigg) \end{aligned} \tag{32}$$

where $s = 1, 2, \dots$.

Theorem 4 *Let all conditions of Theorem 1 hold. Then, the sequence $\{u_{2cs}\}_{s=0}^{\infty}$ defined by (32) is decreasing and convergent and the sequence $\{u_{2bs}\}_{s=0}^{\infty}$ defined by (31) is increasing and convergent. Moreover, the inequality*

$$u_{2cs} > u_{2bs}$$

holds for every $s = 0, 1, \dots$ and for the limits c_2^*, b_2^*, where

$$c_2^* := \lim_{s\to\infty} u_{2cs}, \quad b_2^* := \lim_{s\to\infty} u_{2bs}, \tag{33}$$

we have $c_2^* \geq b_2^*$.

Proof. Let us show that the sequence $\{u_{2cs}\}_{s=0}^{\infty}$ is decreasing, i.e., that $u_{2cs} > u_{2c,s+1}$ for each $s = 0, 1, \dots$. It means that the following inequality, as it follows from formula (32), must be valid:

$$\begin{aligned} \alpha_3^{-1}\Bigg(&\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\Bigg(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots \\ &\cdots\Bigg(\alpha_3^{-1}\Bigg(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))} \end{aligned}$$

$$
\begin{aligned}
&\cdot\ \alpha_3^{-1}\left(\frac{c_2(k_0+s)}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right)\Bigg)\Bigg)\Bigg)\Bigg)\\
>&\alpha_3^{-1}\left(\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots\right.\right.\\
&\cdots\left(\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right.\right.\right.\\
&\cdot\ \alpha_3^{-1}\left(\frac{c_2(k_0+s+1)}{q_2(k_0+s)\alpha_2(u_1^*(k_0+s))}\right)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg). \qquad (34)
\end{aligned}
$$

Applying the function α_3 to the left-hand and the right-hand side of (34), we get

$$
\begin{aligned}
&\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots\right.\\
&\cdots\left(\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))}\alpha_3^{-1}\left(\frac{c_2(k_0+s)}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right)\right)\right)\Bigg)\\
>&\frac{1}{q_2(k_0)\alpha_2(u_1^*(k_0))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots\right.\\
&\cdots\left(\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right.\right.\right.\\
&\alpha_3^{-1}\left(\frac{c_2(k_0+s+1)}{q_2(k_0+s)\alpha_2(u_1^*(k_0+s))}\right)\Bigg)\Bigg)\Bigg)\Bigg).
\end{aligned}
$$

or, because the value $q_2(k_0)$ is positive,

$$
\begin{aligned}
&\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0))}\cdots\right.\\
&\cdots\left(\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))}\alpha_3^{-1}\left(\frac{c_2(k_0+s)}{q_1(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right)\right)\right)\Bigg)\\
>&\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u_1^*(k_0+1))}\cdots\right.\\
&\cdots\left(\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-2)\alpha_2(u_1^*(k_0+s-2))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right.\right.\right.\\
&\cdot\ \alpha_3^{-1}\left(\frac{c_2(k_0+s+1)}{q_2(k_0+s)\alpha_2(u_1^*(k_0+s))}\right)\Bigg)\Bigg)\Bigg)\Bigg).
\end{aligned}
$$

After repeating this procedure in much the same way $(s-2)$-times, we obtain

$$
\begin{aligned}
&\alpha_3^{-1}\left(\frac{c_2(k_0+s)}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\right)\\
&\qquad > \alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\alpha_3^{-1}\left(\frac{c_2(k_0+s+1)}{q_2(k_0+s)\alpha_2(u_1^*(k_0+s))}\right)\right). \qquad (35)
\end{aligned}
$$

Applying the function α_3 to the left-hand and right-hand sides of (35), we see that inequality

$$\frac{c_2(k_0+s)}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))} > \frac{1}{q_2(k_0+s-1)\alpha_2(u_1^*(k_0+s-1))}\alpha_3^{-1}\left(\frac{c_2(k_0+s+1)}{q_2(k_0+s)\alpha_2(u_1^*(k_0+s))}\right), \quad (36)$$

must hold. From (36) we deduce

$$c_2(k_0+s) > \alpha_3^{-1}\left(\frac{c_2(k_0+s+1)}{q_2(k_0+s)\alpha_2(u_1^*(k_0+s))}\right),$$

or, applying the function α_3 to the left-hand and right-hand sides,

$$q_2(k_0+s)\alpha_2(u_1^*(k_0+s))\alpha_3\left(c_2(k_0+s)\right) > c_2(k_0+s+1),$$

Because u_1^* is a (b_1, c_1)-bounded solution, we have $u_1^*(k_0+s) > b_1(k_0+s)$. From inequality (9) we deduce

$$q_2(k_0+s)\alpha_2(b_1(k_0+s))\alpha_3(c_2(k_0+s)) > c_2(k_0+s+1)$$

and because of α_2 being monotone,

$$q_2(k_0+s)\alpha_2(u_1^*(k_0+s))\alpha_3\left(c_2(k_0+s)\right) > q_2(k_0+s)\alpha_2(b_1(k_0+s))\alpha_3(c_2(k_0+s)) > c_2(k_0+s+1).$$

Therefore, the sequence $\{u_{2cs}\}_{s=0}^{\infty}$ is decreasing. In much the same way, it can be proved that the sequence $\{u_{2bs}\}_{s=0}^{\infty}$ is decreasing. Because

$$b_2(k_0+s) < c_2(k_0+s), \quad s = 0, 1, \dots,$$

it follows from (31) and (32) that $u_{2cs} > u_{2bs}$. Therefore, the limits c_2^* and b_2^*, defined by (33) exist and the the inequality $c_2^* \geq b_2^*$ holds. □

Lemma 3 *Assume that all conditions of Theorem 1 hold and that the sequences* $\{u_{2bs}\}_{s=0}^{\infty}$, $\{u_{2cs}\}_{s=0}^{\infty}$ *are defined by* (31) *and* (32). *Let, for a fixed* $s \in \{1, 2, \dots\}$, $u_{2bs}(k)$ *be the solution of initial problem*

$$u_2(k_0) = u_{2bs},$$

and $u_{2cs}(k)$ *be the solution of initial problem*

$$u_2(k_0) = u_{2cs}.$$

Then,

$$u_{2bs}(k_0+s) = b_2(k_0+s)$$

and

$$u_{2cs}(k_0+s) = c_2(k_0+s).$$

Proof. The proof can be done in much the same way as the one of Lemma 1 and is, therefore, omitted. □

Lemma 4 *Assume that all conditions of Theorem 1 hold and that the sequences* $\{u_{2bs}\}_{s=0}^{\infty}$, $\{u_{2cs}\}_{s=0}^{\infty}$ *are defined by* (31) *and* (32). *Let a number* $s \in \{1, 2, \dots\}$ *be fixed. Consider an initial problem*

$$u_2(k_0 + s) = u_{2s},$$

where

$$u_{2s} \in (b_2(k_0 + s), c_2(k_0 + s)).$$

Then,

$$u_{2bs} < u^*_{2s} < u_{2cs}$$

where

$$u^*_{2s} = \alpha_3^{-1}\left(\frac{1}{q_2(k_0)\alpha_2(u^*_1(k_0))}\alpha_3^{-1}\left(\frac{1}{q_2(k_0+1)\alpha_2(u^*_1(k_0+1))}\cdots\right.\right.$$
$$\cdots\left(\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-2)\alpha_2(u^*_1(k_0+s-2))}\right.\right.$$
$$\left.\left.\left.\left.\left.\left.\alpha_3^{-1}\left(\frac{1}{q_2(k_0+s-1)\alpha_2(u^*_1(k_0+s-1))}\alpha_3^{-1}\left(\frac{u_{2s}}{q_2(k_0+s)\alpha_2(u^*_1(k_0+s))}\right)\right)\right)\right)\right)\right)\right).$$

Moreover, initial problem

$$u_2(k_0) = u^*_{2s},$$

defines a solution $u = u_2(k)$ *to equation* (2)*, where* $u_1(k) := u^*_1(k)$*, satisfying*

$$u_2(k_0 + s) = u_{2s}.$$

Proof. The proof can be done in much the same way as the one of Lemma 2 and is omitted. □

In the constructions and results of this section, we assumed that all conditions of Theorem 1 hold, $u^*_1 \in I_1$ and the initial problem (3) defines, by formula (30), a (b_1, c_1)-bounded solution $u^*_1(k)$, $k \in \mathcal{N}(k_0)$ of equation (1). It means that the limits c^*_2, b^*_2, defined by (33), depend on this choice. Therefore, to underline this dependence, we will write $c^*_2(u^*_1)$ instead of c^*_2 and $b^*_2(u^*_1)$ instead of b^*_2. For a given $u^*_1 \in I_1$, define an interval

$$I_2 = I_2(u^*_1) := [b^*_2(u^*_1), c^*_2(u^*_1)].$$

Theorem 5 *Let all conditions of Theorem 1 be valid. Consider initial problem* (4) *where* $u^*_2 \in I_2(u^*_1)$*. Then, this initial problem defines a solution* $u = u^*_2(k)$*,* $k \in \mathcal{N}(k_0)$ *of equation* (2) *such that the inequality*

$$b_2(k) < u^*_2(k) < c_2(k) \ \ \forall k \in \mathcal{N}(k_0)$$

holds, that is, this solution is (b_2, c_2)*-bounded. Moreover, this solution is expressed by the formula*

$$u^*_2(k) = q_2(k-1)\alpha_2(u^*_1(k-1))\alpha_3\left(q_2(k-2)\alpha_2(u^*_1(k-2))\alpha_3\left(q_2(k-3)\alpha_2(u^*_1(k-3))\dots\right.\right.$$
$$\left.\left.\dots\left(\alpha_3\left(q_2(k_0+1)\alpha_2(u^*_1(k_0+1))\alpha_3(q_2(k_0)\alpha_2(u^*_1(k_0))\alpha_3(u^*_2))\right)\right)\right)\right) \quad (37)$$

where $k \in \mathcal{N}(k_0)$.

Proof. The proof is similar to the proof of Theorem 3 and is omitted. □

2.3 Initial data generating (b, c)-bounded solutions to system (1), (2)

Our final result of sets of initial data generating (b, c)-bounded solutions to system (1), (2) combines the conclusions of statements given by Theorem 3 and Theorem 5 and is formulated as follows.

Theorem 6 *Let all conditions of Theorem 1 hold. Let, moreover $u_1^* \in I_1$ and $u_2^* \in I_2$ be fixed. Then, the initial problem* (3), (4) *generates a (b, c)-bounded solution $u = (u_1^*(k), u_2^*(k))$, $k \in \mathcal{N}(k_0)$ of the system* (1), (2) *and representations* (30), (37) *and hold, that is*

$$u_1^*(k) = q_1(k-1)\alpha_1\left(q_1(k-2)\alpha_1\left(q_1(k-3)\ldots\left(\alpha_1\left(q_1(k_0+1)\alpha_1(q_1(k_0)\alpha_1(u^*))\right)\right)\right)\right)$$

and

$$\begin{aligned} u_2^*(k) = q_2(k-1)\alpha_2(u_1^*(k-1))\alpha_3\left(q_2(k-2)\alpha_2(u_1^*(k-2))\alpha_3\left(q_2(k-3)\alpha_2(u_1^*(k-3))\ldots\right.\right.\\ \left.\left.\ldots\left(\alpha_3\left(q_2(k_0+1)\alpha_2(u_1^*(k_0+1))\alpha_3(q_2(k_0)\alpha_2(u_1^*(k_0))\alpha_3(u_2^*))\right)\right)\right)\right) \end{aligned}$$

where $k \in \mathcal{N}(k_0)$.

Proof. Being a simple consequence of Theorem 3 and Theorem 5 and the proof is omitted. □

CONCLUSION

Research presented in the paper is a continuation of previous investigations [1]–[7]. A similar scalar problem for a nonlinear scalar equation is considered in [13] where the bisections method is recommended to define the desired initial data. We refer to [9, 10, 11, 12] as well, where some methods are developed to investigate the asymptotic properties of solutions to discrete systems. To the basics of the theory of difference equations and the asymptotic behavior of solutions, we refer, e.g., to [8, 14] and to the references therein.

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GONIOMETRIC FUNCTIONS IN SCHOOL MATHEMATICS PROBLEMS

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Abstract: *The article contains a set of problems solved with the help of goniometric functions. These problems are not typical for school mathematics and while their solving it is necessary to use the connections with other branches of mathematics (solving binomial and reciprocal equations, Moivre's theorem, Ptolemy's theorem, arithmetic sequence, the theory of rational and irrational numbers, etc.). Solving these problems is the measure for deepening and widening of the secondary school and university students'knowledge about trigonometric functions and it contributes to their general mathematic overview.*

Keywords: Goniometric function; rational and irrational numbers.

INTRODUCTION

Goniometric functions belong to the basic teaching matter at secondary schools. Although these functions are paid great attention while teaching mathematics, the students' knowledge is rather formal. Most students are able to solve typical problems while simplifying goniometric identities and trigonometric problems from textbooks. When they meet something atypical, they are usually in trouble. However, this situation does not apply only for secondary schools; solving more complicated problems aimed at goniometric functions applications seems troublesome for university students, too. This article offers a number of problems where goniometric functions are used while solving. In some of the problems the use of the goniometric identities is evident, but in some of them the use is surprising. Wherever it is possible, the connection with other branches of mathematics is shown. Therefore the analysis of the given problems can be contributing for the secondary school students, especially for those who participate in higher categories of Mathematical Olympiad. The problems which are presented in the problems package are adopted from [2], [3], [7], and [10]. The theory for the introduction and use of goniometric functions is given in [1], [5], [6], [8], [9], [11], including the general didactic theory of teaching mathematics at secondary schools and universities.

PROBLEMS PACKAGE

Problem 1. (See [3]) Prove that there holds the equality $2.(\sin 54^\circ - \sin 18^\circ) = 1$.

Solution: This problem seems simple because finding the artificial step is not trivial.

$$2.(\sin 54^\circ - \sin 18^\circ) = 2\,.\,2.\cos 36^\circ \sin 18^\circ.\frac{\cos 18^\circ}{\cos 18^\circ} = 2\,.\,\cos 36^\circ\,\frac{\sin 36^\circ}{\cos 18^\circ} = \frac{\sin 72^\circ}{\cos 18^\circ} = \frac{\sin 72^\circ}{\sin 72^\circ} = 1\,.$$

Problem 2. (See [7]) Prove that for each acute angle $\alpha \in (0, \frac{\pi}{2})$ there holds the inequality:

$$(\sin\alpha + \cos\alpha)^2 + (\sin\alpha + 1)^2 + (\cos\alpha + 1)^2 > 6\,.$$

Solution: Let us conduct the proof according to [7]. As α is an acute angle, there exists a right-angled triangle with legs of lengths *a*, *b* and with a hypotenuse of the length *c*, where one of its acute angles is α. Without detriment to generality, it is possible to assume that $\sin \alpha = \frac{a}{c}$, $\cos \alpha = \frac{b}{c}$. We will substitute $\sin \alpha$, $\cos \alpha$ by these expressions in the proven inequality and then simplify it. While doing it we will use Pythagoras's theorem and the triangle inequality $a + b > c$.

$$\left(\frac{a+b}{c}\right)^2 + \left(\frac{a}{c}+1\right)^2 + \left(\frac{b}{c}+1\right)^2 = \frac{(a+b)^2+(a+c)^2+(b+c)^2}{c^2} > \frac{c^2+(a+c)^2+(b+c)^2}{c^2} = \frac{3c^2+a^2+b^2+2\mathrm{ac}+2\mathrm{bc}}{c^2} = \frac{4c^2+2c(a+b)}{c^2} > \frac{4c^2+2c\cdot c}{c^2} = \frac{6c^2}{c^2} = 6\,.$$

Problem 3. (See [3]) Prove that for all real numbers *x* with the property $|x| \le 1$ there holds inequality $|4x^3 - 3x| \le 1$.

Solution: It is possible to simplify this inequality without goniometric functions, but the condition for *x* directly invites to use them. Let φ be such a chosen angle for which there holds $\cos \varphi = x$. Then there holds:

$$4x^3 - 3x = 4\cos^3 \varphi - 3\cos \varphi = \cos \varphi\,(2\cos^2\varphi - 1) - 2(1-\cos^2\varphi)\cos \varphi =$$

$$\cos \varphi\,(\cos^2\varphi - \sin^2\varphi) - 2\sin^2\varphi \cos \varphi = \cos \varphi \cos 2\varphi - \sin \varphi \sin 2\varphi = \cos(\varphi + 2\varphi) = \cos 3\varphi\,.$$

For $\cos 3\varphi$ there holds the inequality $-1 \le \cos 3\varphi \le 1$, so the proven inequality holds too.

Problem 4: (See [10]) Calculate the value of $\sin 36°$.

Solution: We will show four possibilities. First two were adopted from [10].

a) According to Moivre's theorem there holds:

$$\cos 5x + i \sin 5x = \cos^5 x + 5i \cos^4 x \sin x - 10 \cos^3 x \sin^2 x - 10i \cos^2 x \sin^3 x + 5 \cos x \sin^4 x + i \sin^5 x\,.$$

Let us equate the imaginary parts from the previous relation and let us substitute $1 - \sin^2 x$ for $\cos^2 x$. We will get after simplification

$$\sin 5x = 16 \sin^5 x - 20 \sin^3 x + 5 \sin x\,.$$

Now let $x = 36°$. Then the left side of the last equation equals zero. If we denote $\sin 36° = y$, this equation will change to the form $16y^5 - 20y^3 + 5y = 0$. This algebraic equation has 5 real roots in the variable *y*:

0, $\frac{+1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$, $\frac{-1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$, $\frac{+1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$, $\frac{-1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$. Considering the notation $\sin 36° = y$, it is evident that the root 0 does not suit. If we further realize that $0 < \sin 36° < \frac{\sqrt{2}}{2} = \sin 45°$, the only root which suits is $\frac{+1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$. Therefore there holds $\sin 36° = \frac{+1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$.

b) Let *ABCDE* be a regular pentagon with the side of the length 1. Let us denote *x* as the length of its diagonal *AC*. Obviously the tetragon *ABCD* is chordal (it is a isosceles trapezoid where a circumscribed circle can be drawn). We will use Ptolemy's theorem according to which for every chordal tetragon *ABCD* there olds

$$|\mathrm{AB}| \cdot |\mathrm{CD}| + |\mathrm{AD}| \cdot |\mathrm{BC}| = |\mathrm{AC}| \cdot |\mathrm{BD}|\,.$$

With the given notation and premise there holds the relation $1 + x = x^2$. As *x* denotes the length of the diagonal, only the positive root $x = \frac{1+\sqrt{5}}{2}$ suits. Now let us discuss the triangle *ABD*. This triangle is an isosceles one with the base of the length 1, the legs of the lengths *x*, and its internal angle at vertex D is 36° (the inscribed angle related to the central angle 72°). According to Cosine Rule there holds

$$1^2 = x^2 + x^2 - 2x^2 \cos 36^\circ.$$

From here after substituting for x we will calculate $\cos x = \frac{1+\sqrt{5}}{4}$, and finally, with respect to the fact that $\sin 36^\circ$ is a positive number, we will calculate $\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \frac{+1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$. This method is numerically quite simple, but its use is rather debatable. The idea how to use a pentagon and its elements requires routine skills and experience with solving similar problems, but these ones students usually lack. Let us present two further methods which reflect the thought process closer to both secondary school and university students.

c) This way of solution uses formulas from trigonometry and solving algebraic equations; the use of computer programs is also possible. First let us simplify the relation for $\sin 36^\circ$.

$$\sin 36^\circ = 2 \sin 18^\circ \cos 18^\circ = 2 \cos 72^\circ \cos 18^\circ = 2 (\cos^2 36^\circ - \sin^2 36^\circ) \sin 72^\circ =$$

$$= 2 (1 - 2 \sin^2 36^\circ)\, 2 \sin 36^\circ \cos 36^\circ = (4 \sin 36^\circ - 8 \sin^3 36^\circ)\sqrt{1-\sin^2 36^\circ}.$$

Now let us introduce the substitution $\sin 36^\circ = x$. After substituting we will get the equation $x = (4x - 8x^3)\sqrt{1-x^2}$. The only root is number 0, which does not suit with respect to the introduced substitution. After dividing by x and further simplification we will get an algebraic equation $64 x^6 - 128 x^4 + 80 x^2 - 15 = 0$. Its solution can be performed either with the help of any from a number of mathematics programs, or through a direct calculation. At the direct calculation we will use the substitution $y = x^2$ and find the roots of the 3rd degree polynomial with the variable y ($y_1 = \frac{3}{4}$, $y_{2,3} = \frac{5\pm\sqrt{5}}{8}$). After the return to the variable x , we will get six different real solutions ($x_{1,2} = \pm\frac{\sqrt{3}}{2}$, $x_{3,4} = \pm\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$, $x_{5,6} = \pm\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$). With respect to the substitution $\sin 36^\circ = x$,the only solution which suits is $\frac{+1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$.

d) Finally we will show the solution which uses two ways of analytical notation of vertices of the regular decagon which is inscribed in a unit circle. These two ways will then be equated. First we will solve a binomial equation $x^{10} = 1$. The general solution (and therefore the analytical expression of the regular decagon vertices) is as follows:

$$x_k = \cos\frac{2k\pi}{10} + i \sin\frac{2k\pi}{10},\ k = 0, 1, 2, ..., 9.$$

For $k = 1$ the value of the argument in the general solution is just 36°. Therefore for our use the only "interesting" root is $x_1 = \cos 36^\circ + i \sin 36^\circ$. Now we will solve the binomial equation $x^{10} = 1$ as if it were an algebraic one. Because ± 1 are its roots, we can write

$$x^{10} - 1 = (x^5 + 1)(x^5 - 1) = (x + 1)(x - 1)(x^4 - x^3 + x^2 - x + 1)(x^4 + x^3 + x^2 + x + 1)\,.$$

The last two equations of the 4th order can be solved as reciprocal ones (the details are not given here) and we will obtain two tetrads of real roots which together with roots ± 1 represent all ten root of the equation $x^{10} = 1$. These roots are given below. Their notation corresponds to the previous notation of roots while solving the equation with the help of the binomial formula.

$$x_0 = 1,\ x_{1,9} = \frac{1+\sqrt{5}\pm i\sqrt{10-2\sqrt{5}}}{4},\quad x_{2,8} = \frac{-1+\sqrt{5}\pm i\sqrt{10+2\sqrt{5}}}{4},$$
$$x_{3,7} = \frac{1-\sqrt{5}\pm i\sqrt{10+2\sqrt{5}}}{4},\quad x_{4,6} = \frac{-1-\sqrt{5}\pm i\sqrt{10-2\sqrt{5}}}{4},\quad x_5 = -1.$$

As was stated before, we are only "interested" in the root x_1 . By comparing its real and imaginary parts we will get the equality $\cos 36^{o} + i \sin 36^{o} = \frac{1+\sqrt{5}+i\sqrt{10-2\sqrt{5}}}{4}$. From this follows $\sin 36^{o} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$. As the "secondary result" we get the relation $\cos 36^{o} = \frac{1+\sqrt{5}}{4}$; with the help of other roots of the binomial equation we can find relations for calculating values of functions sines and cosines of all multiples of 36^{o}, e.g. with the help of the root x_2 we will get $\sin 72^{o} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$, $\cos 72^{o} = \frac{\sqrt{5}-1}{4}$. When we compare all four solution methods of the Problem 3, we can state that the last solution method brings the most favourable responses from students.

Problem 5. (See [2]) Let $a_0 \in (-1,1)$. Let us define in a recurrent way the sequence of real numbers $\{a_n\}_{n\in N}$ as follows: $a_n = \sqrt{\frac{1+a_{n-1}}{2}}$. Find the relation for determining the n-th member of this sequence. Let $\{b_n\}_{n\in N}$ be a sequence defined with the help of the sequence $\{a_n\}_{n\in N}$ as follows: $b_n = 4^n(1-a_n)$. Decide if these two sequences are convergent and set their limits.

Solution: The inductive method gets us nowhere. The clue for the solution is the thought (based on the premise that $a_0 \in (-1,1)$) that there exists the only angle $\varphi \in (0, \pi)$ with the property $a_0 = \cos\varphi$. For this angle φ there holds: $a_1 = \sqrt{\frac{1+\cos\phi}{2}} = \cos\frac{\phi}{2}$. As the previous relation applies generally in the form $\sqrt{\frac{1+\cos\omega}{2}} = \cos\frac{\omega}{2}$ for any angle ω, we can represent the members of the sequence $\{a_n\}_{n\in N}$ as follows: $\cos\varphi, \cos\frac{\phi}{2}, \cos\frac{\phi}{4}, \cos\frac{\phi}{8}, \cos\frac{\phi}{16}, \ldots$, i.e. there holds $a_n = \cos\frac{\phi}{2^n}$. For the n-th member of the sequence b_n then there holds $b_n = 4^n(1-\cos\frac{\phi}{2^n})$. Now let us consider the problem of the convergence. Obviously the sequence $\{a_n\}_{n\in N}$ is convergent and its limit equals 1. Finding the limit of the sequence $\{b_n\}_{n\in N}$ is more complicated because it requires a suitable notation and formulation. Let us gradually calculate $\lim\limits_{n\to\infty} 4^n\left(1-\cos\frac{\phi}{2^n}\right) = \lim\limits_{n\to\infty}\frac{4^n\left(1-\cos\frac{\phi}{2^n}\right)\cdot\left(1+\cos\frac{\phi}{2^n}\right)}{1+\cos\frac{\phi}{2^n}} = \lim\limits_{n\to\infty}\frac{4^n\sin^2\frac{\phi}{2^n}}{1+\cos\frac{\phi}{2^n}}$. Let us expand thee fraction in the last limit by φ^2 and let us continue: $\lim\limits_{n\to\infty}\frac{\phi^2 4^n\sin^2\frac{\phi}{2^n}}{\phi^2\left(1+\cos\frac{\phi}{2^n}\right)} = \lim\limits_{n\to\infty}\frac{\phi^2\sin^2\frac{\phi}{2^n}}{\phi^2\left(\frac{1}{2^n}\right)^2\left(1+\cos\frac{\phi}{2^n}\right)} = \lim\limits_{n\to\infty}\left(\frac{\varphi^2}{1+\cos\frac{\varphi}{2^n}}\right)\cdot\left(\frac{\sin\frac{\varphi}{2^n}}{\frac{\varphi}{2^n}}\right)^2$. As there holds the limit known from mathematical analysis $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$, then the sequence $\{b_n\}_{n\in N}$ is also convergent and its limit equals the value $\lim\limits_{n\to\infty}\left(\frac{\phi^2}{1+\cos\frac{\phi}{2^n}}\right) = \frac{\phi^2}{2}$ (where $\varphi = \arccos a_0$).

Problem 6. (See [10]) Prove that there holds: $tg\, 20^{o} .\ tg\, 40^{o} .\ tg\, 60^{o} .\ tg\, 80^{o} = 3$.

Solution: The clue for the solution is the relation $tg\, 60^{o} = \sqrt{3}$. Further we have to realize that all values of the function tangent for all values of arguments have to be expressed as the tangent of the only argument. As we will show further, it is useful to choose the argument 20^{o}.

From trigonometry there are familiar following relations $tg\,(x+y) = \frac{\mathrm{tg}x+\mathrm{tg}y}{1-\mathrm{tg}x\cdot\mathrm{tg}y}$, $tg\,(x-y) = \frac{\mathrm{tg}x-\mathrm{tg}y}{1+\mathrm{tg}x\cdot\mathrm{tg}y}$. According to them and using $tg\, 60^{o} = \sqrt{3}$ we can write:

$tg\,40^\circ . tg\,80^\circ = tg\,(60^\circ - 20^\circ)\,.\,tg\,(60^\circ + 20^\circ) = \frac{\mathrm{tg}60^0 - \mathrm{tg}20^0}{1+\mathrm{tg}60^0\cdot\mathrm{tg}20^0} . \frac{\mathrm{tg}60^0 + \mathrm{tg}20^0}{1-\mathrm{tg}60^0\cdot\mathrm{tg}20^0} = \frac{3-\mathrm{tg}^2 20^0}{1-3\mathrm{tg}^2 20^0}$. When using the addition formula for $tg(2x + x)$ and the formula $tg\,(x + y) = \frac{\mathrm{tg}x+\mathrm{tg}y}{1-\mathrm{tg}x\cdot\mathrm{tg}y}$ there can be derived the relation for the tangent of the triple angle: $tg\,3x = tg\,x\,\frac{3-\mathrm{tg}^2 x}{1-3\mathrm{tg}^2 x}$. Let us substitute 20° for x and we will get:

$tg\,60^\circ = tg\,(3\,.\,20^\circ) = tg\,20^\circ \frac{3-\mathrm{tg}^2 20^0}{1-3\mathrm{tg}^2 20^0} = tg\,20^\circ .\ tg\,40^\circ .\ tg\,80^\circ$ (using the above given relation). Then there holds:

$tg\,20^\circ .\ tg\,40^\circ .\ tg\,60^\circ .\ tg\,80^\circ = (tg\,20^\circ .\ tg\,40^\circ .\ tg\,80^\circ)\,.tg\,60^\circ = tg^2\,60^\circ = 3.$

Thus the proof is finished.

Problem 7. (See [10]) Let the function of the variable φ be given as follows:

$$f(\varphi) = \sin\varphi + \sin 2\varphi + \sin 3\varphi + ... + \sin 1000\,\varphi .$$

Prove that the sum of the roots of this function on the interval $\langle 0, 2\pi\rangle$ equals 2001π.

Solution: As values $0, 2\pi$ are evidently the roots of the given function (its zero points), we can examine the given function only in the interval $(0, 2\pi)$. The clue for the solution is to modify the formula of the function φ. Let us use the inductive method:

$\sin\varphi + \sin 2\varphi = \frac{\sin\frac{3\phi}{2}\cdot\sin\phi}{\sin\frac{\phi}{2}}$ (with the restriction to the interval $(0, 2\pi)$ obviously $\sin\frac{\phi}{2} \neq 0$), further we will get

$$\sin\varphi + \sin 2\varphi + \sin 3\varphi = \frac{\sin\frac{3\phi}{2}\cdot\sin\phi}{\sin\frac{\phi}{2}} + \sin 3\varphi = \frac{\sin\frac{3\phi}{2}\cdot\sin\phi+\sin 3\phi\cdot\sin\frac{\phi}{2}}{\sin\frac{\phi}{2}} =$$
$$\frac{\sin\frac{3\phi}{2}\cdot\sin\phi+2\sin\frac{3\phi}{2}\cdot\cos\frac{3\phi}{2}\cdot\sin\frac{\phi}{2}}{\sin\frac{\phi}{2}} = \frac{\sin\frac{3\phi}{2}\left(\sin\phi+2\cos\frac{3\phi}{2}\sin\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} = \frac{\sin\frac{3\phi}{2}(\sin\phi+\sin 2\phi-\sin\phi)}{\sin\frac{\phi}{2}} = \frac{\sin\frac{3\phi}{2}\cdot\sin 2\phi}{\sin\frac{\phi}{2}} =$$
$$\frac{\sin\frac{3\phi}{2}\cdot\sin\frac{4}{2}\phi}{\sin\frac{\phi}{2}} .$$

In the same way we can calculate $\sin\varphi + \sin 2\varphi + \sin 3\varphi + \sin 4\varphi$. Finally we will arrive at the hypothesis:

$$\sin\varphi + \sin 2\varphi + \sin 3\varphi + ... + \sin n\,\varphi = \frac{\sin\frac{n\phi}{2}\cdot\sin\frac{(n+1)\phi}{2}}{\sin\frac{\phi}{2}} .$$

We will prove this hypothesis using the mathematical induction.
Obviously, the relation holds for $n = 1$. Now let us assume that the hypothesis holds for $n = k$ and we will prove its validity for $n = k + 1$.

$$\sin\varphi + \sin 2\varphi + \sin 3\varphi + ... + \sin (k+1)\varphi = = \frac{\sin\frac{k\phi}{2}\cdot\sin\frac{(k+1)\phi}{2}}{\sin\frac{\phi}{2}} + \sin (k+1)\varphi =$$
$$= \frac{\sin\frac{k\phi}{2}\cdot\sin\frac{(k+1)\phi}{2}}{\sin\frac{\phi}{2}} + 2\sin\frac{(k+1)\varphi}{2}\cos\frac{(k+1)\varphi}{2} =$$
$$= \frac{\sin\frac{(k+1)\phi}{2}}{\sin\frac{\phi}{2}}\left(\sin\frac{k\varphi}{2} + 2\cos\frac{(k+1)\varphi}{2}\sin\frac{\varphi}{2}\right) = \frac{\sin\frac{(k+1)\varphi}{2}\cdot\sin\frac{(k+2)\phi}{2}}{\sin\frac{\phi}{2}},$$

using the equality $2\cos x\sin y = \sin (x + y) - \sin (x - y)$.

We will get for $n = 1000$ the following

$$\sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots + \sin 1000\varphi = \frac{\sin\frac{1000\phi}{2}\cdot\sin\frac{1001\phi}{2}}{\sin\frac{\phi}{2}}.$$

It is evident that except the previously mentioned roots $0, 2\pi$, other roots are all values of arguments for which there holds $\sin\frac{1000\phi}{2} = 0$ or $\sin\frac{1001\phi}{2} = 0$. From here there follows $\varphi = \frac{2k\pi}{1000}$ or $\varphi = \frac{2l\pi}{1001}$. As we are searching for roots only in the interval $(0, 2\pi)$, it is necessary to select $k = 1, 2, \ldots, 999$ a $l = 1, 2, \ldots, 1000$. Now it is enough to sum all the roots (using the formula for first n members of the arithmetic sequence). Further we calculate:

$$\sum_{k=1}^{999}\frac{2k\pi}{1000} = \frac{2\pi}{1000}\frac{999}{2}(1 + 999) = 999\,\pi;\ \sum_{l=1}^{1000}\frac{2l\pi}{1001} = \frac{2\pi}{1001}\frac{1000}{2}(1 + 1000) = 1000\,\pi.$$

As the values 0 a 2π belong among the roots, then the sum of all roots in the interval $\langle 0, 2\pi\rangle$ is really 2001π, which is what had to be proven.

Problem 8. (See [10]) Let $n \in \boldsymbol{N}$ be an arbitrary natural number. Find the polynomial with integer coefficients whose one root is $\sin\frac{\pi}{n}$.

Solution: The proof of the existence of the required polynomial consists in the statement that with the help of $\sin\frac{\pi}{n}$ a $\cos\frac{\pi}{n}$ and thanks to angle addition formulas it is possible to express the value of $\sin\pi = 0$ and then we can use conveniently the relation $\cos^2 x = 1 - \sin^2 x$. First we will prove a supportive lemma.

Lemma. For every natural number n there exist polynomials $p_n(t), q_n(t), p'_n(t), q'_n(t)$ with integer coefficients such that there holds

$$\sin(nx) = p_n(\sin x) + \cos x\, q_n(\sin x),$$

$$\cos(nx) = p'_n(\sin x) + \cos x\, q'_n(\sin x).$$

Proof. We will use the mathematical induction. We will repeatedly use the angle addition formulas for sine a cosine functions and the relation $\cos^2 x = 1 - \sin^2 x$.

For $n = 1$ we will certainly chose $p_1(t) = t, q_1(t) = 0, p'_1(t) = 0$ a $q'_1(t) = 1$.

For the second induction step let us assume that the lemma statement holds for $n = k$, and we will prove it for $n = k + 1$. Let us simplify:

$$\sin((k + 1)x) = \sin(kx + x) = \sin(kx)\cos x + \cos(kx)\sin x =$$

$$= \cos x\, p_k(\sin x) + \cos^2 x\, q_k(\sin x) + \sin x\, p'_k(\sin x) + \cos x \sin x\, q'_k(\sin x) =$$

$$= (1 - \sin^2 x)\, q_k(\sin x) + \sin x\, p'_k(\sin x) + \cos x\,[p_k(\sin x) + \sin x\, q'_k(\sin x)],$$

$$\cos((k + 1)x) = \cos(kx + x) = \cos(kx)\cos x - \sin(kx)\sin x =$$

$$= \cos x\, p'_k(\sin x) + \cos^2 x\, q'_k(\sin x) - \sin x\, p_k(\sin x) - \cos x \sin x\, q_k(\sin x) =$$

$$= (1 - \sin^2 x)\, q'_k(\sin x) - \sin x\, p_k(\sin x) + \cos x\,[p'_k(\sin x) - \sin x\, q_k(\sin x)].$$

Then there suffices to select

$$p_{k+1}(t) = (1 - t^2)\, q_k(t) + t\, p'_k(t), \qquad q_{k+1}(t) = p_k(t) + t\, q'_k(t),$$

$$p'_{k+1}(t) = (1 - t^2)\, q'_k(t) - t\, p_k(t), \qquad q'_{k+1}(t) = p'_k(t) - t\, q_k(t),$$

which are polynomials with integer coefficients. Thus the lemma has been proven.

Now we can solve the problem quite easily. It is enough to realize that

$$0 = \sin\pi = \sin\left(n\frac{\pi}{n}\right) = p_n\left(\sin\frac{\pi}{n}\right) + \cos\left(\frac{\pi}{n}\right) q_n\left(\sin\frac{\pi}{n}\right),$$

so $$p^2_n\left(sin\frac{\pi}{n}\right) = (1 - sin^2\frac{\pi}{n})\, q^2_n\left(sin\frac{\pi}{n}\right).$$

From here follows that $sin\frac{\pi}{n}$ is the root of the polynomial $r_n(t) = p^2_n(t) - (1 - t^2)\, q^2_n(t)$, which has integer coefficients.

Note: The proof was performed generally for any natural number n. When searching practically for the given polynomial for a certain value n it is possible to proceed in a simpler way. The best one is to start from Moivre's formula $(cos\, x + i\, sin\, x)^n = cos\, nx + i\, sin\, nx$. We will simplify the left side according to the binomial theorem, use the equality $cos^2 x = 1 - sin^2 x$ and set $x = sin\frac{\pi}{n}$ Then we will get the required polynomial. For $n = 3$ and $n = 4$ the polynomials are set in the following problem.

Problem 9. (See [10]) Find a polynomial with integer coefficients whose one root is

a) $sin\frac{\pi}{3}$, b) $sin\frac{\pi}{4}$.

Solution: a) From trigonometry we know the formulas for the sine and cosine of a double angle. With their help we can derive the relation for $sin\, 3x$ which contains only $sin\, x$ as a variable. We can use the addition formula in the form $sin\, 3x = sin(2x + x)$, while the function $cos\, x$ will be modified with the help of the formula $cos\, x = \sqrt{1-\sin^2 x}$. The simplifications are not difficult, so we will present only the result (we can count it also with the help of Moivre's theorem):

$sin\, 3x = 3\, sin\, x - 4\, sin^3 x$. Let us substitute $\frac{\pi}{3}$ for x and we will get: $0 = sin\,(3\cdot\frac{\pi}{3}) = 3\, sin\frac{\pi}{3} - 4\, sin^3\frac{\pi}{3}$. From here then follows that the required polynomial is $f(x) = 3x - 4x^3$.

b) Let us proceed similarly as in case a). With the help of above given methods we will get the relation $sin\, 4x = cos\, x\,(4sin\, x - 8sin^3 x)$. Now we cannot replace the function cosine by the function sine omitting square roots when we use the relation $cos^2 x = 1 - sin^2 x$. Nevertheless we can for $x = \frac{\pi}{4}$ write $0 = sin\,(4\,.\frac{\pi}{4}) = cos\frac{\pi}{4}\,(4sin\frac{\pi}{4} - 8sin^3\frac{\pi}{4})$. As $cos\frac{\pi}{4} \neq 0$, there has to hold $4\, sin\frac{\pi}{4} - 8\, sin^3\frac{\pi}{4} = 0$. From here follows similarly as above that the required polynomial is in the form $f(x) = 4x - 8x^3$.

Problem 10. (See [10]) Prove that $\sum_{k=1}^{n} tg^2\frac{k\pi}{2n+1} = n\,(2n + 1)$.

Solution: We will use Moivre's theorem and with its help we will derive the relation

$$sin\,(2n + 1)\,\alpha = \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}cos^{2n-2k}\alpha\, sin^{2k+1}\alpha\,. \qquad (1)$$

Next we will consider only those values of the angle α, for which $cos\,\alpha \neq 0$.
We will divide the equation (*) by $cos^{2n+1}\alpha$ and substitute $\alpha_k = \frac{k\pi}{2n+1}$, $k = 1, . . . , n$. Let us remark that for all these values of the angle α there holds $sin\,(2n + 1)\,\alpha = 0$. After the substitution we get

$$0 = \binom{2n+1}{1}tg\alpha_k - \binom{2n+1}{3}tg^3\alpha_k + \ldots + (-1)^n\binom{2n+1}{2n+1}tg^{2n+1}\alpha_k\,. \quad (2)$$

For all natural numbers $k = 1, . . . , n$ there holds $tg\alpha_k \neq 0$. We will divide the last equation by $tg\alpha_k$ and we will get

$$0 = \binom{2n+1}{1} - \binom{2n+1}{3} tg^2\alpha_k + \ldots + (-1)^n \binom{2n+1}{2n+1} tg^{2n}\alpha_k \,. \qquad (3)$$

It is evident from the equation (3) that mutually different numbers $tg^2\alpha_k$, $k = 1, \ldots, n,$ are all the roots of the polynomial

$$x^n \binom{2n+1}{2n+1} - x^{n-1}\binom{2n+1}{2n-1} + \ldots + (-1)^n \binom{2n+1}{1} = 0 \,. \qquad (4)$$

Now we will use Viète's relations and we will add up all roots of the polynomial (4). We will get

$$\sum_{k=1}^{n} tg^2 \frac{k\pi}{2n+1} = \binom{2n+1}{2n-1} = n(2n+1).$$

Thus the proof has been finished.

Problem 11. (See [10]) Prove that the number $\frac{\arccos\frac{1}{3}}{\pi}$ is irrational.

Solution: We will use the proof by contradiction. Let us denote $arccos\,\frac{1}{3} = \alpha$. Then there holds $cos\alpha = \frac{1}{3}$. We will suppose that number $\frac{\alpha}{\pi}$ is a rational number, so we can write $\frac{\alpha}{\pi} = \frac{p}{n}$, where $p \in \mathbf{Z}$, $n \in \mathbf{N}$. From the previous equality there follows $n\alpha = p\pi$, then $2n\alpha = 2p\pi$, and finally $cos\ 2n\alpha = cos\ 2p\pi$. The value $cos\ 2p\pi$ equals one, therefore also $cos\ 2n\alpha = 1$. We will show that this cannot happen (we know that α is an angle with the property $cos\ \alpha = \frac{1}{3}$). The equality $cos\ 2n\alpha = 1$ applies for $n = 0$, however this is a contradiction to the premise that the number n is the fraction denominator. We will calculate the values of $cos\ 2n\alpha$ for $n = 1, 2, 3$ with the help of the inductive method. We will use the following formula:

$$cos\,(x + y) = 2\,cos\,x.\,cos\,y - cos\,(x - y)\,.$$

We can derive this formula by adding formulas for the cosine of the sums and differences of angles. For $n = 1$ then holds:

$$cos\,2\alpha = cos\,(\alpha + \alpha) = 2\,cos^2\alpha - cos\,(\alpha - \alpha) = \frac{2}{9} - 1 = -\frac{7}{9}\,.$$

For $n = 2$ similarly:

$$cos\,4\alpha = cos\,(2\alpha + 2\alpha) = 2\,cos^2\,2\alpha - cos\,(2\alpha - 2\alpha) = \frac{98}{81} - 1 = \frac{17}{81}\,,$$

for $n = 3$ then holds

$$cos\,6\alpha = cos\,(4\alpha + 2\alpha) = 2\,cos\,4\alpha\ \ cos\,2\alpha - cos\,(4\alpha - 2\alpha) = \frac{-238}{729} + \frac{7}{9} = \frac{329}{729}\,.$$

Now we can state the hypothesis: For every natural number n the value $cos\ 2n\alpha$ equals the fraction $\frac{k_n}{3^{2n}}$, where k_n is an integer indivisible by three. We will prove this hypothesis with the help of the mathematical induction.

For $n = 1$ the hypothesis holds without any calculation. Let us assume its validity for $n = 2, 3, \ldots, t - 1$ and we will prove it for $n = t$.

$$cos\,2t\alpha = cos\,[(2t-2)\alpha + 2\alpha] = 2\,cos\,(2t-2)\alpha\,.\,cos\,2\alpha - cos\,(2t-4)\alpha =$$

$$2\,.\,\frac{k_{t-1}}{3^{2t-2}}.\left(-\frac{7}{9}\right) - \frac{k_{t-2}}{3^{2t-4}} = \frac{-14k_{t-1} - 81k_{t-2}}{3^{2t}}.$$

As numbers k_{t-1}, k_{t-2} according to the precondition are not divisible by three, then the numerator is not divisible by three either. Thus the proof using the mathematical induction is concluded. From the proven hypothesis there follows that the value $cos\ 2n\alpha$ can never equal one. This is the contradiction to the precondition that $\frac{\arccos\frac{1}{3}}{\pi}$ is a rational number, so it has to be an irrational one. Thus the proof has been performed.

CONCLUSION

The aim of this article was to present a set of problems which can show that apart from "traditional" problems presented in textbooks and problem collections for secondary schools there exist a number of problems which are unusual and of various levels of difficulty. Students are not used to them, so their solution can cause problems. When such problems are introduced to the teaching process, it is possible to deepen the students' knowledge about goniometric functions and show them connections among different branches of school mathematics. Let us mention the performed connections with the theory of rational and irrational numbers, binomial and reciprocal equations, etc. The further possible perspective on goniometric and hyperbolic functions from the point of view of discrete iterative theory can be found in [4]. This topic can also contribute to increase the students' interest in mathematics as the whole. Further, solving such types of problems can be beneficial for students who participate in higher rounds of Mathematical Olympiad and mathematical competition Klokan. Solving non-traditional problems, not only from the area of goniometric functions, has its essential place in teaching mathematics.

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SOLUTIONS OF PERTURBED SECOND-ORDER DISCRETE EMDEN-FOWLER TYPE EQUATION WITH POWER ASYMPTOTICS OF SOLUTIONS

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Abstract: *The paper discusses the existence of solutions to the perturbed Emden-Fowler type difference equation with power asymptotics. To prove the result, the given equation is transformed into an equivalent system of difference equations. Then, a result on the existence of solutions with graphs remaining in a previously given domain is applied.*

Keywords: difference equation, Emden-Fowler type equation, power asymptotics, equivalent system, perturbation.

INTRODUCTION

Let k_0 be a natural number. By $\mathbb{N}(k_0)$ we denote the set of all natural numbers greater than or equal to k_0, that is,

$$\mathbb{N}(k_0) = \{k_0, k_0 + 1, \dots\}.$$

In the paper we study the asymptotic behavior of solutions of a perturbed second-order nonlinear discrete equation of Emden-Fowler type

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = \sigma(k) \tag{1}$$

where $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution, $\Delta u(k)$ is its first-order forward difference, i.e., $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 u(k)$ is its second-order forward difference, i.e.,

$$\Delta^2 u(k) = \Delta(\Delta u(k)) = u(k+2) - 2u(k+1) + u(k),$$

α, m ($m \neq 1$) are real numbers, $\sigma\colon \mathbb{N}(k_0) \to \mathbb{R}$ is a function characterizing the perturbation assumed to be sufficiently small, as it were, in the paper.

A solution $u = u(k)$ of equation (1) is defined as a function $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ satisfying (1) for each $k \in \mathbb{N}(k_0)$.

Below we assume that the sign $+$ in equation (1) is admissible only in the case of m having the form of a ratio of integers p/q, where the difference $p - q$ is odd. Our investigation does not cover the cases $m = 1$, $m = 0$ leading to linear equations.

The equation (1), if $\sigma(k)$ is omitted, can be regarded as a discrete analogue of the second-order Emden-Fowler-type differential equation

$$y''(x) \pm x^{\alpha} y^{m}(x) = 0. \tag{2}$$

Indeed, replacing x by k, $y(t)$ by $u(k)$ and $y''(t)$ by $\Delta^2 u(k)$, we formally arrive at equation (1).

Going back to history, the second-order differential nonlinear equation

$$y'' + x^{\sigma} |y|^{\delta - 1} y = 0,$$

where σ and δ ($\delta \neq 1$) are real numbers, was considered by R. Emden [1]. Later, a generalized n-th order Emden-Fowler type equation

$$y^{(n)}(x) = p(x)|y|^{k} \operatorname{sgn} y$$

was considered, where $n \geq 2$, $n \in \mathbb{N}$, $k > 0$ and p is a continuous function. The asymptotic behaviour of the solutions to this equation has been investigated by many authors. For example, for $n = 2$, by R. Bellman [2], F. V. Atkinson [3] and others. The general cases were studied by I. T. Kiguradze and T. A. Chanturia [4], V. A. Kondratev and V. S. Samovol [5], I. V. Astashova [6] and others. L. Erbe, J. Baoguo and A. Peterson in [7] discussed the asymptotic behavior of the second order Emden-Fowler equation in time scales calculus.

It can be easily verified that, for some α and m, equation (2) has an exact solution of the power form

$$y(x) = \frac{a}{x^{s}}$$

where

$$s = \frac{\alpha + 2}{m - 1} \tag{3}$$

and

$$a = [\mp s(s+1)]^{1/(m-1)} \tag{4}$$

provided that a exists. Unfortunately, the discrete equation (1), if $\sigma(k) = 0$, $k \in \mathbb{N}(k_0)$, has no such exact solution. In other words, there exists no exact solution of the form

$$u(k) = \frac{a}{k^{s}}$$

to equation (1) with $\sigma(k) = 0$, $k \in \mathbb{N}(k_0)$ where a and s are some suitable constants. Nevertheless, in the paper we prove that there exists a solution $u = u(k)$ of (1) with an asymptotic behaviour

$$u(k) \sim \frac{a}{k^{s}} \tag{5}$$

when $k \to \infty$.

We derive even an asymptotic behaviour sharper than (5) since, from our result, it follows that there exists a solution $u = u(k)$ to (1) with the asymptotic representation

$$u(k) = \frac{a}{k^s} + \frac{b}{k^{s+1}} + O\left(\frac{1}{k^{s+1+\gamma}}\right) \tag{6}$$

if $k \to \infty$, assuming that

$$b = \frac{as(s+2)}{s+2-ms} \tag{7}$$

exists, γ is a fixed number from interval $(0,1)$ and the symbol O means the well-known Landau order symbol "big O". When we use this symbol we assume that the relevant relation holds for all $k \in \mathbb{N}(k_0)$ where k_0 is sufficiently large.

1 APPROXIMATIVE POWER SOLUTION TO EQUATION (1)

In this part we show that there exists an approximative solution to equation (1) in the form

$$u_{app}(k) \propto \frac{a}{k^s} + \frac{b}{k^{s+1}} \tag{8}$$

where the coefficients a, s and b are determined by formulas (4), (7), $k \in \mathbb{N}(k_0)$ and k_0 is sufficiently large.

Lemma 1. *Let*

$$\sigma(k) = O\left(\frac{1}{k^{s+4}}\right) \tag{9}$$

where s is defined by (4)*. Then, the function $u_{app}(k)$ given by formula* (8) *is an approximate solution to equation* (1)*.*

Proof. We are looking for the solution of the form (8). Substituting $u(k) = u_{app}(k)$ in equation (1), we get

$$\frac{a}{(k+2)^s} - \frac{2a}{(k+1)^s} + \frac{a}{k^s} + \frac{b}{(k+2)^{s+1}} - \frac{2b}{(k+1)^{s+1}} + \frac{b}{k^{s+1}} \pm k^{\alpha}\left(\frac{a}{k^s} + \frac{b}{k^{s+1}}\right)^m = \sigma(k)$$

or, equivalently,

$$\frac{a}{k^s}\left[\left(1+\frac{2}{k}\right)^{-s} - 2\left(1+\frac{1}{k}\right)^{-s} + 1\right] + \frac{b}{k^{s+1}}\left[\left(1+\frac{2}{k}\right)^{-(s+1)} - 2\left(1+\frac{1}{k}\right)^{-(s+1)} + 1\right]$$
$$\pm \frac{a^m}{k^{ms-\alpha}}\left[1+\frac{b}{ak}\right]^m = \sigma(k). \tag{10}$$

Assuming k_0 sufficiently large and using asymptotic decomposition of terms in square brackets, we derive

$$\frac{a}{k^s}\left[1 - \frac{2s}{k} + \frac{2s(s+1)}{k^2} - \frac{4s(s+1)(s+2)}{3k^3} + O\left(\frac{1}{k^4}\right)\right] - \frac{2a}{k^s}\left[1 - \frac{s}{k} + \frac{s(s+1)}{2k^2}\right.$$
$$\left. - \frac{s(s+1)(s+2)}{6k^3} + O\left(\frac{1}{k^4}\right)\right] + \frac{a}{k^s} + \frac{b}{k^{s+1}}\left[1 - \frac{2(s+1)}{k} + \frac{2(s+1)(s+2)}{k^2} + O\left(\frac{1}{k^3}\right)\right]$$
$$-\frac{2b}{k^{s+1}}\left[1 - \frac{s+1}{k} + \frac{(s+1)(s+2)}{2k^2} + O\left(\frac{1}{k^4}\right)\right] + \frac{b}{k^{s+1}} \pm \frac{a^m}{k^{ms-\alpha}}\left[1 + \frac{bm}{ak} + O\left(\frac{1}{k^2}\right)\right] = \sigma(k).$$

It is easy to see that coefficients of the terms k^{-s} and k^{-s-1} are equal to zero and the last equality reduces to

$$\frac{a}{k^s}\left[\frac{2s(s+1)}{k^2}-\frac{4s(s+1)(s+2)}{3k^3}+O\left(\frac{1}{k^4}\right)\right]-\frac{2a}{k^s}\left[\frac{s(s+1)}{2k^2}\right.$$
$$\left.-\frac{s(s+1)(s+2)}{6k^3}+O\left(\frac{1}{k^4}\right)\right]+\frac{b}{k^{s+1}}\left[-\frac{2(s+1)}{k}+\frac{2(s+1)(s+2)}{k^2}+O\left(\frac{1}{k^3}\right)\right]$$
$$-\frac{2b}{k^{s+1}}\left[-\frac{s+1}{k}+\frac{(s+1)(s+2)}{2k^2}+O\left(\frac{1}{k^4}\right)\right]\pm\frac{a^m}{k^{ms-\alpha}}\left[1+\frac{bm}{ak}+O\left(\frac{1}{k^2}\right)\right]=\sigma(k).$$

Next, assume that the powers $-(s+2)$ and $-(ms-\alpha)$ are equal. Then, the equation

$$ms-\alpha=s+2$$

implies formula (3), that is,

$$s=\frac{\alpha+2}{m-1}\,.$$

Then,

$$\frac{1}{k^{s+2}}\left[as(s+1)\pm a^m\right]+\frac{1}{k^{s+3}}\left[-as(s+1)(s+2)+b(s+1)(s+2)\pm bma^{m-1}\right]$$
$$+O\left(\frac{1}{k^{s+4}}\right)=\sigma(k).$$

As σ satisfies (9), we derive

$$\frac{1}{k^{s+2}}\left[as(s+1)\pm a^m\right]+\frac{1}{k^{s+3}}\left[-as(s+1)(s+2)+b(s+1)(s+2)\pm bma^{m-1}\right]$$
$$+O\left(\frac{1}{k^{s+4}}\right)=0.$$

If

$$as(s+1)\pm a^m=0,$$

then we get formula (4), that is,

$$a=\left[\mp s(s+1)\right]^{1/(m-1)}.$$

Assuming also

$$-as(s+1)(s+2)+b(s+1)(s+2)\pm bma^{m-1}=0,$$

we get

$$b=\frac{as(s+2)}{s+2-ms}$$

and formula (7) is proved as well. Therefore, if $u(k)$ in equation (1) is replaced by the approximate solution $u_{app}(k)$ given by formula (8), then the coefficients of terms k^{-s}, k^{-s-1}, k^{-s-2} and k^{-s-3} in the left-hand side of (10) will be eliminated. $\square$

2 AUXILIARY LEMMA

To prove that there exists a solution to equation (1) with an asympotic behavior described by formula (6), we need some auxiliaries used in [8]. Consider a system of discrete equations

$$\Delta Y(k) = F(k, Y(k)), \;\; k \in \mathbb{N}(k_0) \tag{11}$$

where $F = (F_1, \dots, F_n)^T \colon \mathbb{N}(k_0) \times \mathbb{R}^n \to \mathbb{R}^n$, $Y = (Y_0, \dots, Y_{n-1})^T$. A solution $Y = Y(k)$ of system (11) is defined as a function $Y \colon \mathbb{N}(k_0) \to \mathbb{R}^n$ satisfying (11) for each $k \in \mathbb{N}(k_0)$.

Let $b_i, c_i \colon \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, ..., n$ be given functions satisfying

$$b_i(k) < c_i(k), \;\; i = 1, ..., n, \;\; k \in \mathbb{N}(k_0). \tag{12}$$

Define auxiliary functions $B_i, C_i \colon \mathbb{N}(k_0) \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$,

$$\begin{aligned} B_i(k, Y) &:= -Y_{i-1} + b_i(k), \;\; i = 1, \dots, n, \\ C_i(k, Y) &:= \quad Y_{i-1} - c_i(k), \;\; i = 1, \dots, n \end{aligned}$$

and auxiliary sets

$$\begin{aligned} \Omega_B^i &:= \{(k, Y) \colon k \subset \mathbb{N}(k_0), B_i(k, Y) = 0, B_j(k, Y) \le 0, C_p(k, Y) \le 0, \;\forall\, j, p = 1, \dots, n, \; j \ne i\}, \\ \Omega_C^i &:= \{(k, Y) \colon k \subset \mathbb{N}(k_0), C_i(k, Y) = 0, B_j(k, Y) \le 0, C_p(k, Y) \le 0, \;\forall\, j, p = 1, \dots, n, \; p \ne i\} \end{aligned}$$

where $i = 1, ..., n$.

Playing a crucial role in the proof of the main result and being suitable for applications, the following lemma is a slight modification of [8, Theorem 1].

Lemma 2. *Let the inequality*

$$F_i(k, Y) < b_i(k+1) - b_i(k) \tag{13}$$

hold for every $i = 1, \dots, n$ *and every* $(k, Y) \in \Omega_B^i$. *Let, moreover, inequality*

$$F_i(k, Y) > c_i(k+1) - c_i(k) \tag{14}$$

hold for every $i = 1, \dots, n$ *and every* $(k, Y) \in \Omega_C^i$. *Then, there exists a solution* $u = u(k)$, $k \in \mathbb{N}(k_0)$ *of system* (11) *satisfying the inequalities*

$$b_i(k) < Y_{i-1}(k) < c_i(k) \tag{15}$$

for every $k \in \mathbb{N}(k_0)$ *and* $i = 1, \dots, n$.

3 SYSTEM ASYMPTOTICALLY EQUIVALENT TO EQUATION (1)

Below, instead of equation (1), we will analyze an equivalent system of two difference equations. This system will be constructed using the below auxiliary transformations

$$u(k) = \frac{a}{k^s} + \frac{b}{k^{s+1}}(1 + Y_0(k)), \tag{16}$$

$$\Delta u(k) = \Delta\left(\frac{a}{k^s}\right) + \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)), \tag{17}$$

$$\Delta^2 u(k) = \Delta^2\left(\frac{a}{k^s}\right) + \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_2(k)). \tag{18}$$

where s, a and b are defined by formulas (3), (4) and (7), and $Y_i(k)$, $i = 0, 1, 2$ are new dependent functions. Below we derive relations connecting them.

Taking the first differences of the right-hand side and the left-hand side, respectively, of (16) leads to

$$\Delta u(k) = \Delta\left(\frac{a}{k^s}\right) + \frac{b}{(k+1)^{s+1}}\Delta Y_0(k) + \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_0(k)).$$

Comparing the result with (17) we get the equation

$$\frac{b}{(k+1)^{s+1}}\Delta Y_0(k) + \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_0(k)) = \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)),$$

which is equivalent with

$$\Delta Y_0(k) = (k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right)(-Y_0(k) + Y_1(k)). \tag{19}$$

Next, taking the first differences of the right-hand side and the left-hand side, respectively, of (17) leads to

$$\Delta^2 u(k) = \Delta^2\left(\frac{a}{k^s}\right) + \Delta\left(\frac{b}{(k+1)^{s+1}}\right)\Delta Y_1(k) + \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)).$$

Comparing the result with (18), we get

$$\Delta\left(\frac{b}{(k+1)^{s+1}}\right)\Delta Y_1(k) + \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)) = \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_2(k)),$$

and an equivalent equation is

$$\Delta Y_1(k) = \frac{\Delta^2\left(\dfrac{1}{k^{s+1}}\right)}{\Delta\left(\dfrac{1}{(k+1)^{s+1}}\right)}(-Y_1(k) + Y_2(k)). \tag{20}$$

The derived system of difference equations (19), (20) defines the relationships between $Y_i(k)$, $i = 0, 1, 2$. Next, we will get a system equivalent with equation (1). To do this, we must express $Y_2(k)$

in (20) in terms of $Y_0(k)$ using initial equation (1). We can substitute (16) – (18) into equation (1). Then,

$$\Delta^2\left(\frac{a}{k^s}\right)+\Delta^2\left(\frac{b}{k^{s+1}}\right)(1+Y_2(k))\pm k^\alpha\left(\frac{a}{k^s}+\frac{b}{k^{s+1}}(1+Y_0(k))\right)^m=\sigma(k),$$

or,

$$\frac{a}{(k+2)^s}-\frac{2a}{(k+1)^s}+\frac{a}{k^s}+\left(\frac{b}{(k+2)^{s+1}}-\frac{2b}{(k+1)^{s+1}}+\frac{b}{k^{s+1}}\right)(1+Y_2(k))$$
$$\pm\frac{a^m}{k^{ms-\alpha}}\left(1+\frac{b}{ak}(1+Y_0(k))\right)^m=\sigma(k).$$

Assume that $Y_0(k)=O(1)$ and $Y_2(k)=O(1)$. Condition $Y_0(k)=O(1)$ will be satisfied if we prove the main result. Condition $Y_2(k)=O(1)$ will be the consequence of the previous one. Expressing asymptotically each of the expressions in the previous equation, we obtain

$$\begin{aligned}
&\frac{a}{k^s}\left(1-\frac{2s}{k}+\frac{s(s+1)}{2}\frac{4}{k^2}-\frac{s(s+1)(s+2)}{6}\frac{8}{k^3}+O\left(\frac{1}{k^4}\right)\right)\\
&\quad-\frac{2a}{k^s}\left(1-\frac{s}{k}+\frac{s(s+1)}{2}\frac{1}{k^2}-\frac{s(s+1)(s+2)}{6}\frac{1}{k^3}+O\left(\frac{1}{k^4}\right)\right)+\frac{a}{k^s}\\
&\quad+(1+Y_2(k))\left[\frac{b}{k^{s+1}}\left(1-\frac{2(s+1)}{k}+\frac{(s+1)(s+2)}{2}\frac{4}{k^2}+O\left(\frac{1}{k^3}\right)\right)\right.\\
&\quad\left.-\frac{2b}{k^{s+1}}\left(1-\frac{(s+1)}{k}+\frac{(s+1)(s+2)}{2}\frac{1}{k^2}+O\left(\frac{1}{k^3}\right)\right)+\frac{b}{k^{s+1}}\right]\\
&\quad\pm\frac{a^m}{k^{s+2}}\left(1+\frac{mb}{ak}(1+Y_0(k))+O\left(\frac{1}{k^2}\right)\right)=O\left(\frac{1}{k^{s+4}}\right).
\end{aligned}$$

Carefully collecting the coefficients multiplying the power functions, we simplify this relation to

$$\begin{aligned}
&\frac{1}{k^s}(a-2a+a)+\frac{1}{k^{s+1}}(-2as+2as+(1+Y_2(k))(b-2b+b))\\
&\quad+\frac{1}{k^{s+2}}(2as(s+1)-as(s+1)+(1+Y_2(k))(-2b(s+1)+2b(s+1))\pm a^m)\\
&\quad+\frac{1}{k^{s+3}}\left[-as(s+1)(s+2)\frac{4}{3}+as(s+1)(s+2)\frac{1}{3}+(1+Y_2(k))(2b(s+1)(s+2)\right.\\
&\quad\left.-b(s+1)(s+2))\pm mba^{m-1}(1+Y_0(k))\right]+O\left(\frac{1}{k^{s+4}}\right)=0.
\end{aligned}$$

Hence, we have arrived at the equation

$$-as(s+1)(s+2)+b(s+1)(s+2))+Y_2(k)b(s+1)(s+2)$$
$$\pm mba^{m-1}(1+Y_0(k))+O\left(\frac{1}{k}\right)=0.$$

Because

$$-as(s+1)(s+2) + b(s+1)(s+2) \pm mba^{m-1} = 0,$$

we have

$$Y_2(k)b(s+1)(s+2) - mbs(s+1)Y_0(k) + O\left(\frac{1}{k}\right) = 0.$$

Hence,

$$Y_2(k) = \frac{ms}{s+2}Y_0(k) + O\left(\frac{1}{k}\right). \tag{21}$$

System of equations (19), (20), if in (20) $Y_2(k)$ is replaced by formula (21), takes the form

$$\Delta Y_0(k) = (k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right)\left(-Y_0(k) + Y_1(k)\right), \tag{22}$$

$$\Delta Y_1(k) = \frac{\Delta^2\left(\dfrac{1}{k^{s+1}}\right)}{\Delta\left(\dfrac{1}{(k+1)^{s+1}}\right)}\left(\frac{ms}{s+2}Y_0(k) - Y_1(k) + O\left(\frac{1}{k}\right)\right). \tag{23}$$

It is easy to verify that

$$(k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right) = -\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)$$

and

$$\frac{\Delta^2\left(\dfrac{1}{k^{s+1}}\right)}{\Delta\left(\dfrac{1}{(k+1)^{s+1}}\right)} = -\frac{s+2}{k} + O\left(\frac{1}{k^2}\right).$$

Applying these formulas to (22), (23), we have

$$\Delta Y_0(k) = \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-Y_0(k) + Y_1(k)\right), \tag{24}$$

$$\Delta Y_1(k) = \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k) - Y_1(k) + O\left(\frac{1}{k}\right)\right). \tag{25}$$

The system (24), (25) is asymptotically equivalent with the equation (1). We will use (24), (25) in the proof of the main result.

4 MAIN RESULT

Theorem 1. *Let s, a and b be defined by formulas* (3), (4) *and* (7). *Let, moreover, $-2 < s < -1$ and $m < 0$. Assume that the perturbation σ satisfies condition* (9) *and that there exists a constant*

γ, satisfying $0 < \gamma < 1$ and positive numbers ε_i, $i = 1, 2, 3, 4$ such that

$$\varepsilon_4 < \varepsilon_1 \frac{\gamma + s + 1}{-(s+1)}\,, \tag{26}$$

$$\varepsilon_3 < \varepsilon_2 \frac{\gamma + s + 1}{-(s+1)}\,, \tag{27}$$

$$\varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms}\,, \tag{28}$$

$$\varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms}\,. \tag{29}$$

Then, for sufficiently large fixed k_0, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1) *such that, for every $k \in \mathbb{N}(k_0)$,*

$$-\frac{\varepsilon_3}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_4}{k^\gamma}\,, \tag{30}$$

$$-\frac{\varepsilon_1}{k^\gamma} < \left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} < \frac{\varepsilon_2}{k^\gamma}\,, \tag{31}$$

$$-\frac{\varepsilon_3}{k^\gamma} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) - \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1} < \frac{\varepsilon_4}{k^\gamma} + O\left(\frac{1}{k}\right). \tag{32}$$

Proof. To prove the result we use the auxiliary system (24), (25) and Lemma 2. In system (11) we set $n = 2$ and

$$F_1(k, Y_0, Y_1) := \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-Y_0(k) + Y_1(k)\right),$$
$$F_2(k, Y_0, Y_1) := \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k) - Y_1(k) + O\left(\frac{1}{k}\right)\right).$$

Let $\varepsilon_i > 0$, $i = 1, \dots, 4$, $\gamma > 0$ and $\beta > 0$ be fixed. Assuming $k_0 > 0$ and sufficiently large such that the asymptotic computations in the proof are correct for every $k \in \mathbb{N}(k_0)$, define functions b_i, c_i, $b_i(k) < c_i(k)$, $i = 1, 2$, satisfying (12), by formulas

$$b_1(k) := -\frac{\varepsilon_1}{k^\gamma}\,, \quad c_1(k) := \frac{\varepsilon_2}{k^\gamma}\,,$$
$$b_2(k) := -\frac{\varepsilon_3}{k^\beta}\,, \quad c_2(k) := \frac{\varepsilon_4}{k^\beta}\,.$$

Then,

$$\begin{aligned}
B_1(k, Y) &:= -Y_0 + b_1(k) = -Y_0 - \frac{\varepsilon_1}{k^\gamma}\,,\\
B_2(k, Y) &:= -Y_1 + b_2(k) = -Y_1 - \frac{\varepsilon_3}{k^\beta}\,,\\
C_1(k, Y) &:= \quad Y_0 - c_1(k) = \quad Y_0 - \frac{\varepsilon_2}{k^\gamma}\,,\\
C_2(k, Y) &:= \quad Y_1 - c_2(k) = \quad Y_1 - \frac{\varepsilon_4}{k^\beta}
\end{aligned}$$

and

$$\Omega_B^1 = \left\{(k,Y)\colon k\in\mathbb{N}(k_0),\;\; Y_0 = -\frac{\varepsilon_1}{k^\gamma}\,,\;\; -\frac{\varepsilon_3}{k^\beta} \le Y_1 \le \frac{\varepsilon_4}{k^\beta}\right\}, \tag{33}$$

$$\Omega_B^2 = \left\{(k,Y)\colon k\in\mathbb{N}(k_0),\;\; Y_1 = -\frac{\varepsilon_3}{k^\beta}\,,\;\; -\frac{\varepsilon_1}{k^\gamma} \le Y_0 \le \frac{\varepsilon_2}{k^\gamma}\right\}, \tag{34}$$

$$\Omega_C^1 = \left\{(k,Y)\colon k\in\mathbb{N}(k_0),\;\; Y_0 = \frac{\varepsilon_2}{k^\gamma}\,,\;\; -\frac{\varepsilon_3}{k^\beta} \le Y_1 \le \frac{\varepsilon_4}{k^\beta}\right\}, \tag{35}$$

$$\Omega_C^2 = \left\{(k,Y)\colon k\in\mathbb{N}(k_0),\;\; Y_1 = \frac{\varepsilon_4}{k^\beta}\,,\;\; -\frac{\varepsilon_1}{k^\gamma} \le Y_0 \le \frac{\varepsilon_2}{k^\gamma}\right\}. \tag{36}$$

4.1 Estimation of functions F_1 and F_2 - inequalities (13) and (14)

To apply Lemma 2, inequalites (13) and (14) must hold. Since inequality (13) assumes $(k,Y)\in\Omega_B^i$, $i=1,\dots,n$ and inequality (14) assumes $(k,Y)\in\Omega_C^i$, $i=1,\dots,n$, we need to verify (taking into account specifications (33)-(36)) the following:

$$F_1(k,b_1(k),Y_1) = F_1\left(k,-\frac{\varepsilon_1}{k^\gamma},\quad Y_1\right) < b_1(k+1)-b_1(k) = -\frac{\varepsilon_1}{(k+1)^\gamma}+\frac{\varepsilon_1}{k^\gamma}, \tag{37}$$

$$F_1(k,c_1(k),Y_1) = F_1\left(k,\quad \frac{\varepsilon_2}{k^\gamma},\quad Y_1\right) > c_1(k+1)-c_1(k) = \frac{\varepsilon_2}{(k+1)^\gamma}-\frac{\varepsilon_2}{k^\gamma}, \tag{38}$$

$$F_2(k,Y_0,b_2(k)) = F_2\left(k,\quad Y_0,-\frac{\varepsilon_3}{k^\beta}\right) < b_2(k+1)-b_2(k) = -\frac{\varepsilon_3}{(k+1)^\beta}+\frac{\varepsilon_3}{k^\beta}, \tag{39}$$

$$F_2(k,Y_0,c_2(k)) = F_2\left(k,\quad Y_0,\quad \frac{\varepsilon_4}{k^\beta}\right) > c_2(k+1)-c_2(k) = \frac{\varepsilon_4}{(k+1)^\beta}-\frac{\varepsilon_4}{k^\beta} \tag{40}$$

whenever

$$-\frac{\varepsilon_3}{k^\beta} \le Y_1 \le \frac{\varepsilon_4}{k^\beta} \tag{41}$$

in (37), (38) and

$$-\frac{\varepsilon_1}{k^\gamma} \le Y_0 \le \frac{\varepsilon_2}{k^\gamma} \tag{42}$$

in (39), (40). Ranges (41) for Y_1 and (42) for Y_0 we will apply below without any special comment.

4.1.1 Verification of inequality (37)

In this case we can estimate $F_1(k,b_1(k),Y_1)$ as follows:

$$\begin{aligned}
F_1(k,b_1(k),Y_1) &= \left(-\frac{s+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\frac{\varepsilon_1}{k^\gamma}+Y_1(k)\right)\\
&< \max F_1(k,b_1(k),Y_1) = \left(-\frac{s+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\frac{\varepsilon_1}{k^\gamma}+\frac{\varepsilon_4}{k^\beta}\right)\\
&< b_1(k+1)-b_1(k) = -\frac{\varepsilon_1}{(k+1)^\gamma}+\frac{\varepsilon_1}{k^\gamma} = \frac{\varepsilon_1\gamma}{k^{\gamma+1}}\left(1+O\left(\frac{1}{k}\right)\right).
\end{aligned}$$

The latter inequality will hold if

$$\frac{\varepsilon_4}{k^\beta} < \frac{\varepsilon_1}{k^\gamma}\frac{\gamma+s+1}{-(s+1)}\,. \tag{43}$$

Inequality (43) holds if either

$$\beta > \gamma > -(s+1) \tag{44}$$

or

$$\gamma = \beta, \ \ \varepsilon_4 < \varepsilon_1 \frac{\gamma + s + 1}{-(s+1)} \,. \tag{45}$$

4.1.2 Verification of inequality (38)

In this case we can estimate $F_1(k, c_1(k), Y_1)$ as follows:

$$\begin{aligned} F_1(k, c_1(k), Y_1) &= \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-\frac{\varepsilon_2}{k^\gamma} + Y_1(k)\right) \\ &> \min F_1(k, c_1(k), Y_1) = \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-\frac{\varepsilon_2}{k^\gamma} - \frac{\varepsilon_3}{k^\beta}\right) \\ &> c_1(k+1) - c_1(k) = \frac{\varepsilon_2}{(k+1)^\gamma} - \frac{\varepsilon_2}{k^\gamma} = -\frac{\varepsilon_2 \gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned}$$

The latter inequality will hold if

$$\frac{\varepsilon_3}{k^\beta} < \frac{\varepsilon_2}{k^\gamma} \frac{\gamma + s + 1}{-(s+1)} \,. \tag{46}$$

Inequality (46) holds if either

$$\beta > \gamma > -(s+1) \tag{47}$$

or

$$\gamma = \beta, \ \ \varepsilon_3 < \varepsilon_2 \frac{\gamma + s + 1}{-(s+1)} \,. \tag{48}$$

4.1.3 Verification of inequality (39)

In this case we can estimate $F_2(k, Y_0, b_2(k))$ as follows:

$$\begin{aligned} F_2(k, Y_0, b_2(k)) &= \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2} Y_0(k) + \frac{\varepsilon_3}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\ &< \max F_2(k, Y_0, b_2(k) = \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-\frac{ms}{s+2}\frac{\varepsilon_1}{k^\gamma} + \frac{\varepsilon_3}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\ &< b_2(k+1) - b_2(k) = -\frac{\varepsilon_3}{(k+1)^\beta} + \frac{\varepsilon_3}{k^\beta} = \frac{\varepsilon_3 \beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned}$$

The latter inequality will hold if

$$\frac{\varepsilon_1}{k^\gamma} < \frac{\varepsilon_3}{k^\beta} \frac{2(\beta + s + 2)}{ms} \,. \tag{49}$$

Inequality (49) holds if either

$$\beta > \gamma \tag{50}$$

or

$$\gamma = \beta, \ \ \varepsilon_1 < \varepsilon_3 \frac{\beta + s + 2}{ms} \,. \tag{51}$$

4.1.4 Verification of inequality (40)

In this case we can estimate $F_2(k, Y_0, c_2(k)$ as follows:

$$\begin{aligned}
F_2(k, Y_0, c_2(k) &= \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k) - \frac{\varepsilon_4}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\
&> \min F_2(k, Y_0, c_2(k) = \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{\varepsilon_2}{k^\gamma} - \frac{\varepsilon_4}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\
&> c_2(k+1) - c_2(k) = \frac{\varepsilon_4}{(k+1)^\beta} - \frac{\varepsilon_4}{k^\beta} = -\frac{\varepsilon_4 \beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right).
\end{aligned}$$

The latter inequality will hold if

$$\frac{\varepsilon_2}{k^\gamma} < \frac{\varepsilon_4}{k^\beta}\frac{\beta+s+2}{ms}\,. \tag{52}$$

Inequality (52) holds if either

$$\gamma > \beta \tag{53}$$

or

$$\gamma = \beta, \;\; \varepsilon_2 < \varepsilon_4\frac{\beta+s+2}{ms}\,. \tag{54}$$

4.1.5 Summary of partial constraints

In this part we summarize all constraints involved in parts 4.1.1–4.1.4. First, inequality (53) contradicts inequalities (44), (47) and (50). Therefore, only values $\beta = \gamma$ are admissible. Analyzing inequalities (45), (48), (51) and (54), we derive inequalities (26)–(29), that is

$$\varepsilon_4 < \varepsilon_1\frac{\gamma+s+1}{-(s+1)}\,,$$

$$\varepsilon_3 < \varepsilon_2\frac{\gamma+s+1}{-(s+1)}\,,$$

$$\varepsilon_1 < \varepsilon_3\frac{\gamma+s+2}{ms}\,,$$

$$\varepsilon_2 < \varepsilon_4\frac{\gamma+s+2}{ms}\,.$$

All assumptions of Lemma 2 are fulfiled and, therefore, there exists a solution

$$Y = Y(k) = (Y_1(k), Y_2(k))^T$$

of system (24), (25) satisfying the inequalities

$$b_i(k) < Y_{i-1}(k) < c_i(k), \;\; i = 1, 2$$

for every $k \in \mathbb{N}(k_0)$, that is, by (41), (42),

$$-\frac{\varepsilon_3}{k^\beta} \le Y_1(k) \le \frac{\varepsilon_4}{k^\beta}\,, \tag{55}$$

$$-\frac{\varepsilon_1}{k^\gamma} \le Y_0(k) \le \frac{\varepsilon_2}{k^\gamma} \tag{56}$$

for every $k \in \mathbb{N}(k_0)$. We conclude, by equations (16), (17), (18) and (21), that there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1) such that, for every $k \in \mathbb{N}(k_0)$,

$$\left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} = Y_0(k), \tag{57}$$

$$\left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} = Y_1(k), \tag{58}$$

$$\left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) + \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\right]^{-1} = Y_2(k), \tag{59}$$

$$Y_2(k) = \frac{ms}{s+2} Y_0(k) + O\left(\frac{1}{k}\right). \tag{60}$$

From (55)–(60) inequalities (30)–(32) follow. The theorem is proved. □

Remark 1. *Theorem 1 and inequality* (30)*, that is,*

$$-\frac{\varepsilon_3}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_4}{k^\gamma}$$

obviously imply

$$-\frac{\varepsilon_3 b}{k^{\gamma+s+1}} < u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}} < \frac{\varepsilon_4 b}{k^{\gamma+s+1}}.$$

Therefore, for $k \to \infty$,

$$u(k) \sim \frac{a}{k^s}$$

and (5) *holds, or*

$$u(k) = \frac{a}{k^s} + \frac{b}{k^{s+1}} + O\left(\frac{1}{k^{s+1+\gamma}}\right)$$

and (6) *holds as well.*

Example 1. *Consider equation of the type* (1) *where $\alpha = -1/4$, $m = -1/6$ and $\sigma(k) = k^{-3}$, that is, the equation*

$$\Delta^2 u(k) \pm \frac{1}{k^{1/4}} u^{-1/6}(k) = k^{-3}. \tag{61}$$

Then, by formula (3)*,*

$$s = \frac{\alpha + 2}{m - 1} = \frac{(-1/4) + 2}{(-1/6) - 1} = -\frac{3}{2},$$

by formula (4)*,*

$$a = [\mp s(s+1)]^{1/(m-1)} = [\mp(-3/2)((-3/2)+1)]^{1/((-1/6)-1)} = \frac{1}{(\mp 3/4)^{6/7}}$$

and, by formula (7)*,*

$$b = \frac{as(s+2)}{s+2-ms} = \frac{(1/(\pm 3/4)^{6/7})(-3/2)((-3/2)+2)}{(-3/2)+2-(-1/6)(-3/2)} = -\frac{3}{(\pm 3/4)^{6/7}}.$$

Set $\gamma = 3/4$, $\varepsilon_1 = \varepsilon_2 = 4$ and $\varepsilon_3 = \varepsilon_4 = 1$. Then all conditions of Theorem 1 are fulfilled, since

$$1 = \varepsilon_4 < \varepsilon_1 \frac{\gamma + s + 1}{-(s+1)} = 4 \frac{(3/4) + (-3/2) + 1}{-((-3/2) + 1)} = 2$$

and inequality (26) *(as well as inequality* (27)*, which is the same in this case) holds,*

$$4 = \varepsilon_1 < \frac{(3/4) + (-3/2) + 2}{(-1/6)(-3/2)} = 5$$

and inequality (28) *(as well as inequality* (29)*, which is the same in this case) holds. Then, the equation* (61) *has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites* (30)–(32)*, that is,*

$$-\frac{1}{k^{3/4}} < \left[u(k) \pm \frac{1}{(3/4)^{6/7}} k^{3/2} \mp \frac{3}{(3/4)^{6/7}} k^{1/2}\right] \left[\pm \frac{3}{(3/4)^{6/7}} k^{1/2}\right]^{-1} < \frac{1}{k^{3/4}}\,, \tag{62}$$

$$-\frac{4}{k^{3/4}} < \left[\Delta u(k) \pm \frac{1}{(3/4)^{6/7}} \Delta\left(k^{3/2}\right) \mp \Delta\left(\frac{3}{(3/4)^{6/7}} k^{1/2}\right)\right] \left[\Delta\left(\pm \frac{3}{(3/4)^{6/7}} k^{1/2}\right)\right]^{-1} < \frac{4}{k^{3/4}}\,, \tag{63}$$

$$-\frac{1}{k^{3/4}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \frac{1}{(3/4)^{6/7}} \Delta^2\left(k^{3/2}\right) \mp \Delta^2\left(\frac{3}{(3/4)^{6/7}} k^{1/2}\right)\right] \left[\Delta^2\left(-\frac{3}{(\mp 3/4)^{6/7}} k^{1/2}\right)\frac{1}{2}\right]^{-1} < \frac{1}{k^{3/4}} + O\left(\frac{1}{k}\right). \tag{64}$$

Using inequality (62)*, we can derive a simplified formula*

$$u(k) = \mp \left(\frac{4}{3}\right)^{6/7} \left(k^{3/2} - 3k^{1/2}\right) + O\left(\frac{1}{k^{1/4}}\right).$$

Since $\Delta k^{1/2} = O\left(k^{-1/2}\right)$, $\Delta^2 k^{1/2} = O\left(k^{-3/2}\right)$, from (63) *and* (64) *we derive*

$$\Delta u(k) = \mp \left(\frac{4}{3}\right)^{6/7} \Delta\left(k^{3/2} - 3k^{1/2}\right) + O\left(\frac{1}{k^{5/4}}\right)$$

and

$$\Delta^2 u(k) = \mp \left(\frac{4}{3}\right)^{6/7} \Delta^2\left(k^{3/2} - 3k^{1/2}\right) + O\left(\frac{1}{k^{9/4}}\right).$$

Open Problem 1. *Theorem 1 is, among others, formulated in terms of inequalities* (26)–(29) *between positive numbers ε_i, $i = 1, 2, 3, 4$. It is an open problem to find conditions for these inequalities to have a solution. This problem can also be formulated as follows. Let s, a and b be defined by formulas* (3), (4) *and* (7)*. Let, moreover, $-2 < s < -1$, $m < 0$ and let γ be a fixed constant such that $0 < \gamma < 1$. Find sufficient conditions for the solvability of inequalities* (26)–(29) *with respect to ε_i, $i = 1, 2, 3, 4$.*

CONCLUSION

This paper discusses the asymptotic behaviour of solutions to a class of perturbed discrete second-order Emden-Fowler type equations. Under certain assumptions it is proved that this equations has a power-type solution and this solution is asymptotically similar to a solution of the second-order differential Emden-Fowler type equation. Some preliminary results related to Emden-Fowler type equations were published in [9].

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LOCAL AUTOMORPHISMS OF CERTAIN TREE FRAGMENTS OF NEURAL NETWORKS

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Abstract: *In the structure of the most used artificial neural network - multilayer perceptron and functionality of artificial neuron, there is possibility using certain analogy with relations between descriptions of differential equations certain quality, in this paper is developed new insight into these subjects. In deeper view of point there is used some concepts of locale finite trees of fragments of artificial neural networks and further to research by mappings among these structures and their properties and their characteristics with description and modeling systems of investigated structures of time-varying artificial neurons including local automorphisms of trees of artificial neurons, which forms considered fragments. There are found numbers of local automorphisms of special fragments with the root and two layers.*

Keywords: Neural network, transposition hypergroups, linear ordinary differential operators, groups of neurons.

INTRODUCTION

A neuron called also as artificial or formal neuron is the basic stone of the mathematical model of any neural network. Its design and functionality are derived from observation of a biological neuron that is basic building block of biological neural networks (systems) which includes the brain, spinal cord and peripheral ganglia. In case of artificial neuron the information comes into the body of an artificial neuron via inputs that are weighted (i.e. each input can be individually multiplied with a weight). The body of an artificial neuron then sums the weighted inputs, bias and ”processes” the sum with a transfer function. At the end an artificial neuron passes the processed information via outputs.

Neuron activity can be described mathematically: Capturing signal and transmission in neurons, there is created potential P:

$$P = w_1 * x_1 + w_2 * x_2 + ... + x_n * w_n n$$

If the potential is sufficiently large, the neuron transmits a signal y:

$$y = 1, if\ P > w_0,\ otherwise\ y = 0.$$

The condition that $P > w_0$ can be overridden by activating function $f(P)$. The entire activity of neurons can then enroll in one mathematical relationship where w_0 is a negative number that represents the threshold that shall overcome potential. Formally, the transfer function can have a zero threshold and a neural boundary with the negative sign being understood as the weight, the so-called bias $w_0 = {}^{-}\theta$ of another formal input $x_0 = 1$ with a constant unit value. The value of the internal potential y_in,where

$$\mathbf{y\_in} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{w_i x_i}.$$

after reaching the value $\mathbf{b}(\mathbf{w0} = \mathbf{b} - \mathbf{bias})$ it invokes the output state $\mathbf{y}$ of the neuron $\mathbf{Y}$, axon pulse. Increasing the output values $y = y\_in$ when the potential value of $\mathbf{b}$ is given by the activating (transfer) function f. In general, a single-layer neural network is not capable of solving all tasks. Therefore the most commonly used type is a multilayer feedforward network with a back-propagation learning method. From paper focus is interesting the linear transfer function. The output of a linear transfer function is equal to its input:

$$a = n,$$

as illustrated in Figure.

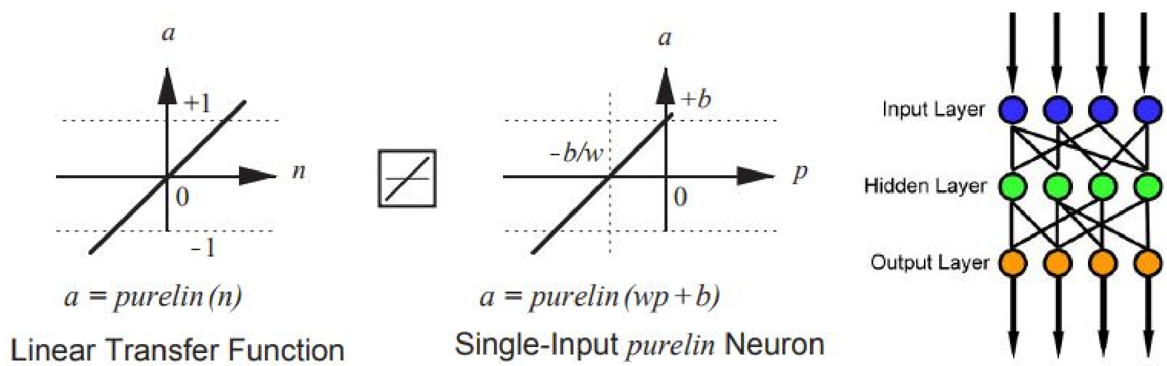

Figure 1: Feedforward neural network and linear transfer function.

Neurons with this transfer function are used in the ADALINE networks. The output neuron expression can be written in matrix form:

$$\mathbf{y} = \mathbf{W} * \mathbf{x} + \mathbf{b},$$

where the matrix for the single neuron case has only one row. Output and input product of artificial neurons can be the same way interpreted as vectors of input or output linear vector spaces. Now the neuron output of multilayer neural network can be written according terms of matrix form description as:

$$\mathbf{y^n} = \mathbf{f^n}(\mathbf{W^n f^{n-1}}(\mathbf{W^{n-1} f^{n-2}} ...(\mathbf{W^1 x} + \mathbf{b^1}) + \mathbf{b^2}) ... + \mathbf{b^n})$$

for n-layer neural network.

The beginning of the establishment of neural networks is considered to work Warren McCulloch and Walter Pitts [13] of 1943, which created a very simple mathematical model of a neuron, which is the basic cell of the nervous system [2]. The numerical values of the parameter in this model were predominantly bipolar, i.e. from the set $\{-1, 0, 1\}$. They showed that the simplest types of neural networks can in principle compute any arithmetic or logic function.

Artificial neural networks can be viewed as a weighted directed graphs in which artificial neurons are nodes and directed edges with weight are connections between neuron outputs and neuron inputs.

Recall that in the framework of Artificial neural networks perceptrons is a network of simple neurons called perceptrons. The basic concept of a single perceptron was introduced by Rosenblatt in the year 1958. The perceptron computes a single output from multiple real-valued inputs by forming a linear combination according to its input weights and then possibly putting the output through some nonlinear activation function. As usually mathematically this can be written as:

$$y = \varphi\left(\sum_{i=1}^{n} w_i x_i + b\right) = \varphi(\vec{w}^T \vec{x} + b),$$

where $\vec{w} = (w_1, \ldots, w_n)$ denotes the vector of weights, $\vec{x} = (x_1, \ldots, x_n)$ is the vector of inputs, b is the bias and φ is the activation function.

Other informations concerningneural networks can be found e.g. in [2, 3, 6, 9, 10, 11, 13, 16, 18, 19]

PRELIMINARIES

In the paper there are investigated locally finite trees and forests with regular, inverse, complete regular (and possessing another properties) semigroups (monoids in fact) of local automorphisms with respect to some modifications of the transitivity of the action of these semigroups on carrier sets of mentioned posets. Considerations are based on some results of L. A. Skornjakov [17] concerning endomorphism semigroups of monounary algebras which are close to mentioned questions.in the connection with generalized transitive actions.

Local isomorphisms or local automorphisms of locally finite trees introduced by Charles Wells [20] are useful transformations of finite trees of neurons which are fragments of considered neural nets. In this contribution, it is calculated their cardinal number$|\mathrm{LA}(T, \leq_T)|$ which allows to analyze the structure of monoids $\mathrm{LA}(T, \leq_T)$.

An isotone selfmap f of a locally finite forest $(T, \leq)$ is said to be a local automorphism of $(T, \leq)$ (cf. [20]) if for any pair of elements $s, t \in T$ such that $s < t$ the restriction $f|[s, t]$ is an order isomorphism of the interval $[s, t]$ onto the interval $[f(s), f(t)]$. The monoid of all local automorphisms of $(T, \leq)$ will be denoted by $\mathrm{LA}(T, \leq)$. The full transformation monoid of a set X (i.e. X^X endowed with the binary operation of the composition of mappings) is denoted by $M(X)$.

Basic notions from the algebraic theory of semigroups, which can be used as a powerful tool to investigate of algebraic structures based on time-varying neurons of perceptron, can be found in [5], [12] or [15]. Other investigations devoted to algebraic approach to this matter can be found in [1, 4, 7, 15, 17, 21].

MAIN RESULTS

In this contribution we consider finite rooted trees which serve as fragments of neural networks and derive the number of local automorphisms of considered trees. Let us begin with five-element trees.

Example. Let us denote $T = \{t_0, t_1, t_2, t_3, s\}$, i.e. $|T| = 5$ and define the ordering $\leq_T$ of the set T in this way:

$$\leq_T = \{[t_0, t_k]; k = 1, 2, 3\} \cup \{[t_0, t_3], [t_3, s], [t_0, s]\} \cup \{[t_k, t_k]; k = 0, 1, 2, 3\} \cup \{[s, s]\}.$$

Here the ordered pair $[t_i, t_j]$ means $t_i \leq_T t_j$ and to is the root of this tree. Evidently for each local automorphism $f \in \mathrm{LA}(T, \leq_T)$ we have

$$f(t_0) = t_0, f(t_3) = t_3 \text{ (because } t_3 \leq_T s) \text{ and } f(s) = s.$$

Further, the restrictions of transformation under local automorphism on the set $\{t_1, t_2, t_3\}$ are the following:

(i) $f(t_i) = t_i$ for $i = 1, 2, 3$,

(ii) $f(t_1) = t_1,\ f(t_2) = t_1,\ f(t_3) = t_3$,

(iii) $f(t_1) = t_2 = f(t_2),\ f(t_3) = t_3$,

(iv) $f(t_1) = t_2,\ f(t_2) = t_1,\ f(t_3) = t_3$,

(v) $f(t_1) = t_3 = f(t_3),\ f(t_2) = t_2$,

(vi) $f(t_1) = t_3 = f(t_3),\ f(t_2) = t_1$,

(vii) $f(t_i) = t_3$ for $i = 1, 2, 3$,

(viii) $f(t_1) = t_2,\ f(t_2) = t_3 = f(t_3)$,

(ix) $f(t_i) = t_i$ for $i = 1, 3$ and $f(t_2) = t_3$.

Thus the set $\mathrm{LA}(T, \leq_T)$ is formed by these transformations:

$$f_1 = id_T, \qquad f_2 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_1 & t_1 & t_3 & s \end{pmatrix},$$

$$f_3 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_2 & t_2 & t_3 & s \end{pmatrix}, \qquad f_4 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_2 & t_1 & t_3 & s \end{pmatrix},$$

$$f_5 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_3 & t_2 & t_3 & s \end{pmatrix}, \qquad f_6 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_3 & t_1 & t_3 & s \end{pmatrix},$$

$$f_7 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_3 & t_3 & t_3 & s \end{pmatrix}, \qquad f_8 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_2 & t_3 & t_3 & s \end{pmatrix},$$

$$f_9 = \begin{pmatrix} t_0 & t_1 & t_2 & t_3 & s \\ t_0 & t_1 & t_3 & t_3 & s \end{pmatrix}.$$

Hence $|\mathrm{LA}(T, \leq_T)| = 9$.

Another motivating example, which is leading to the final results is the following one.

Example. Suppose $T = \{t_0,\ t_1,\ t_2\} \cup \{s_k; k = 1, \cdots, m\},\ n \in \mathbb{N}$, i.e. $|T| = m + 3$ and define

$$\leq_T = \{[t_0, t_1], [t_0, t_2]\} \cup \{[t_0, s_k]; k = 1, \cdots, m\} \cup \{[t_2, s_k]; k = 1, \cdots, m\} \cup \{[t, t]; t \in T\}.$$

Here the root of the tree $(T, \leq_T)$ is t_0, the hidden layer is two/element $\{t_1, t_2\}$ and the output layer is $\{s_1, \cdots, s_m\}$. For any transformation $f \in \mathrm{LA}(T, \leq_T)$ we have $f(t) = t$, for $t \in \{t_0, t_2\}$ and either $f(t_1) = t_1$ or $f(t_1) = t_2$. Then we obtain $|\mathrm{LA}(T, \leq_T)| = m^m + m^m = 2m^m$.

Now we will consider the fragment of a neural network formed by the root t_0 and by two layers of neurons having n and m neurons. So, suppose $T_1 = \{t_1, \cdots, t_n\},\ T_2 = \{s_1, \cdots, s_m\}$ and

$$\leq_T = \{[t_0, t_k]; k = 0, 1, \cdots, n\} \cup \{[t_0, s_k]; k = 1, 2, \cdots, m\} \cup \{[t_1, s_k]; k = 1, 2, \cdots, m\} \cup$$

$$\cup \{[t_k, t_k]; k = 1, 2, \cdots, n\} \cup \{[s_k, s_k]; k = 1, 2, \cdots, m\}.$$

Now, for any local automorphisms $f \in \mathrm{LA}(T, \leq_T)$ there is $f(t_0) = t_0,\ f(t_1) = t_1$. Further, for any pair of neurons $t, u \in T_1,\ t \neq t_1$ we have $f(t) = u$ and for any pair of neurons $s, v \in T_2$ similarly $f(s) = v$. Denote

$$M_1 = \{f : T \to T; f \in \mathrm{LA}(T, \leq_T), s \in T_2 \Rightarrow f(s) = s\}.$$

Since $f(t_1) = t_1$, the set M_1 is equivalent to the set of all mappings of the set $T_1 \setminus \{t_1\}$ into the set T_1, thus

$$|M_1| = |T_1^{T_1 \setminus \{t_1\}}| = |T_1|^{|T_1 \setminus \{t_1\}|} = n^{n-1}.$$

Now denote

$$M_2 = \{f : T \to T; f \in \mathrm{LA}(T, \leq_T), t \in T_1 \Rightarrow f(t) = t\}.$$

Then similarly as above we have that sets $M_2,\ T_2^{T_2}$ are equivalent. Then

$$|M_2| = |T_2|^{|T_2|},$$

consequently $|\mathrm{LA}(T, \leq_T)| = m^m \cdot n^{n-1}$.

Remark 1. *If the both layers $T_1,\ T_2$ have the same number of neurons, say $|T_1| = |T_2| = n$, then, of course, $|\mathrm{LA}(T, \leq_T)| = n^{2n-1}$.*

Remark 2. *Let us apply the just derived formula on cases contained in the above examples.In the first example we have $n = 3,\ m = 1$ thus $|\mathrm{LA}(T, \leq_T)| = 1 \cdot 3^2 = 9$, in the second example $n = 2$, hence $|\mathrm{LA}(T, \leq_T)| = m^m \cdot 2^{2-1} = 2m^m$.*

CONCLUSION

The contribution is mainly based on investigations of properties of monoids of local automorphisms of locally finite trees, with respect to number of automorphisms among special fragments of feed-forward neural network. Their possible link with states fragments of neural network and at the same time reflection neural network in these algebraic structures is further way of research aimed on operations among systems of trees and their generalizations of artificial neural networks. Trees and forests are special ordered sets and authors are prepared to use some results concerning algebraic properties of these mentioned structures, in particular we are interested in their transformations. These ideas give us new insight to dynamical structure of perceptron and as that it can provide new view of point to functionality of these structures. Thus the presented contribution forms a base for further investigations.

Author Contributions: Contributions of both authors of this paper are equal.

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FEATURES OF THE LINEAR OPERATOR IN A PROCESS OF FALSE POSITIVES DETECTION IN WHITE MATTER TRACTOGRAPHY

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Abstract: *To date, tractography using magnetic resonance data is the only way how to map white matter structure in living human brain. However, current tractography algorithms carry a burden of high false positive detection rate. COMMIT algorithm reduces the implausible positive connections by incorporating microstructural information into the analysis. In this paper, features of the COMMIT formulation are explored and hypothesis for further exploration is proposed.*

Keywords: magnetic resonance, white matter, brain, tracts, tractography, COMMIT.

INTRODUCTION

Magnetic resonance imaging (MRI), concretely diffusion weighted magnetic resonance imaging, seems to be a promising method of brain white matter mapping. Since the white matter consist of axons (nerve fibres) and diffusion follows these fibres, we can get the information about structure by observing diffusion [1]. Information about direction of diffusion is collected from every voxel (elementary volume unit). It is important to keep in mind that MRI does not reach such a spatial resolution to capture diffusion direction in every single axon. However, axons tend to cluster into bundles where linear homogeneous structure is assumed, thus information about one prevailing direction may be roughly sufficient [1].

Process of connecting voxels with subsequent diffusion information is called tractography [1]. Many tractography algorithms were invented, but it becomes obvious that current tractography algorithms carry a burden of high false positive detection rate [2] since 60-90% of the voxels contain nonlinear conformation of fibers [3].

Modern approaches tend to minimize this problem by including extra information to the tracking algorithm. Such an information may be axon density, axon diameter, etc. Term established for such methods is microstructure informed tractography.

COMMIT algorithm [4] incorporate microstructural properties of the tissue into the analysis and detect false positive tracts. Task is solved by optimization technique. In this paper, features of the task are described and hypothesis for the further exploration is proposed.

1 METHODS

1.1 COMMIT task formulation

COMMIT [4], an acronym of Convex Optimization Modeling for Microstructure Informed Tractography, is a top-down tractography algorithm. The input is a set of candidate tracts and a diffusion data, output is a set of anatomically plausible tracts. The only necessary feature of the input tractogram is it contains true positive and possibly many false positive tracts. For easier imagination, such a tractogram may look like is shown in figure 1. Goal is to filter out false positives.

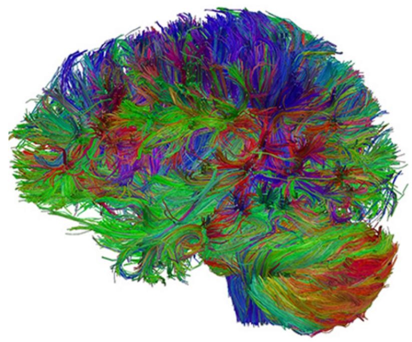

Figure 1: Set of candidate fibres, hence tractogram, of the brain white matter. Directionality of the tracts is color coded, red color is for sagital plane, blue for frontal plane and green for transversal plane [5]. Many of the tracts are probably false positive detections.

Core of the COMMIT is a global convex optimization problem.

In figure 2, we can see a principle of COMMIT algorithm. Acquired diffusion data is simplified by diffusion profile. Situation shown is illustrative. We can see two true positive tracts (blue and purple) and one false positive (red) tract in the tractogram.

From figure 2 we have:

- to obtain the first value of $\mathbf{y}$, the first two entries of $\mathbf{x}$ must be equal to 1, the rest to 0
- to obtain the second value of $\mathbf{y}$, the first entry of $\mathbf{x}$ must be equal to 1, the rest to 0
- to obtain the third value of $\mathbf{y}$, the second entry of $\mathbf{x}$ must be equal to 1, the rest to 0
- to obtain the fourth value of $\mathbf{y}$, the last entry of $\mathbf{x}$ must be equal to 1, the rest to 0

By this procedure we can detect that only the first (blue) and second (purple) tracts are correct and red tract is false positive.

In detail, COMMIT may be described by equation (2), $\mathbf{y} \in \mathbf{R_+}^{n_d n_v}$ is a vector containing n_d diffusion MRI data in all voxels n_v, η is acquisition noise and modeling error. $\mathbf{A} \in R^{n_d n_v \times n_c}$ is observation matrix, result of multi compartment modelling for each voxel, $\mathbf{x} \in R^{n_c}$ is a vector of positive weights, contributions of the η_c basis functions in $\mathbf{A}$ [4].

Matrix $\mathbf{A}$ is a block matrix, equation (1). Submatrix $\mathbf{A^{IC}}$ is a contribution of the signal from intraaxonal diffusion, each column refers to each tract in the set of candidate tracts. Every row refers to contribution of intraaxonal signal for one voxel and one diffusion gradient [4].

Submatrix $\mathbf{A^{EC}}$ contains data from extraaxonal signal contribution. It is possible to include only one extraaxonal compartment for every unique fiber population in each voxel [4].

Submatrix $\mathbf{A^{ISO}}$ contains data from isotropic contribution of the signal.

$$\mathbf{A}=\left[\mathbf{A}^{IC}\mathbf{A}^{EC}\mathbf{A}^{ISO}\right] \tag{1}$$

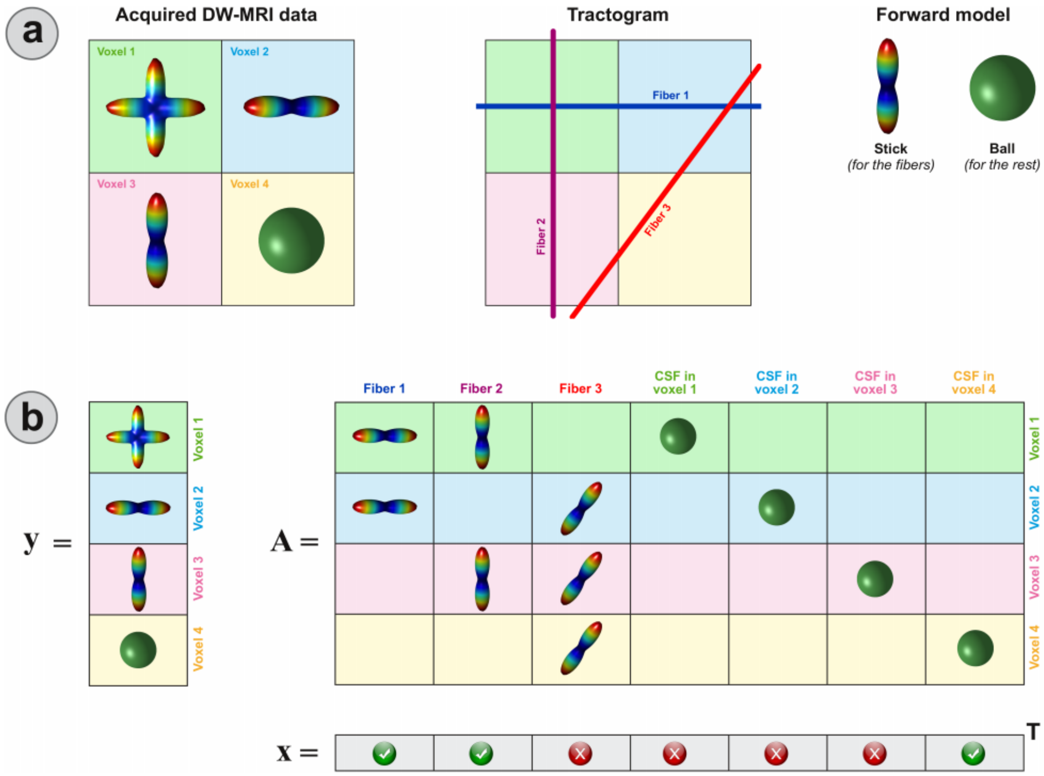

Figure 2: Simplified scheme of the COMMIT algorithm [6]

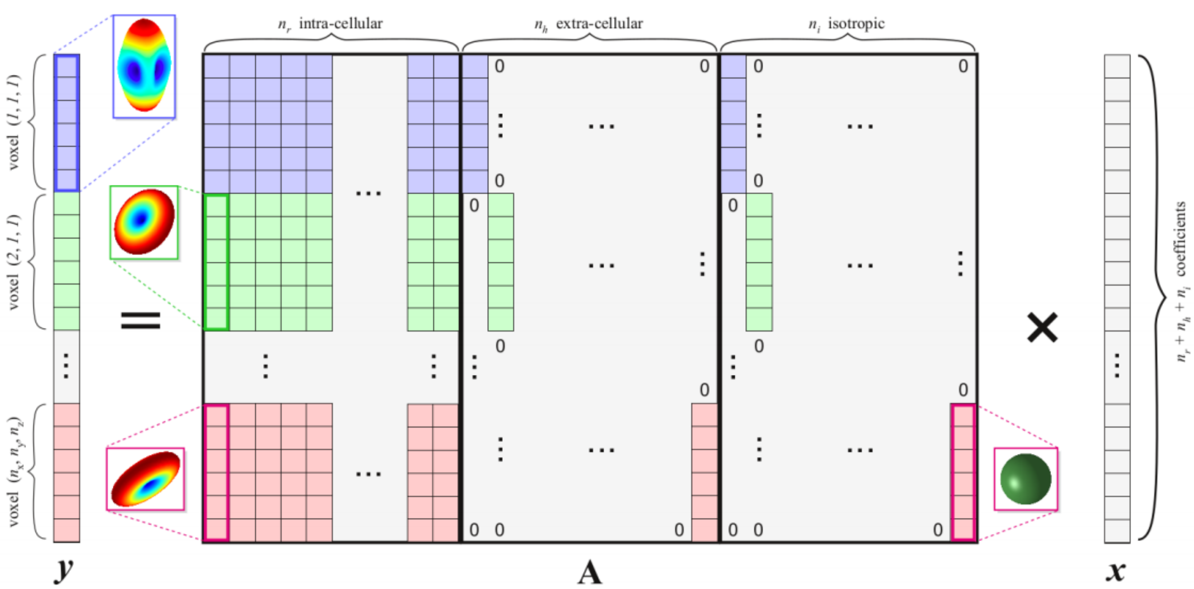

Figure 3: Schematic representation of the COMMIT [4].

$$y = \mathbf{A}\mathbf{x} + \eta \tag{2}$$

The goal is to find the optimal vector $\mathbf{x}$, which may be done by minimization, for example in a meaning of least squares, equation (3) [4]. $||\cdot||_2$ is a vector norm.

$$\arg\min_{\mathbf{x}\geq 0} ||\mathbf{A}\mathbf{x} - \mathbf{y}||_2^2 \tag{3}$$

In recent time, COMMIT2 was introduced. Even if the paper [6] is still in review, it shows surprising improvement of reducing false positive tracts. Idea behind is an injection of anatomical priors into the original COMMIT analysis by clustering bundles into groups. This paper may be understood as a motivation for exploring possibilities of COMMIT algorithm.

2 RESULTS

2.1 Features of the linear operator A

Even if COMMIT analysis is performed on a small dataset (compared to human brain data), the linear operator $\mathbf{A}$ reaches tremendous dimensions. Static storage of the data is problematic unless special approaches are used. However, since each tract crosses only few voxels within dataset, and submatrices for extraaxonal and isotropical compartments are diagonal, matrix $\mathbf{A}$ is sparse.

Another feature of the matrix is that it is poorly conditioned. Definition of condition number C_p of a $\mathbf{A}$ is defined as in equation (4), where $||\cdot||$ is a norm of a matrix [9].

$$C_p = ||\mathbf{A}||\ ||\mathbf{A^{-1}}|| \tag{4}$$

Matrices with high condition number are prone to roundoff errors, which may lead to nonsense solutions [8]. Example of the poorly conditioned system is in equations (5) and (6).

$$\begin{gathered} x + y = 2 \\ x + 1.0001y = 2.0001 \end{gathered} \tag{5}$$

$$\begin{gathered} x + y = 2 \\ x + 1.0001y = 2.0002 \end{gathered} \tag{6}$$

Solution of the system 5 is $x = 1$ and $y = 1$, solution of the system 6 is $x = 0$ and $y = 2$, condition number $C_p \approx= 4004$. Condition number of the matrix $\mathbf{A}$ is approximately 10^{30}.

Generally, matrix $\mathbf{A}$ is rectangular, the shape is dependent on acquisition scheme and modeling technique used.

2.2 Search for solution

Due to the size, shape and condition number of the matrix, all direct methods of solution are not applicable. For example as shown in [10], such a problem is not solvable by pseudoinversion [7].

The solution $\mathbf{x}$ from eqation $\mathbf{y} = \mathbf{A}\mathbf{x}$ may be achieved by pseudoinversion using Moore-Penrose pseudoinverse, labeled $\mathbf{A}^+$, equation (7).

$$\begin{gathered} \mathbf{x_p} = \mathbf{A}^+\mathbf{y} \\ \mathbf{x_p} = (\mathbf{A^T}\mathbf{A})^{-1}\mathbf{A^T}\mathbf{y} \end{gathered} \tag{7}$$

The solution vector $\mathbf{x_p}$ is a vector of approximate solution close to $\mathbf{x}$, equation (8).

$$\lim_{p\to\infty}(\mathbf{x}-\mathbf{x_p})=0 \tag{8}$$

2.3 Achieving new information about matrix A

Our hypothesis is that it could be possible to significantly reduce the dimensionality of the problem in case of using multicompartment model, equation (4).

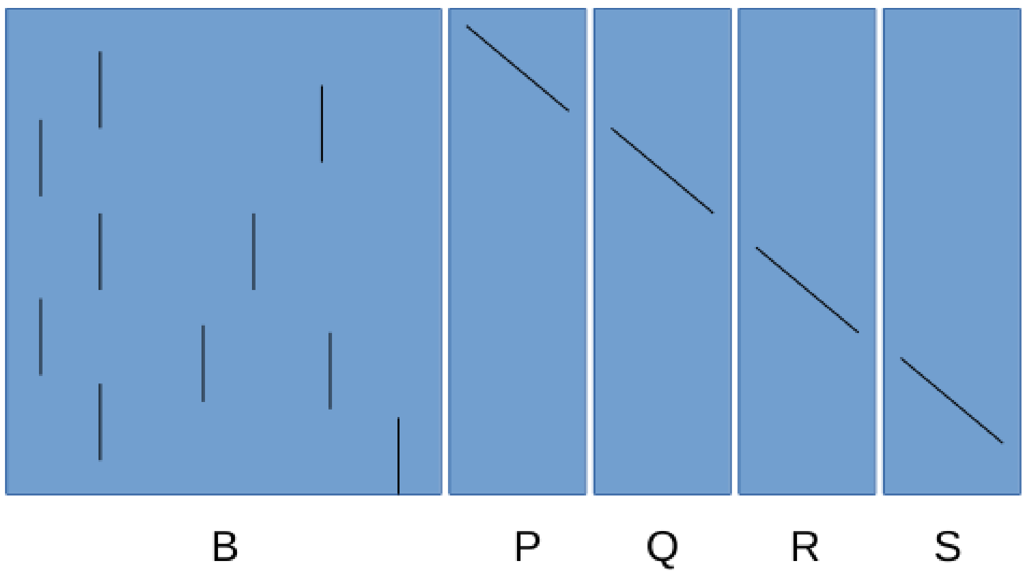

Figure 4: Visualization of the matrix A structure.

Our hypothesis is it could be possible to detect whether other than intraaxonal compartments carry crucial information for the analysis.

Lets divide matrix $\mathbf{A}$ into sumatrices B, P, Q, R, S. Then we can merge sumbatrix $\mathbf{B}$ with each other, respectively. Than we can compare created submatrices as follows in equation (9).

$$||\mathbf{B}\cup\mathbf{P}|| \doteq ||\mathbf{B}\cup\mathbf{Q}|| \doteq ||\mathbf{B}\cup\mathbf{R}|| \doteq ||\mathbf{B}\cup\mathbf{S}|| \tag{9}$$

If other than intraaxonal compartments (matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$) are not crucial, norms of the sumbatrices created by combination of intraaxonal and each other compartments may be approximately equal to each other. This could be a reason for dropping such data, hence reduction of the dimensionality and improvement of the behavior of the problem significantly, which may allow usage of various methods widely used in engineering, such as methods exploiting structure and sparsity of the linear operator [8].

CONCLUSION

In the paper, some features of the linear operator $\mathbf{a}$ within a COMMIT, specific tractography algorithm for brain white matter tractography, are explored. Hypothesis about properties of the linear operator $\mathbf{A}$ is proposed.f this hypothesis is true, it would be possible to simplify analysis and use commonly used methods of solution exploiting beneficial features of the problem.

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Capturing the Shape of Tusks of Wild Boars (Sus scropha L.) using different Types of Curves

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Abstract: *The paper aims to analyse the shape of wild boars tusks, which are apparently curved. The samples of tusks from hunted individuals were collected and compared with several models of spirals and the ellipse as a complementary curve. The fitting of the shape was solved via nonlinear optimization in SAS/STAT software using the trust region method. Except one, the rest of the samples did not form the whole screw-thread. In these cases, both the logarithmic spiral and the ellipse were chosen as the best fitting curves. One extra sample with almost whole screw-thread shows the logarithmic spiral as the best choice for fitting the shape of tusks.*

Keywords: tusks, wild boar, ellipse, logarithmic spiral, nonlinear optimization.

INTRODUCTION

Adult wild boars have 44 teeth in the complete dental formula and in the case of both sexes tend to produce distinctive canines (tusks) in both the upper and lower jaws (Fig. 1). Moreover, the permanent tusks of boars are significantly larger than those in sows. In males, the lower tusks usually grow to 20 cm, but in exceptional cases, they can grow to a length of 30 cm. Approximately two-thirds of the total length of the boar's lower tusks are contained within the tooth's socket in the lower jaw (Fig. 2). The uncovered part of the tusk, which is outside the jaw, is typically coloured. The lower tusks are locally called cutters, because these teeth tend to be very sharp at the tips, and as such, could potentially be used by the animal for cutting a rival or as a defence against a predator. The upper tusks are referred to as whetters, because these teeth primarily function to sharpen the lower tusks [9].

The shape of tusks is usually described as semicircular in the longitudinal direction and roughly trapezoidal in cross-section [9]. The analogously shaped structures can be found in animal and human anatomy (horns, seashells, and bones), as well as in botany (formation of leaves, flowers, fruit, and tree trunks). Many of these natural curves are described to be similar to the shape of the logarithmic spiral [2,3,8]. In numerous cases are these curves not only planar (2D) but space curves (3D). The logarithmic spirals also were used for the construction of some technical devices, such as cams, which rock climbers call friends. Their shape is based on the logarithmic spiral and allows securing ropes to cracks in a rock face and fits into a range of crack sizes [1].

In light of the above, the article aims to verify the presumption that the outer shape of arches made by the tusks of wild boars could be approximated by a part of a logarithmic spiral. At the same time, each data set will also be compared with the Archimedean and Fermat's spiral, and the ellipse.

The rest of this paper is organized as follows: Sec. 1 discusses the model curves used for optimization. In the next part, we describe both the methodology of data collection and the principles of shape optimization. In the section focused on the results, we evaluate the degree of conformity of individual data sets with model curves.

Fig. 1. The upper tusk (left) and the lower tusk (right) of wild boar
Source: own

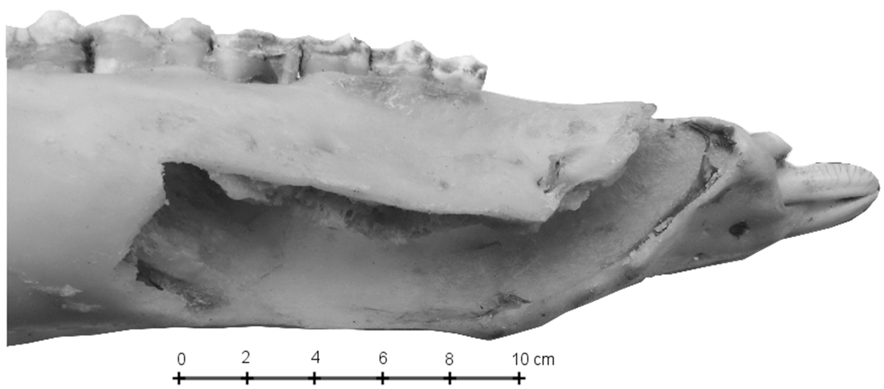

Fig. 2. The lower jaw of wild boar after tusk extraction with a well visible canal (the tooth's socket) in which approximately two-thirds of the total tooth length lies
Source: own

1 The model curves

Due to the computer modelling, the parametric equations for all investigated curves (logarithmic, Archimedean, and Fermat's spiral, and ellipse) were going to be used. The parametric equations of the spiral (logarithmic, Archimedean, and Fermat's) are

$$\begin{aligned} x &= x_0 + r(t)\cos(t), \\ y &= y_0 + r(t)\sin(t), \end{aligned} \tag{1}$$

where $t \geq 0$, and $r(t)$ is a continuous function. By an appropriate choice of function $r(t)$, we can obtain

- logarithmic spiral, where $r(t) = ae^{bt}$,

- Archimedean spiral, where $r(t) = at$,
- Fermat's spiral, where $r(t) = a\sqrt{t}$.

The parametric equations of the ellipse are

$$\begin{aligned} x &= x_0 + a\cos(t), \\ y &= y_0 + b\sin(t), \end{aligned} \tag{2}$$

where a, b are the lengths of the semi-axes of the ellipse.

2 Materials and methods

The assessed tusks came from the collections of the Czech Forestry Academy in Trutnov, and from the exhibits of the Regional Trophies Breeding Show held in Trutnov.

The photos of tusks with the scale applied were used to find the sequence of coordinates of the points located on the arches, which were a projection of the edge of the tusk on a planar. The angular distance of the observed points from the edge was approximately 5°. The data from abraded or damaged parts of tusks were omitted (Fig. 3). When comparing with the model of the curves under consideration (Geogebra, Fig. 4), the input values of the curve parameters (the coordinates of the curve centres, coefficient values, the lengths of the semi-axes etc.) were estimated for subsequent optimization via SAS/STAT software.

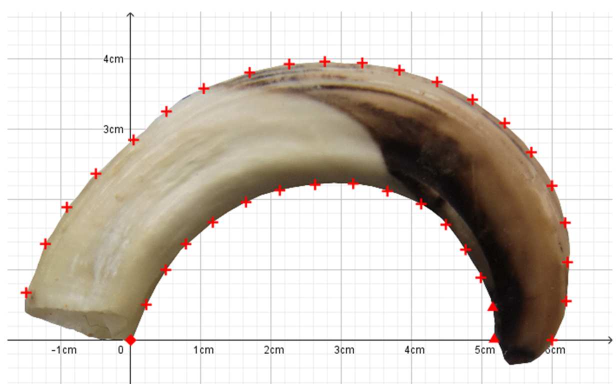

Fig. 3. Upper tusk of wild boar with the observed points
Source: own

The spirals (Archimedean, logarithmic, Fermat's), which best fit the observed data were looked for by minimizing functions that computed the sum of the squared distances (in the radial direction) between the observed data and the mathematical model of the curve. When fitting the ellipse, the SAS program was written down using the definition of this curve as a set of all points in a plane, where the sum of the two distances to the focal points is a constant. The function searched for the optimal length of the major axis of the ellipse was minimized.

The parameters of all models were estimated using the trust-region method (nonlinear optimization method, NLPTR call) in SAS/IML (SAS Institute). This procedure computes an optimal value of the function using the initial guess of parameters. The algorithm used for the calculations requires that the objective function F has continuous first- and second-order derivatives inside the feasible region (for more details see [10]). The observed data were consequently overlaid with the obtained mathematical models, both spirals and the ellipse.

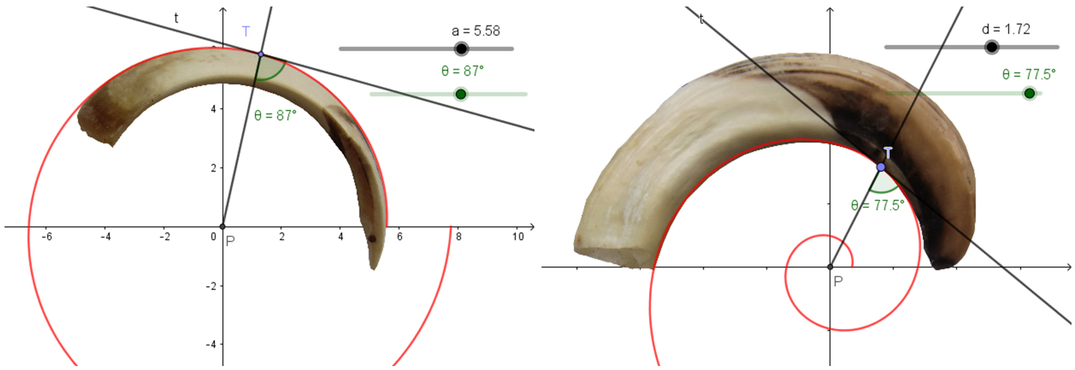

Fig. 4. The comparison of the shape of the lower tusk (left), and the upper tusk (right) with a logarithmic spiral (spirals created by Geogebra)
Source: own

2.1 Spirals

The objective function expressing the sum of the squared distances between the observed $[x_i, y_i]$ and predicted values $[u_i, v_i]$ along the arch is described with the equation [5,7]

$$F_{x_0,y_0,a,b} = \sum_{i=1}^{n} (u_i - x_i)^2 + (v_i - y_i)^2,$$

where

$$u_i = x_0 + r(t_i)\cos(t_i),$$
$$v_i = y_0 + r(t_i)\sin(t_i),$$

and $r(t)$ is a function with parameters a, b defining the type of spiral. The parameter b applies only to the logarithmic spiral. The values of t_i have been determined from the condition

$$\tan(t_i) = \frac{y_i - y_0}{x_i - x_0}.$$

2.2 Ellipse

Ellipse is a plane curve surrounding two focal points, where for all points on the curve is the sum of the two distances to the focal points constant. When fitting the ellipse, the objective function

$$F_{x_e,y_e,x_f,y_f,a} = \sum_{i=1}^{n} \left[\sqrt{(x_i - x_e)^2 + (y_i - y_e)^2} + \sqrt{(x_i - x_f)^2 + (y_i - y_f)^2} - 2a \right]^2 \tag{3}$$

was minimized.
With respect to the necessary condition for the existence of an extremum (minimum)

$$\frac{\partial F}{\partial a} = 0,$$

we can simplify the equation (3) and get

$$2a = \frac{1}{n} \sum_{i=1}^{n} \sqrt{(x_i - x_e)^2 + (y_i - y_e)^2} + \sqrt{(x_i - x_f)^2 + (y_i - y_f)^2}. \tag{4}$$

From the equation (4) implies that the optimization can be reduced to searching for the coordinates $[x_e, y_e]$, $[x_f, y_f]$ of both estimated focal points, which refer to the average value of the sum of the distances observed points to both focal points of the predicted ellipse. The coordinates of the ellipse in a normal position have been calculated according to the equation (2).
To investigate the compliance of the model and the observed data, the angle φ

$$\varphi = \arctan\left(-\frac{y_e - y_0}{x_e - x_0}\right),$$

formed by the x-axis and the major axis of the normal ellipse was calculated. Using the angle φ, the coordinates of the normal ellipse were transformed (Appendix A) into the coordinates of the general ellipse (Figure 6, Figure 8)

$$u_i = \frac{x_e + x_f}{2} + x_i \cos(\varphi) + y_i \sin(\varphi),$$
$$v_i = \frac{y_e + y_f}{2} + y_i \sin(\varphi) - x_i \cos(\varphi).$$

3 Results

In this study, in total 14 samples of cutters and 9 samples of whetters were investigated. One extra sample of the cutter was rated from the photo (provided by V. Badr). The reliability of the created models depended on an appropriate estimation of the input parameters of the optimized curves. At the same time, two initial values (lambda $\lambda = 0$, radius $\Delta = 1$) of the used optimization method TRM were chosen (for more details see [10]). In the case of optimization problems, the input values of the estimated curve parameters were adjusted.

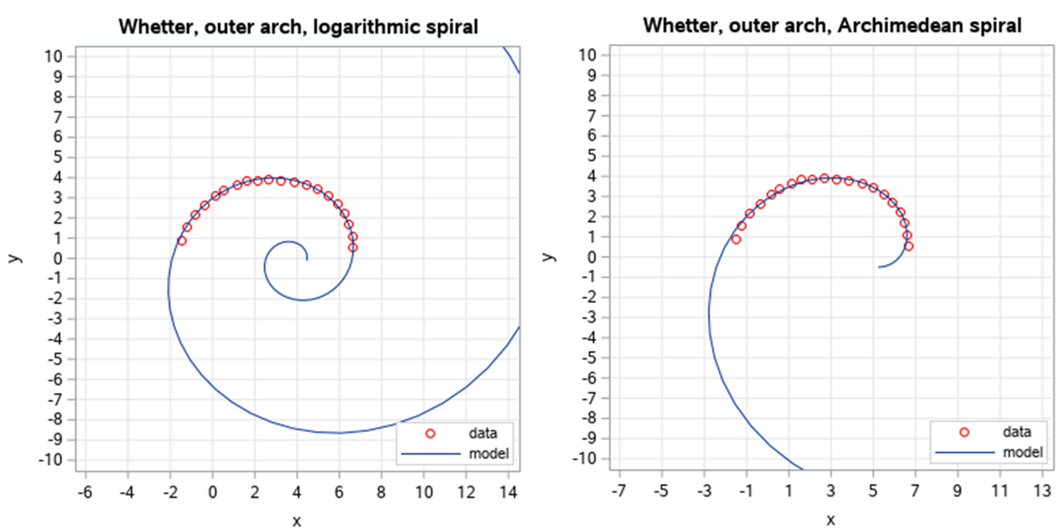

Fig. 5. Fitting results to the upper tusk (whetter), logarithmic spiral (left), Archimedean spiral (right)
Source: own

The tusks of the upper jaw are shorter and more curved. When fitting their shape with the logarithmic spiral (Figure 1, Figure 5, Figure 6), the magnitudes of the pitch angles ranged from ca

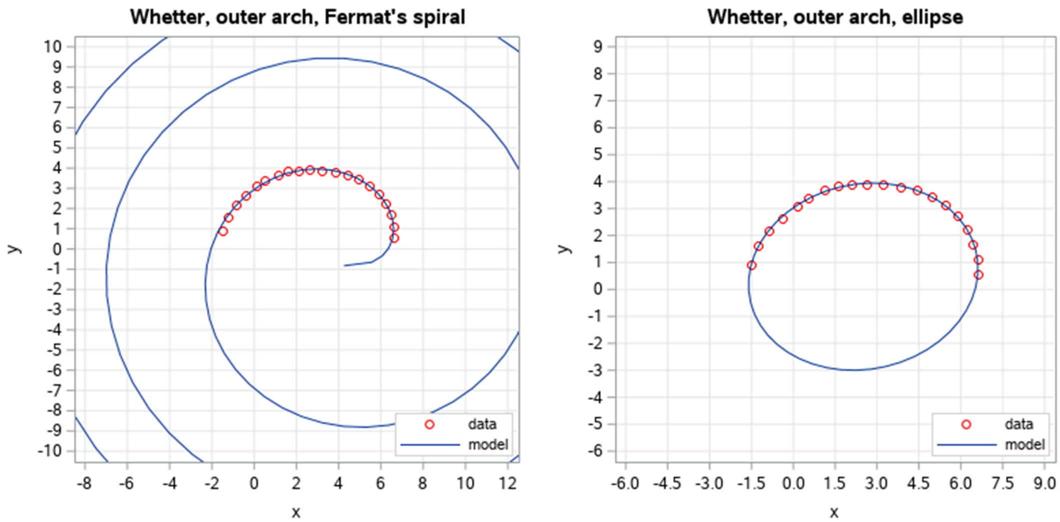

Fig. 6. Fitting results to the upper tusk (whetter), Fermat's spiral (left), ellipse (right)
Source: own

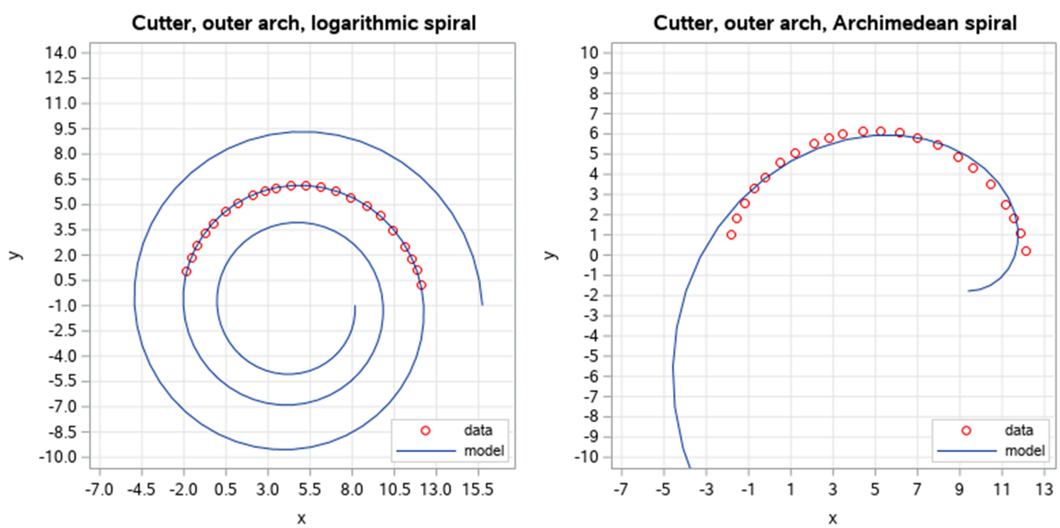

Fig. 7. Fitting results to the lower tusk (cutter), logarithmic spiral (left), Archimedean spiral (right)
Source: own

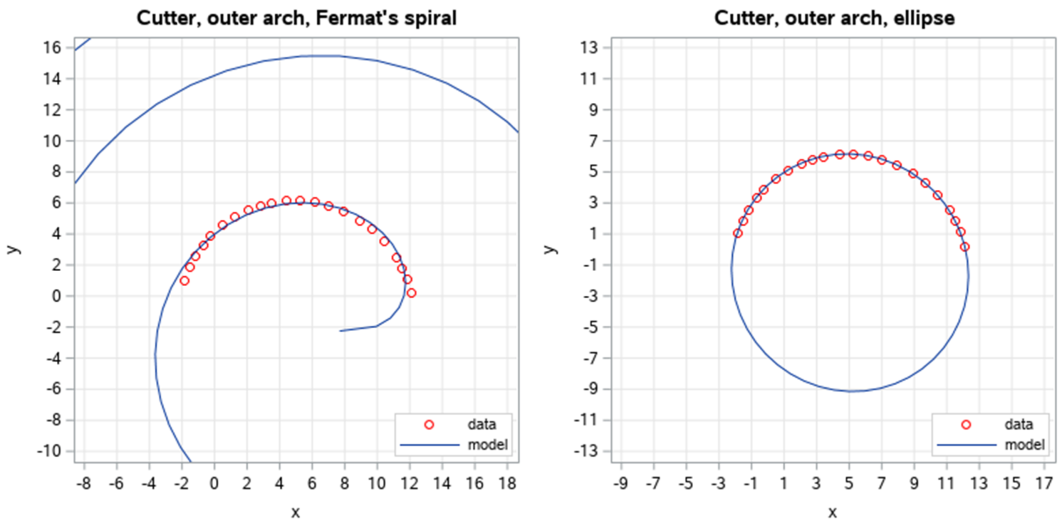

Fig. 8. Fitting results to the lower tusk (cutter), Fermat's spiral (left), ellipse (right)
Source: own

71° to 86° . The tusks of the lower jaw (cutters, Figure 1, Figure 7, Figure 8) have significantly less curvature, the calculated pitch angles were in the range of approximately 80° to 90°. The whetters have a greater numerical eccentricity (0.33–0.67) than the cutters (0.15–0.48).

Supplement

SAS programs for fitting the ellipse and spirals can be found online at:
https://www.claTrutnov.cz/index.php/cs/skola/dokumenty/category/78-spichal-ludek-mgr

CONCLUSION

The investigated examples of tusks have been modeled using three types of spirals and the ellipse. The best conformity between the data and models was observed in both the logarithmic spiral (Figure 5, Figure 7) and the ellipse (Figure 6, Figure 8), the other spirals (Archimedean, Fermat's) did not provide a good match with the data sets when fitting (Figure 5, Figure 6, Figure 7, Figure 8). However, due to the limited span of the arches of the tusks under consideration (except the sample in Figure 9), it is not possible to reliably distinguish which curve is closer to reality.

In fact, it is quite difficult to obtain samples of tusks forming more than one screw-thread, even when searching on the internet. Moreover, such a shape can be considered as an abnormality caused due to the lack of sufficient grinding wear between the upper and lower tusk.[1] On the other hand, it can be shown on isolated available examples that the logarithmic spiral is the best fitting choice (Fig. 9).

If we accept the hypothesis that the shape of tusks corresponds to the logarithmic spiral, then the associated presence of a constant pitch angle could in a certain way be related to the mechanical properties of long teeth (e.g. stiffness). This assumption can be compared, e.g., with the geometry of human ribs where the pattern of the logarithmic spiral was also found [3].

[1] https://feralhogs.extension.org/feral-hog-tusk-characteristics/

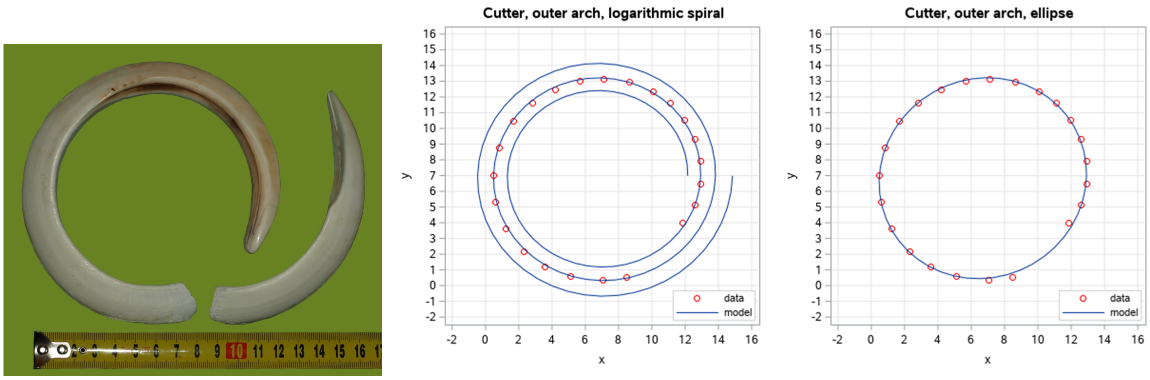

Fig. 9. The lower tusk and fitting results
Source: photo (V. Badr), fitting results (own)

Acknowledgement: I would like to thank Mr V. Badr for providing a photo in Figure 9.

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Appendix A

Transformation of the ellipse coordinates

The ellipse points $[x, y]$ in the normal position form with the x-axis angle φ (Fig. 10)

$$x = r\cos\varphi,$$
$$y = r\sin\varphi.$$

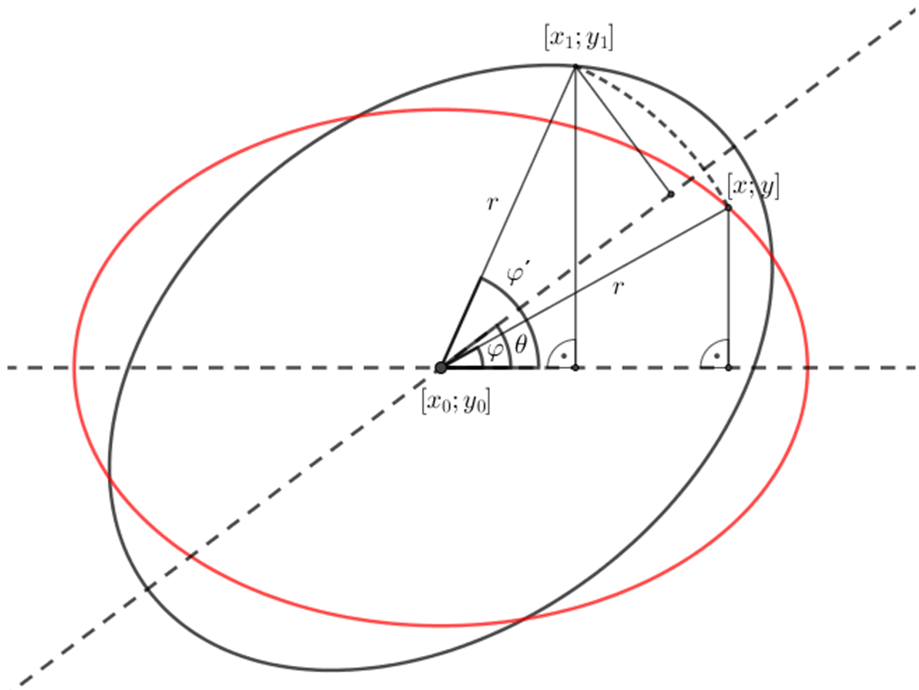

Fig. 10. The transformation of the ellipse coordinates
Source: own

The ellipse points $[x_1, y_1]$ in the general position form with the x-axis angle $\varphi´$

$$x_1 = r\cos\varphi´ \to \cos\varphi´ = \frac{x_1}{r},$$
$$y_1 = r\sin\varphi´ \to \sin\varphi´ = \frac{y_1}{r}.$$

Suppose that the major axis of the general ellipse forms an angle ϑ (the angle of rotation of the major axis of the general ellipse) with the x-axis

$$\vartheta = \varphi´ - \varphi \to \varphi = \varphi´ - \vartheta,$$

then the normal ellipse coordinates can be transformed using the equations

$$\begin{aligned} x &= r\cos(\varphi´ - \vartheta) = r(\cos\varphi´\cos\vartheta + \sin\varphi´\sin\vartheta) = \\ &= r\left(\frac{x_1}{r}\cos\vartheta + \frac{y_1}{r}\sin\vartheta\right) = x_1\cos\vartheta + y_1\sin\vartheta, \\ y &= r\sin(\varphi´ - \vartheta) = r(\sin\varphi´\cos\vartheta - \sin\vartheta\cos\varphi´) = \\ &= r\left(\frac{y_1}{r}\cos\vartheta - \frac{x_1}{r}\sin\vartheta\right) = y_1\cos\vartheta - x_1\sin\vartheta. \end{aligned}$$

EDUCATIONAL DATA MINING TECHNIQUES FOR AN ELECTRONIC COURSE: A CASE STUDY

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Abstract: *The paper presents a practical experience of detecting students' educational strategies within an electronic course in the LMS Moodle. In the study, different types of analytic tools for this purpose were used - standard Moodle reports (Logs, Activity report, Statistics), external tools for analysing data directly from Moodle (KEATS analytics), and other multifunctional data analysis programs (Excel, Weka, Deductor). These instruments and the educational data mining techniques helped to clarify students' online behaviour and to identify typical groups of students in terms of information preferences.*

Keywords: educational data mining, e-learning, digital learning environment, electronic course, students, information preferences, learning behaviour, educational strategies, LMS Moodle

INTRODUCTION

Decision making in the field of education is a complex multi-faceted process with a large circle of stakeholders involved. An important criterion for making an effective decision is the analysis of information received from participants of the educational process at its various stages (Angeli et al, 2017). Digitalisation of education has led to the fact that various training systems and environments accumulate today a large number of educational data. The examples of data sources are students' behaviour and performance during classes, test and survey results, data stored in the log files of a Learning Management System (LMS) of online courses, thematic forums, and others. A new challenge is to implement a high-quality educational process in the digital learning environment, it is necessary to be able to analyse the accumulated information about the learning process and make justified adjustments on its basis. Methods of data mining allow us to analyse data at a new level of effectiveness. A separate scientific direction known as Educational Data Mining (EDM) was highlighted in 2008. This multidisciplinary direction developing with applied statistics, artificial intelligence, and educational technologies, accumulates and develops new methods for the analysis and solution of educational problems.

In the modern educational system, blended learning takes an increasingly strong place when the traditional and remote and forms of learning are merging, and the possibilities for distance education are expanding. The number of universities that offer e-learning is growing; the number of different distance learning courses is rapidly increasing; digital platforms are actively developing (such as Coursera, EdX, Open Education, etc.). These platforms are providing free access to lectures and courses of various institutions. Blended learning is often supported through LMS (Moodle, Blackboard, E-stage, etc.). In this regard, an urgent task is to study the analytics capabilities of these digital platforms and LMSs, to help teachers in identifying various patterns of student learning behaviour in the digital environment.

The main objective of the case study presented in the paper is to identify opportunities for the available data mining techniques that could contribute to the identification of students' learning behaviour patterns. Several research questions arise in this context. What are the LMS Moodle capabilities for the educational data mining? What features of learning behaviour can be most clearly determined with the selected techniques? What grounds for improving the course can a teacher find in the obtained data?

1 EDUCATIONAL DATA MINING FOR PEDAGOGICAL DECISION SUPPORT IN THE DIGITAL LEARNING ENVIRONMENT

1.1 Educational data mining for personalised learning

Today one of the relevant teachers' objectives is the personalisation of learning (Han 2020). From this perspective, several areas of educational data mining can be proposed (Romero 2010, Aldowah 2019) (Table 1).

Table 1 - Examples of educational data mining in the digital environment

Research objective	Educational data mining tools
1 - Providing feedback to support teachers	
• Organising course materials, assigning homework of various difficulty levels, evaluating the effectiveness of online courses; • Extracting opinions from web pages of the LMS, analysing reviews in a natural language; • Extracting the main correlations from the score matrix to create an extended final test.	Clustering algorithms, classifications, associative rules, analysis of sequential rules, dependency modeling, correlation mining, text mining.
2 - Developing an individual educational route	
• Providing recommendations (which next task to solve, which links to view, etc.) for each specific user by personal activity to adapt the learning content, interface, and sequence of actions; • Providing recommendations on the most expected links that a student should visit in a web-adaptive educational system; • Providing recommendations on the content acquisition and online discussions participation based on the analysis of tests results; • Identifying sequences of activities that indicate problems or successes of students to assist and early recognition of students' difficulties; • Developing a model of recommendations for students in similar situations with the use of clustering and a method of collective filtering.	Search for sequential patterns, associative rules, clustering, collective filtering.
3 - Predicting students' performance	
• Evaluating an unknown variable that describes a student (e.g., a point or grade, academic performance, knowledge); • Researching students' self-regulated learning behaviors such as goal setting and monitoring, identifying successful self-learning strategies;	Regression, correlation analysis, classification, Bayess networks, neural networks, outlier detection

• Forecasting results of exams at the end of the year through the activity of students within the LMS, predicting mistakes made by students, as well as revealing candidates for admission to the university; • Classifying students: low risk of failure, medium risk, and high risk.	

1.2 LMS Moodle capabilities for the educational data mining

We analysed available tools for educational data mining based on records of students' activities in the LMS Moodle and divided all existing capabilities into four categories:

1. Standard system tools,
2. Additional plugins for the system,
3. External tools for analysing data directly from Moodle,
4. Various multifunctional data analysis programs.

The first group includes several standard Moodle reports (competency breakdown report, logs, participation report, activity report, and statistics) and for last system versions - learning analytics built-in models [Moodle Analytics]. These tools allow collecting data on users' activity, identifying the most and the least popular elements of an e-course, accumulating information about views of the course items by each student, and get predictions about students at risk of dropping out or upcoming activities.

The second group includes plugins from both Moodle developers and third-party researchers. Plugins represent an extension of the Moodle functionality, which is not available in the basic version. The catalogue currently contains 1677 plugins. One of the examples of plugins for data analysis is Inspire Analytics. Among the capabilities of this plugin is predicting students risks of not completing the Moodle course because of the low engagement. One more example is SmartKlassTM Learning Analytics Moodle - the technology uses machine learning algorithms to build a dashboard with consolidated information about users with learning difficulties. The Heatmap tool represents a "heat map", which allows a visual tracking of users' activity within the course. Warm colours on the map correspond to the areas with frequent clicks, cold colours - to the areas with rare clicks.

The external tools group includes stand-alone Moodle data analysis solutions. For example, the Institute of Distance Education of Tomsk State University has developed a system for monitoring content and user activity in Moodle (Monitoring System), which is intended for technical stuff. Many existing Moodle data analysis solutions are based on additional analysis of event log data. The freely distributed KEATS analytics solution (Konstantinidis 2013), which is quite easy to use, is sufficient for analysing Moodle log files based on Excel macros and Visual Basic; to use this tool only Excel is required.

The group of various multifunctional data analysis programs requires specialised knowledge and skills - statistical methods, data mining methods, programming languages, data pre-processing. Examples of such tools are presented in Figure 1.

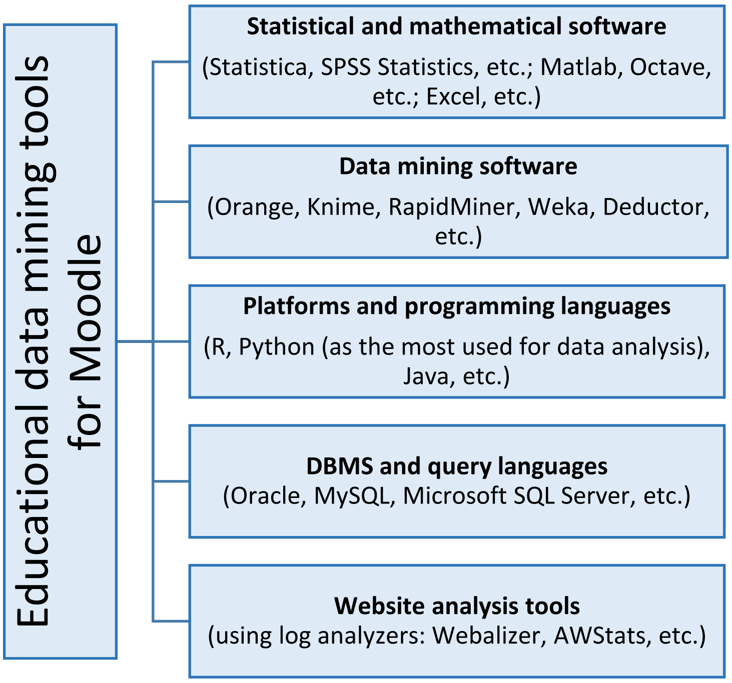

Fig. 1. Multifunctional data analysis programs
Source: own work

2 A PILOT STUDY OF THE LEARNING ANALYTICS CAPABILITIES FOR THE PEDAGOGICAL MONITORING OF AN E-COURSE

2.1 General characteristic of students' activity in the electronic course

The experimental work was carried out in the frame of the "Infocommunication Technology" course for the first-year bachelor students of the Herzen University. The course restarts every semester. 400 students studied this course in the first semester of the 2019-2020 academic year. Following the psychodidactic approach, as well as the typology of digital educational resources (Noskova, 2018, 2019 DIVAI) an electronic course should comprise three types of digital educational resources: digital content (information resources), digital resources for the organisation of interactions and communication (communication resources), and digital resources for the educational and cognitive activities management (management resources).

Information resources of the course were presented in the format of videos, collections of hyperlinks to open Internet resources, a series of problematic questions, assignments, methodological recommendations for completing assignments and assessment criteria, files with additional information, various graphics (visualisation and infographics). Thus, these resources made it possible to achieve several important learning objectives - acquiring new knowledge, accessing additional reference information, providing a practical component of training, and monitoring students' activities. Communication resources provided users' interactions in a discussion forum. Management resources included express polls, reflective questionnaires, and assignments.

The Moodle report "Statistics" shows that every semester students have similar educational behaviour and activity within the course (Fig. 2). During the first two academic months of a semester, students' activity increases. In the third month, it decreases and then it strongly increases in the fourth month. This trend can be explained both by the psychological features of the educational activity as a whole and by the features of the particular course design. From the course content and design point of view, at the beginning of the semester, activity growth is associated with the need to understand the logic of work in the course. Students register; create their websites as portfolios of future assignments. Besides, they have to understand how the course services function, for example, an electronic grade book, forums, assignments, reflexive pools, and teachers' feedback. By the middle of the semester, the logic of the work in the course becomes clear. Nevertheless, here the traditional risks of any prolonged training begin to appear - a decrease in activity, missed classes, etc. The increase of activity at the end of the semester is due to the upcoming credit - students must undertake a series of tests, and pass assignments for the autonomous work.

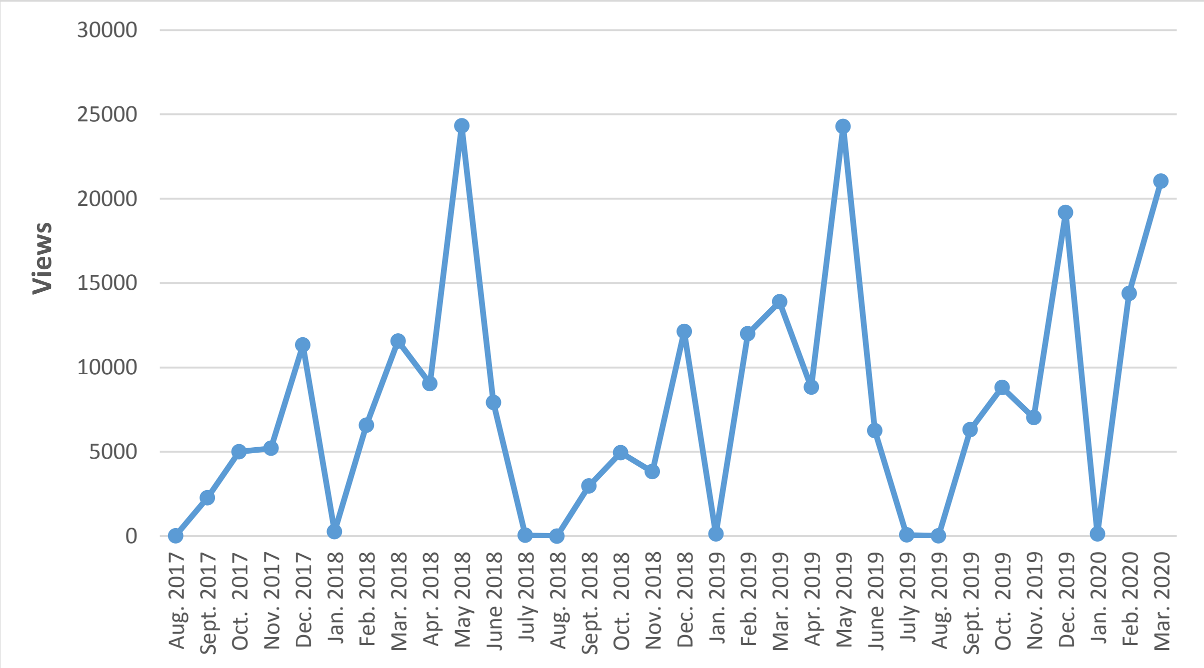

Fig. 2. Cyclic recurrence of activity in the e-learning course
Source: own work

A significant difference between the values reflecting the views and changes made by students is natural for this course since the changes reflect only the performance of tests and questionnaires. The main work on the implementation of their learning projects students carry out outside the LMS (e.g., Google Drive, mind mapping resources, online presentations, web search, etc.); this activity does not leave digital footprints in the LMS system and is not reflected in the activity graphs.

Course analysis with the use of Moodle participation reports proved that students' educational activity changed throughout the course. For example, the number of students who viewed the tasks of the 2nd and 3rd practical classes turned out to be 18% less than the number of students who watched the tasks to the 1st practical class. Then by the 4th practical class this number was restored almost to the initial level. This is explained by the fact that the 1st and the 4th

practical works suggested a greater variety of educational activity, while the 2nd and 3rd practical tasks had rather definite instructions. For example, at the 1st practical lesson, students were asked to get acquainted with the basic concepts, pass a test for self-control, choose a research topic, get acquainted with the evaluation criteria of the final project, and complete the task of preparing a portfolio. In the 4th practical lesson, students were asked to choose one of the tasks that involved the use of different digital tools. Accordingly, they paid more attention to the acquaintance with the contents of the tasks to decide on further actions.

Analysis of the time of student activities in the course with KEATS analytics tool [Konstantinidis 2013] showed that the greatest activity on the Moodle course is on Thursday, which corresponds to the scheduled classes (Fig. 3). As you can see, some activity is also observed on weekends and a little more on Wednesday evening before the class. Students are likely to finalising their assignments at this time.

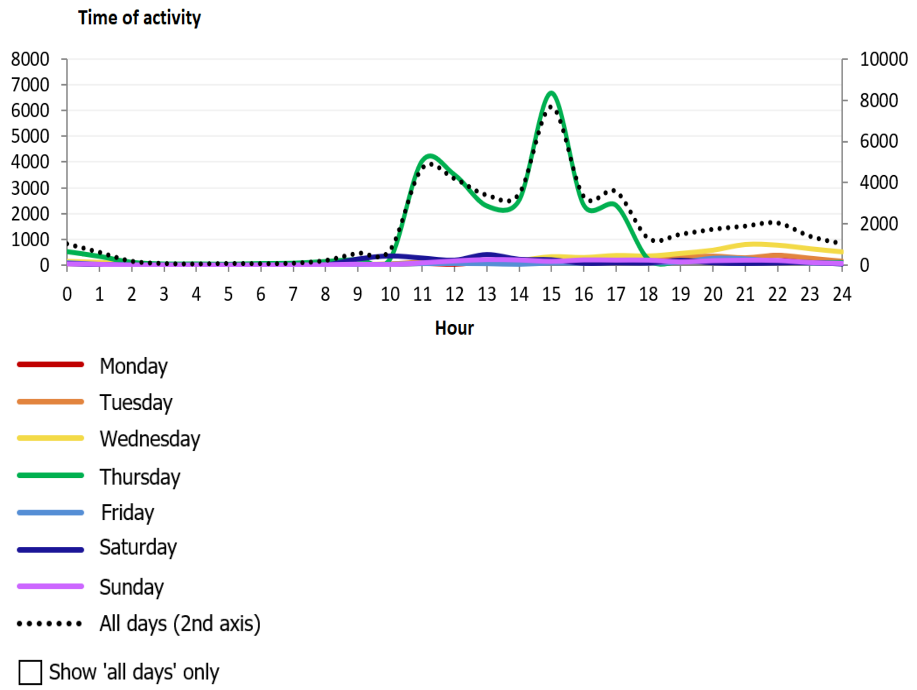

Fig. 3. Temporal features of activity in the course
Source: own work

A correlation analysis was also performed, which showed that there was no linear relationship between students' academic performance (points scored) and online activity on the course. However, at the same time, because of such activity in the course, we can conclude that this is a course that provides mainly informational support for traditional teaching and is not focused on the implementation of the blended learning model, which involves the active autonomous study of theoretical materials.

2.2 Analysis of students' activity and information preferences in the digital environment

For a detailed analysis of students' activity and information preferences in the digital environment, the results of an entrance survey, which preceded the course, were processed.

These results can be used in the future to monitor changes in students' educational behaviour strategies within the electronic course.

The questionnaire consisted of 42 questions, which can be divided into 8 groups: Presentation of theoretical material, Memorisation of information, Control of understanding, Application of knowledge, MOOC, Collaboration, Gamification, and Pedagogical management. In the question students had to relate their attitude to the element of a group to a five-point scale (1-5). For example, the group "Control of understanding" contains such elements as mobile polls, mindmaps, interactive educational games, etc.

For each student, an average score for each section of the questionnaire was calculated. An average value (from 1 to 5) for the section shows the degree of variability of the student's preferences in a certain direction (ways of memorisation, ways of checking understanding, ways of collaboration, etc.). For example, the average value in the "Collaboration" section shows the degree of variability for collaborative activities. The higher the indicator, the more means/tools/methods for collaborative activities a student perceives positively. Correlation analysis between these averaged values was performed (Table 2).

Table 2 - Correlation analysis of students' activity preferences in the digital environment

	Presentation of theoretical material	Memorisation of information	Control of understanding	Application of knowledge	MOOC	Collaboration	Gamification	Pedagogical management
Presentation of theoretical material	1,00							
Memorisation of information	0,64	1,00						
Control of understanding	0,65	**0,74**	1,00					
Application of knowledge	0,69	0,63	**0,74**	1,00				
MOOC	0,53	0,55	0,64	0,52	1,00			
Collaboration	0,61	0,55	**0,73**	**0,73**	0,57	1,00		
Gamification	0,61	0,61	**0,74**	0,71	0,44	**0,73**	1,00	
Pedagogical management	0,63	0,68	**0,78**	0,69	0,65	**0,73**	0,71	1,00

Analysis shows that the strongest linear correlation exists between "Control of understanding" and "Pedagogical management" (r=0,78). This indicates that the more ways of pedagogical management (reminders, deadlines, self-assessment criteria, visual progress bar, and so on) a student allows, the higher is the number of positively perceived methods of control of understanding (mobile polls, mindmaps, interactive educational games, etc.). In general, students who are most open to various activities (in collaboration, gamification, memorisation, application of knowledge), are ready to test their understanding in various ways.

Another detected correlation is “Gamification” – “Collaboration” (r=0,73). Students supporting various elements of gamification are more ready for various ways of collaboration.

Further, we will describe the result of detailed statistical analysis on some sections of the questionnaire.

Control of understanding. The preferred form of control for students is interactive educational games (4,26 out of 5 on average).

A strong linear relation (r=0,7-1,0) was found for the following points of the “Control of understanding” group:

- Control of understanding (interactive tasks for compiling and matching series, e.g., the Learning Apps service) - Memorisation of information (interactive timelines, services, and applications of the “time graphics” type); r=0,71;
- Control of understanding (interactive educational games) - Gamification (quests); r= 0,75;
- Control of understanding (interactive educational games) - Gamification (educational games with a virtual agent or character); r=0,77;
- Control of understanding (mapping, smart cards for transcoding information) - Control of understanding (mind maps); r=0,73;
- Control of understanding (interactive tasks for the compilation and comparison of series e.g., the Learning Apps service) - Control of understanding (mind maps); r=0,74;

Thus, students show a rather high demand in the use of interactive forms. We can note that the preferences of students are associated with the visualisation and conceptualisation of educational information. Consequently, students do not perceive traditional linear educational texts and the content of lectures as a sufficient tool for the knowledge system formation.

Application of knowledge. As the survey showed, students most of all want to apply knowledge in discussions with other students on forums, in social networks (an average score - 4,15). Students are warier of mutual assessment (3.68) and scribing (3.60) (explanation through drawings, sketches).

Correlation analysis showed the next linear relationship: Application of knowledge (development of explanatory materials - video, infographics, scribing) - Application of knowledge (scribing, explanation through sketches, drawings, e.g., Sparkol service); r=0,74. This shows us that for students to be ready to design various teaching materials (video, infographics, scribing), we need to purposefully teach them how to apply their knowledge in practice using scribing (e.g., Sparkol).

MOOC. Data reflecting students’ attitudes toward using MOOC materials show that students have this practice in non-formal education and self-development and welcome the inclusion of such open resources in the content of disciplines taught at the university. Most often students use lecture fragments for a better understanding of the material being studied and tests for strengthening knowledge and checking understanding. Most students are ready to use MOOCs for self-education.

Collaboration. As shown by correlation analysis if a student is ready to use network discussions in training, then he is also ready to use interaction in digital environments, e.g., virtual laboratories, gaming environments (r = 0,74). This correlation allows us to conclude that students' expectations regarding the organisation of joint activities in the digital environment are much broader than traditional discussions. Taking into account their focus on productive forms of digital educational activity, the preferences regarding gaming techniques identified above, we can conclude that there is a demand for the inclusion of virtual environments in training, together with collaborative design of digital products and joint problem-solving.

Pedagogical management. Direct methods of pedagogical management in the digital environment are difficult to implement. That is why we see from the answers to the questionnaire that students, along with the usual methods of managing their actions, perceive positively the methods of indirect pedagogical management, which promote self-management of educational actions (4,33 out of 5 on the average) and support of learning motivation with gamification (4,08 out of 5 on the average). At the same time, many students expressed a negative attitude towards strict deadlines (3,19 on average) and penalties (3,04 on average). One linear relationship was found between the elements of this group: progress bar or screen – a system of badges or other rewards (r=0,71). Both elements represent the student's attitude towards some visualisation of his progress.

To identify typological groups of students according to activity preferences in the digital learning environment, we applied the cluster analysis method. Clustering involves dividing all students into groups so that similar students are in the same group, and the groups themselves are as distinct as possible from each other.

The EM (Expectation-Maximisation) clustering algorithm, implemented in the Weka data-mining program, distributed 400 respondents into three clusters:
1) 1 cluster – 116 students (29%);
2) 2 cluster – 200 students (50%);
3) 3 cluster – 84 students (21%).

The first cluster included students who gave on average a small score for each item on the questionnaire (an average of 2,92 points). The third cluster comprised those students who assigned high scores for each item of the questionnaire (an average of 4,76 points). Besides, the second cluster included intermediate values (an average of 4,04 points).

Clustering results can only be visualised in a two-dimensional or three-dimensional space, i.e. displaying only two or three characteristics of the data (from the whole set of indicators, and in our case clustering was carried out according to 42 indicators) and their distribution by clusters. An example of students' distribution in clusters depending on the values of "Understanding (Mobile Surveys)" (x-axis) and "Memorisation Methods (Interactive timelines - services and applications of the time graphics type)" (y-axis) are shown in Figure 4.

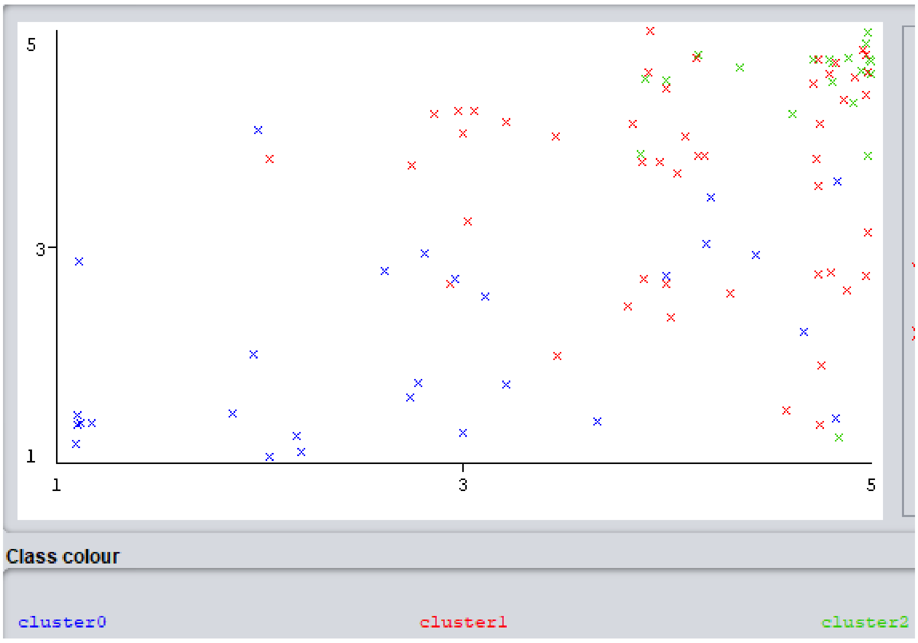

Fig. 4. An example of students' distribution in clusters depending on the selected values
Source: own

The Figure 3 shows that the cluster "2" corresponds to the dots in the upper right corner, which represent students who assigned high scores on both selected indicators. Students who gave small scores for both indicators (bottom left corner) are highlighted in blue and belong to the "0" cluster.

This visualisation additionally shows that the concentration of dots reflecting the positive students' attitude to the indicated innovative methods of interactions in the digital environment is quite high. However, at the same time, some students consciously choose traditional methods. This emphasises the need to create a digital learning environment that is diverse in its functions and information capabilities, providing the personalisation of educational activities through the expansion of educational opportunities and freedom of choice.

In the future, it is supposed to analyse the differences in the achievements of these groups and to trace the dynamics of changes in learning behaviour both for the typological groups and for particular students. The identification of students with low motivation or superficial knowledge can be used in the further organisation of a differentiated learning approach. These students can be offered simpler tasks at the initial stage, as well as additional instructions and tips. However, the more effective is the organisation of joint work - mutual assistance of "stronger" students to "weaker" ones.

CONCLUSION

Educational data mining tools are important for pedagogical monitoring of the digital educational process. Pedagogical monitoring is understood as a diversified process of collecting and analysing information about the activities of educational process participants with the objectives of tracking and predicting future activities and effects (Ovsyanitskaya et al, 2018).

The results of the data analysis described in the paper give grounds for identifying the weaknesses and strengths of students' learning behaviour in the electronic course. The positive trends that a teacher can rely on when developing a course include students' interest in collaboration and in involving, interactive learning content (e.g., gamification techniques). Among the weaknesses of students' educational behaviour, we can name an insufficient knowledge of the new learning algorithms provided by the digital environment capabilities (e.g., digital tools for memorisation support and understanding facilitation – infographics, explanatory videos, digital planning, and scheduling tools). These approaches can be used, for example, to solve the revealed problem of reducing students' activity during the course (reducing the number of views, completing optional tasks, etc.).

Thus, clustering demonstrated that along with students willing to use a variety of digital tools and ways of interacting with the educational content there is a significant number of students not highly motivated to implement new learning approaches. This is due to both a lack of experience and a negative impression of the experience gained in a variety of educational situations. The creation of such an experience is possible through the enrichment of the electronic course with interactive educational elements, communication services, and tools for formative assessment.

Further work in this area includes the following directions:

- recommendations for the Russian-language systems for data analysis in Moodle;
- rules that should be followed when organising an online course to collect appropriate data for further analytics;
- cluster analysis of students' activities for identifying possible behavioural groups;
- study of motivation and approaches to teaching students in each typical group;
- recommendations for working with groups of students with different digital behaviour;
- data mining in a Moodle course with a variety of training elements (video lectures, interactive assignments, etc.).

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