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post-conference proceedings of extended versions of selected papers

Editors:

Jaromír Baštinec and Miroslav Hrubý

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Aims and target group of the conference:

The conference **MITAV 2018** should attract in particular teachers of all types of schools and is devoted to the most recent discoveries in mathematics, informatics, and other sciences, as well as to the teaching of these branches at all kinds of schools for any age group, including e-learning and other applications of information technologies in education. The organizers wish to pay attention especially to the education in the areas that are indispensable and highly demanded in contemporary society. The goal of the conference is to create space for the presentation of results achieved in various branches of science and at the same time provide the possibility for meeting and mutual discussions among teachers from different kinds of schools and focus. We also welcome presentations by (diploma and doctoral) students and teachers who are just beginning their careers, as their novel views and approaches are often interesting and stimulating for other participants.

Organizers:

Union of Czech Mathematicians and Physicists, Brno branch (JČMF),
in co-operation with
Faculty of Military Technology, University of Defence in Brno,
Faculty of Science, Faculty of Education and Faculty of Economics and Administration,
Masaryk University in Brno,
Faculty of Electrical Engineering and Communication, Brno University of Technology.

Venue:

Club of the University of Defence in Brno, Šumavská 4, Brno, Czech Republic
June 14 and 15, 2018.

Conference languages:

Czech, Slovak, English

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Programme of the conference:

Thursday, June 14, 2018

12:00-13:45	Registration of the participants
13:45-14:00	Opening of the conference
14:00-14:50	Keynote lecture No. 1 (Jindřich Bečvář, Czech Republic)
14:50-15:10	Break
15:10-16:00	Keynote lecture No. 2 (Zbyněk Kubáček, Slovakia)
16:00-16:30	Break
16:30-18:00	Presentations of papers
18:00-18:40	Conference dinner
18:45	Bus departure to the Brno lake
19:30-21:30	Social event (evening on the boat)

Friday, June 15, 2015

09:00-09:45	Keynote lecture No. 3 (Miroslava Huclová, Czech Republic)
09:45-10:00	Break
10:00-11:30	Presentations of papers
11:30-11:45	Break
11:45-13:15	Presentations of papers
13:30	Closing

Each MITAV 2018 participant received printed collection of abstracts **MITAV 2018** with ISBN 978-80-7582-040-2. CD supplement of this printed volume contains all the accepted contributions of the conference.

Now, in autumn 2018, this **post-conference CD** was published, containing extended versions of selected MITAV 2018 contributions. The proceedings are published in English and contain extended versions of 15 selected conference papers. Published articles have been chosen from 46 conference papers and every article was once more reviewed.

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REMARK ON THE DOMINANT WEIERSTRASS CRITERION FOR THE UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

Vladimír Baláž, Alexander Maťašovský and Tomáš Visnyai
Faculty of Chemical and Food Technology, Slovak University of Technology in Bratislava
Radlinskho 9, 812 37 Bratislava, vladimir.balaz@stuba.sk, alexander.matasovsky@stuba.sk,
tomas.visnyai@stuba.sk

Abstract: *The condition of the dominant Weierstrass criterion is the sufficient condition for the uniform convergence of series of functions. In this article is shown that this really is not a necessary condition, as at the first sight it seems to be. In the effort to find necessary and sufficient conditions for the uniform convergence of series of functions, it would be enough for a smaller class of sequences of functions, we will investigate two special types of sets of sequences of functions. In this context, there is studied the notion of variation of a sequence of real numbers.*

Keywords: series of functions, uniform convergence, variation of function, variation of sequence.

INTRODUCTION

This article deals with the well-known dominant Weierstrass criterion of uniform convergence of series of functions. We will show that the converse of this criterion is not true. This fact leads us to split the sequences of non-negative functions into two types. We give a simple characterization of these types. Finally, we connect our considerations with the notion of variation of the function.

1 BASIC NOTIONS

We recall some basic notions. Suppose $M \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}$ we have a function $f_n \colon M \to \mathbb{R}$, by $(f_n)_{n=1}^{\infty}$ we denote a *sequence of functions* defined on M. For each fixed $x \in M$, $(f_n(x))_{n=1}^{\infty}$ is a sequence of numbers, and it makes sense to ask whether this sequence converges. If the sequence $(f_n(x))_{n=1}^{\infty}$ converges for each $x \in M$, a function $f \colon M \to \mathbb{R}$ defined by the following way

$$f(x) = \lim_{n \to \infty} f_n(x)$$

is called the *pointwise limit* of the sequence $(f_n)_{n=1}^{\infty}$, it is said f_n *converges pointwise* to f on M. This is abbreviated by $f_n \to f$.

The sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n \colon M \to \mathbb{R}$ $(n = 1, 2, \dots)$ *converges uniformly* to a function $f \colon M \to \mathbb{R}$ on M, if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ so that whenever $n \geq n_0$ and $x \in M$ we have $|f_n(x) - f(x)| < \varepsilon$. In this case, we write $f_n \rightrightarrows f$.

Uniform convergence is a stronger condition than pointwise convergence, if $f_n \rightrightarrows f$ then $f_n \to f$. The converse is not true.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on a set $M \subseteq \mathbb{R}$, and $(s_n)_{n=1}^{\infty}$ be the sequence of partial sums of the first n terms

$$s_n(x) = \sum_{i=1}^{n} f_i(x).$$

Then the series $\sum_{i=1}^{\infty} f_i$ is *pointwise convergent on M to a function s* if and only if for each $x \in M$ the sequence $(s_n(x))_{n=1}^{\infty}$ converges to $s(x)$, and the series $\sum_{i=1}^{\infty} f_i$ is *uniformly convergent on M to s* if and only if the sequence $(s_n)_{n=1}^{\infty}$ converges uniformly to s on M.

Let us also recall the definition of variation of a function. Consider the collection D of ordered $(n+1)$-tuples of numbers $x_0 < x_1 < \cdots < x_n$ belong to M, $M \subseteq \mathbb{R}$, where n is an arbitrary natural number. The *variation of a real-valued function* $f\colon M \to \mathbb{R}$ is given by

$$\mathrm{Var}(f) = \sup_D \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\}.$$

2 MAIN RESULT

The following Theorem is well-known assertion so-called dominant Weierstrass criterion (see [1], [5]). For a shortness of the proof, we remember it.

Theorem 1. *Let $f_n\colon M \to \mathbb{R}$ $(n = 1, 2, \dots)$, $M \subseteq \mathbb{R}$. Suppose that there exist numbers $a_n \geq 0$ $(n = 1, 2, \dots)$ such that*

(1) for all $x \in M$, $|f_n(x)| \leq a_n$ $(n = 1, 2, \dots)$,

(2) $\sum_{n=1}^{\infty} a_n < +\infty$.

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on M.

Proof. We use the Cauchy criterion for uniform convergence. Given $\varepsilon > 0$, we must find a response $n_0 \in \mathbb{N}$, independent of x, such that $n_0 \leq m < n$ implies that

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon.$$

We know that

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} a_k.$$

The convergence of $\sum_{n=1}^{\infty} a_n$ guarantees an n_0, independent of x, for which this sum is less than ε when $n_0 \leq m < n$. $\square$

Consider the following proposition, it is the converse of the previous Theorem 1. There arises a natural question whether this proposition is true.

Proposition 2. *If the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on the set $M \subseteq \mathbb{R}$ then there exist numbers $a_n \geq 0$ $(n = 1, 2, \dots)$ having the properties (1) and (2) of Theorem 1.*

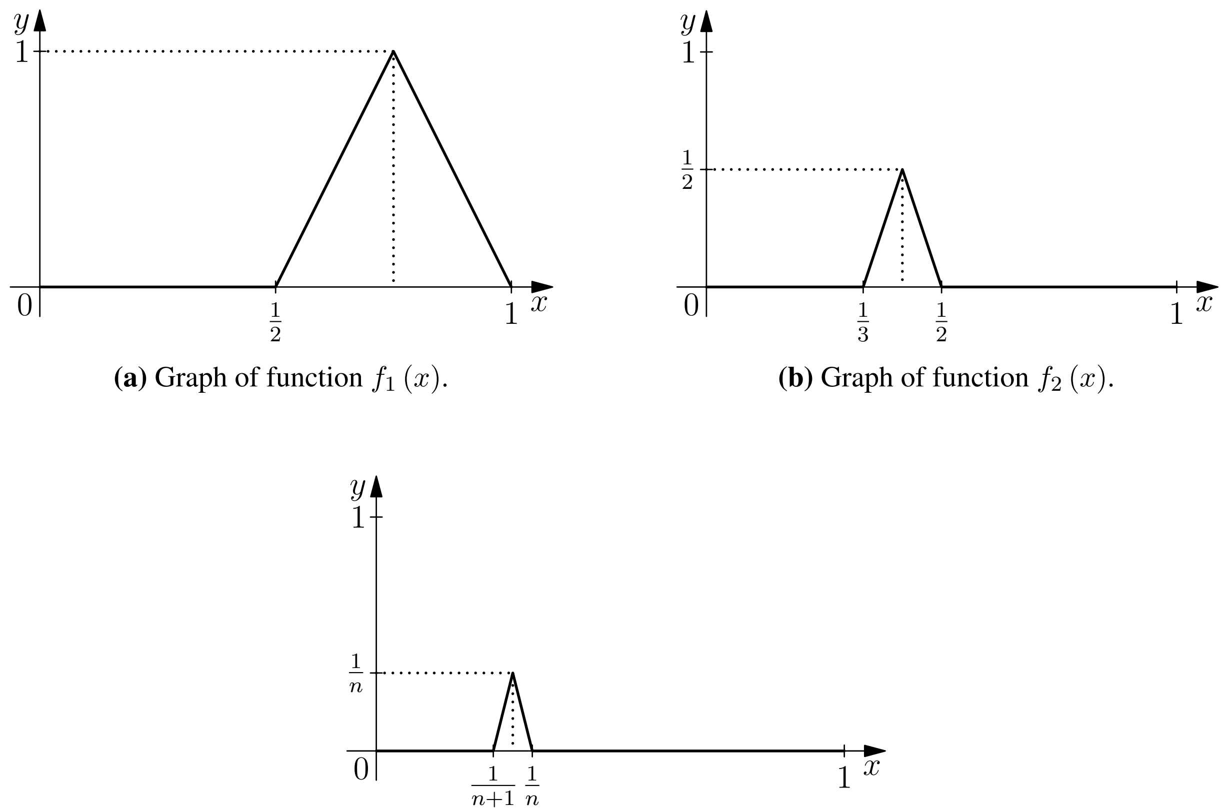
We will show that in generality the Proposition 2 is false. For this is enough to construct a sequence $(f_n)_{n=1}^{\infty}$ of functions such that $\sum_{n=1}^{\infty} f_n$ converges uniformly on M and there do not exist numbers $a_n \geq 0$ having the properties (1) and (2).

We are going to construct such a sequence of functions. We use a construction from P. R. Halmos (see [3]).

Example 3. *Let $M = \langle 0, 1\rangle$, we define a sequence of functions $(f_n)_{n=1}^{\infty}$ for $n = 1, 2, \dots$ by the following way:*

$$f_n(x) = \begin{cases} 2(n+1)x - 2 & \text{if } x \in \left(\frac{1}{n+1}, \frac{2n+1}{2n(n+1)}\right\rangle, \\ -2(n+1)x + \frac{2(n+1)}{n} & \text{if } x \in \left(\frac{2n+1}{2n(n+1)}, \frac{1}{n}\right\rangle, \\ 0 & \text{otherwise}. \end{cases}$$

See the Fig. 1.



(a) Graph of function $f_1(x)$.

(b) Graph of function $f_2(x)$.

(c) Graph of function $f_n(x)$.

Fig. 1. Elements of sequence of functions $(f_n)_{n=1}^{\infty}$.

The m-th partial sum $s_m(x) = f_1(x) + f_2(x) + \dots + f_m(x)$ is the function, which equals to zero on $\left\langle 0, \frac{1}{m+1}\right\rangle$ and elsewhere its graph is created by the legs of isosceles triangles, as it shows Fig. 2.

Let $s(x) = \lim_{m\to\infty} s_m(x)$. Then $s(x)$ is the function defined on $\langle 0, 1\rangle$ as follows: $s(0) = 0$ and $s(x)$ is make up by the legs of isosceles triangles having the altitude equals to $\frac{1}{m}$ on $\left(\frac{1}{m+1}, \frac{1}{m}\right\rangle$ for $m = 1, 2, \ldots$. Clearly $|s_m(x) - s(x)| = 0$ for $x \in \left\langle \frac{1}{m+1}, 1\right\rangle$ and elsewhere we have $|s_m(x) - s(x)| \leq \frac{1}{m+1}$ for the reason that the remaining isosceles triangles having the biggest altitude $\frac{1}{m+1}$.

Let $\varepsilon > 0$. Chose $n_0 = n_0(\varepsilon)$ such that $\frac{1}{n_0+1} < \varepsilon$. Let $m \geq n_0$ then for all $x \in \langle 0, 1\rangle$ we have $|s_m(x) - s(x)| \leq \frac{1}{m+1} \leq \frac{1}{n_0+1} < \varepsilon$. Therefore $(s_m)_{m=1}^{\infty}$ converges uniformly to s on $\langle 0, 1\rangle$.

Now we show that there are no numbers $a_n \geq 0$ $(n = 1, 2, \ldots)$ with the properties (1) and (2) of Theorem 1. If there exist such numbers then from (1) we have that $a_n \geq \frac{1}{n}$ $(n = 1, 2, \ldots)$. Therefore (2) does not hold, because $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

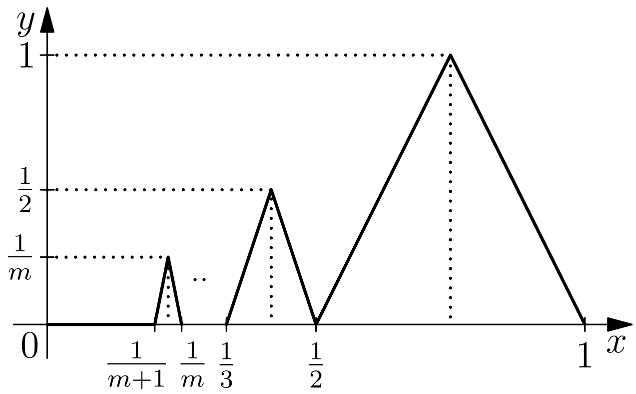


Fig. 2. Graph of function $s_m(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$.

The previous example leads us to study the following situation.

Let f_n $(n = 1, 2, \ldots)$ be *non-negative functions* defined on a set M, $M \subseteq \mathbb{R}$. Split the sequences $(f_n)_{n=1}^{\infty}$ of such functions into two types:

(A) $\displaystyle\sum_{n=1}^{\infty} \sup_{x\in M} f_n(x) < +\infty,$

(B) $\displaystyle\sum_{n=1}^{\infty} \sup_{x\in M} f_n(x) = +\infty.$

It is clear that for sequences of functions of type (A) the dominant Weierstrass criterion is the necessary and sufficient condition for the uniform convergence of series of functions. Our aim is to characterize these two types of sequences of functions.

Let $(f_n)_{n=1}^{\infty}$ be of the type (B). Then for every n there exists $x_n \in M$ such that

$$f_n(x_n) > \sup_{x\in M} f_n(x) - \frac{1}{n^2},$$

therefore

$$\sum_{n=1}^{\infty} f_n(x_n) \geq \sum_{n=1}^{\infty} \sup_{x\in M} f_n\left(x\right) - \sum_{n=1}^{\infty} \frac{1}{n^2} = +\infty.$$

Hence $\sum_{n=1}^{\infty} f_n(x_n) = +\infty$ for the selected sequence $(x_n)_{n=1}^{\infty}$ where $x_n \in M$, $n = 1, 2, \ldots$.

We obtain the following Proposition.

Proposition 4. *If the sequence $(f_n)_{n=1}^{\infty}$ is of the type (B), then there exists a sequence $(x_n)_{n=1}^{\infty}$ of numbers from M such that*

$$\sum_{n=1}^{\infty} f_n(x_n) = +\infty.$$

Another situation is for the sequence $(f_n)_{n=1}^{\infty}$ of the type (A). If the sequence $(f_n)_{n=1}^{\infty}$ is of the type (A) and $(x_n)_{n=1}^{\infty}$ is an arbitrary sequence of numbers from M then the following inequality holds:

$$f_n\left(x_n\right) \leq \sup_{x\in M} f_n\left(x\right), \quad n = 1, 2, \ldots .$$

From this we have

$$\sum_{n=1}^{\infty} f_n(x_n) \leq \sum_{n=1}^{\infty} \sup f_n\left(x\right) < +\infty.$$

Hence

$$\sum_{n=1}^{\infty} f_n(x_n) < +\infty.$$

We proved the following proposition.

Proposition 5. *If the sequence $(f_n)_{n=1}^{\infty}$ is of the type (A), then for every sequence $(x_n)_{n=1}^{\infty}$ of numbers from M holds*

$$\sum_{n=1}^{\infty} f_n(x_n) < +\infty.$$

From Propositions 4 and 5 we get a characterization of sequences $(f_n)_{n=1}^{\infty}$ of the type (B) and (A) respectively.

Theorem 6. *A sequence $(f_n)_{n=1}^{\infty}$ is of the type (B) if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$, $x_n \in M$, $n = 1, 2, \ldots$ such that*

$$\sum_{n=1}^{\infty} f_n(x_n) = +\infty.$$

Theorem 7. *A sequence $(f_n)_{n=1}^{\infty}$ is of the type (A) if and only if for every sequence $(x_n)_{n=1}^{\infty}$, $x_n \in M$, $n = 1, 2, \ldots$ we have*

$$\sum_{n=1}^{\infty} f_n(x_n) < +\infty.$$

For more results concerning $\sum f_n(x_n)$ see for example [2], [4] or [6].

In the following, we will investigate the above-mentioned properties from the point of view variation of a sequence.

Let $\boldsymbol{a} = (a_n)_{n=1}^{\infty}$ be a sequence of real numbers. The *variation of a sequence* $\boldsymbol{a}$ is called the real number

$$\mathrm{Var}(\boldsymbol{a}) = \sum_{j=1}^{\infty} |a_j - a_{j+1}|.$$

In the other case it says that $\boldsymbol{a} = (a_n)_{n=1}^{\infty}$ has infinite variation. It is well known that if a sequence $\boldsymbol{a} = (a_n)_{n=1}^{\infty}$ has finite variation, then it converges. The converse in not true e.g. $\boldsymbol{a} = (a_n)_{n=1}^{\infty} = \left(1, -1, \frac{1}{2}, -\frac{1}{2}, \dots, \frac{1}{n}, -\frac{1}{n}, \dots\right)$.

We are going to study the variation of a sequence $(f_n(x))_{n=1}^{\infty}$ for $x \in M$.

Proposition 8. *If* $\boldsymbol{F} = (f_n)_{n=1}^{\infty}$ *is a sequence of functions of the type (A), then for each* $x \in M$ *we have*

$$\mathrm{Var}\left(\boldsymbol{F}(x)\right) = \sum_{j=1}^{\infty} |f_j(x) - f_{j+1}(x)| < +\infty.$$

Proof. By the assumption the sequence $(f_n)_{n=1}^{\infty}$ of non-negative functions f_n is of the type (A) and

$$\sum_{n=1}^{\infty} |f_n\left(x\right)| = \sum_{n=1}^{\infty} f_n\left(x\right) < +\infty.$$

Let $(a_n)_{n=1}^{\infty}$ is an arbitrary sequence having the property $\sum_{n=1}^{\infty} |a_n| < +\infty$ then we also have $\sum_{j=1}^{\infty} |a_j - a_{j+1}| < +\infty$. For this it is enough to realise that

$$|a_1 - a_2| + |a_2 - a_3| + \cdots \leq |a_1| + 2\,|a_2| + 2\,|a_3| + \cdots < +\infty.$$

Denote by $K = \sum_{n=1}^{\infty} \sup_{x \in M} f_n\left(x\right)$. From the previous it is clear, that for all $x \in M$ we have $\mathrm{Var}\left(\boldsymbol{F}(x)\right) \leq 2K < +\infty$. Moreover $\sup_{x \in M} \mathrm{Var}\left(\boldsymbol{F}(x)\right) \leq 2K < +\infty$. $\square$

How it works with the variation of a sequence $(f_n)_{n=1}^{\infty}$ that is of the type (B). We would expect that there exists such $x \in M$ that

$$\mathrm{Var}\left(\boldsymbol{F}(x)\right) = +\infty, \quad \boldsymbol{F} = (f_n)_{n=1}^{\infty}.$$

We show that the above hypothesis is false. It is enough to take into consideration the sequence $(f_n)_{n=1}^{\infty}$ in the Example 3 where the sequence $(f_n)_{n=1}^{\infty}$ is of the type (B) because

$$\sum_{n=1}^{\infty} \sup_{x \in \langle 0,1 \rangle} f_n\left(x\right) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

and $f_n \geq 0$ $(n = 1, 2, \dots)$ on $\langle 0, 1 \rangle$. Fix $x_0 \in \langle 0, 1 \rangle$. If $x_0 = 0$ or $x_0 = 1$ then $f_n\left(x_0\right) = 0$ for all $n = 1, 2, \dots$ and therefore $\mathrm{Var}\left(\boldsymbol{F}(x_0)\right) = 0$. Let $0 < x_0 < 1$. Then there exists $m \geq 1$ such that

$$\frac{1}{m+1} \leq x_0 \leq \frac{1}{m}.$$

For every $n \neq m$ we have $f_n(x_0) = 0$. Hence

$$\begin{aligned}\operatorname{Var}(\boldsymbol{F}(x_0)) &= \sum_{j=1}^{\infty} |f_j(x_0) - f_{j+1}(x_0)| \\ &\leq |f_{m-1}(x_0) - f_m(x_0)| + |f_m(x_0) - f_{m+1}(x_0)| \\ &= 2f_m(x_0) < +\infty.\end{aligned}$$

Therefore for all $x_0 \in \langle 0, 1\rangle$ the inequality $\operatorname{Var}(\boldsymbol{F}(x_0)) < +\infty$ holds. Moreover it is easy to see that

$$\sup_{x \in \langle 0,1\rangle} \operatorname{Var}(\boldsymbol{F}(x)) < +\infty.$$

A simple estimate gives

$$\operatorname{Var}(\boldsymbol{F}(x)) \leq 2\,|f_n(x_0)| \leq 2f_1\left(\frac{3}{4}\right) = 2.1 = 2.$$

Hence we found a sequence of functions $(f_n)_{n=1}^{\infty}$ being of the type (B) nevertheless

$$\operatorname{Var}(\boldsymbol{F}(x)) < +\infty, \text{ for each } x \in M = \langle 0, 1\rangle.$$

Example 9. *At the same moment we manage to find an example of sequence of functions $\boldsymbol{G} = (g_n)_{n=1}^{\infty}$ defined on the interval $\langle 0, 1\rangle$ with the following properties:*

i) for all $x \in \langle 0, 1\rangle$ the variation $\operatorname{Var}(\boldsymbol{G}(x)) < \infty$,

ii) g_n converges uniformly to g,

iii) for all $n \in \mathbb{N}$ the variation $\operatorname{Var}(g_n(x)) < \infty$ while $\operatorname{Var}(g(x)) = \infty$ and $\operatorname{Var}(g_n - g) = \infty$.

It is enough to take the sequence $\boldsymbol{G} = (g_n)_{n=1}^{\infty}$ where $g_n = f_1 + f_2 + \cdots + f_n$ and functions f_n, $(n = 1, 2, \dots)$ are from the Example 3. It is easy to see that the sequence $\boldsymbol{G} = (g_n)_{n=1}^{\infty}$ fulfils all desired properties.

CONCLUSION

On the basis of the previous text the following two problems arise.

1) What premisses we need to add to the Proposition 2 to be this proposition true, may be for a smaller class of sequences of functions.

2) To find a suitable characterization of sets of sequences of functions of type (A) and (B) respectively.

References

[1] Bressoud, D. M. *A radical approach to real analysis.* 2nd edition, USA: The Mathematical Association of America, 2007, ISBN 13: 978-0-88385-747-2.

[2] Ganguli, P. L., Lahiri, B. K. Some results on certain sets of series. *Czechoslovak Mathematical Journal*, **18**, 1968, 589–594.

[3] Halmos, P. R. *Measure Theory.* 12th printing, Princenton: van Nostrand, 1968.

[4] Lahiri, B. K., Ghosh, D. B. Some properties of infinite series. *Bull. Calcutta Math. Soc.*, **87**, 1995, 247–252.

[5] Šalát, T. *Infinite series.* [in Slovak] Praha: Academia, 1974.

[6] Šalát, T., Vadovič, P. The structure of the Fréchet space s regarding the series $\sum f_n(x_n)$. *Tatra Mountains Mathematical Publications*, **44**, 2009, 1–8.

CONSTRUCTION OF ALGEBRAIC STRUCTURES OF LINEAR SPACES OF TWO-DIMENSIONAL SMOOTH FUNCTIONS

Jaroslav Beránek
Masaryk University, Faculty of Education
Poříčí 7, 603 00 Brno, Czech Republic
beranek@ped.muni.cz

Jan Chvalina
Brno University of Technology
Faculty of Electrical Engineering and Communication
Technická 10, 616 00 Brno, Czech Republic
chvalina@feec.vutbr.cz

Abstract: *In the contribution there is constructed a group of a two-dimensional function solution space of ordinary linear differential equations of the second order. Considerations serve as the preparation for further constructions of related multistructures, for example noncommutative hypergroups.*

Keywords: *Noncommutative group; differential equation; linear space.*

INTRODUCTION

Teaching mathematics to university students of various specializations requires aiming students' efforts at linking up different mathematics theories in order that they could more profoundly understand the relations between studied parts of mathematics. This article offers the possible linkage between teaching materials of mathematical analysis and the theory of algebraic structures, specifically the relation of common differential equations of the second order and the concept of an ordered group. Using the construction of the noncommutative ordered group of linear spaces of these differential equations' solutions there is demonstrated the connection between classical problems of mathematical analysis and the theory of algebraic structures. The applied concept appeared in monographs [3], [4], [11], [12], [13] and articles [7], [8]. The set of all real numbers is denoted as $\boldsymbol{R}$; under $\boldsymbol{C}(J)$ (sometimes also $\boldsymbol{C}^0(J)$) there is understood the ring of all continuous functions on an interval $J \subset \boldsymbol{R}$ with the usual addition and multiplication of functions. Similarly, the ring of all continuous functions on the interval J which have all derivations up to the order n for any natural number n will be denoted as $\boldsymbol{C}^n(J)$.

CONSTRUCTION A NONCOMMUTATIVE GROUP ($G(F),\cdot$)

Let us describe the construction of the group, possibly an ordered group, of linear differential operators of the second order which form left sides of usual homogenous differential equations in the form

$$y'' + p(x)\, y' + q(x)\, y = 0 \ . \tag{1}$$

During the construction we will apply considerations presented in [10]. Let $J \subset \boldsymbol{R}$ be an open interval in the set of real numbers; we do not exclude the case $J = \boldsymbol{R}$. We will habitually denote $\boldsymbol{C}^k(J)$ as the commutative ring of all real functions $f: J \to \boldsymbol{R}$ which have continuous derivations on the interval J up to the order k included, i.e. in all points $x \in J$. Instead of $\boldsymbol{C}^0(J)$ we will use the following: $\boldsymbol{C}(J)$ is the ring of all continuous real functions of one variable defined on the interval J; $\boldsymbol{C}_+(J)$ is the subsemiring of the ring $\boldsymbol{C}(J)$ formed by all positive continuous functions, so

$$\boldsymbol{C}_+(J) = \{ f: J \to \boldsymbol{R}; f(x) > 0 , x \in J \} .$$

The set of all differential equations (1), where $p \in \boldsymbol{C}_+(J)$, $q \in \boldsymbol{C}(J)$, will be $A_2(J)$ in accord with [8]. Further, I_d is an identity operator on $\boldsymbol{C}^2(J)$, i.e. $I_d\, y = y$ for every function $y \in \boldsymbol{C}^2(J)$, and let us set $D = \dfrac{d}{dx}$, so $D_y(x) = y'(x)$ for every function $y \in \boldsymbol{C}^2(J)$. Then $L(p,q)$ will denote the differential operator for the pair of functions $p \in \boldsymbol{C}(J)$, $q \in \boldsymbol{C}(J)$ as follows

$$L(p, q) = D^2 + p(x)\, D + q(x)\, I_d \ .$$

In this notation the equation (1) is represented as $L(p,q)(y) = 0$. Let us further set

$$\boldsymbol{LA}_2(J) = \{L(p, q): \boldsymbol{C}^2(J) \to C(J); [p, q] \in \boldsymbol{C}_+(J) \times \boldsymbol{C}(J)\} \ , \tag{2}$$

i.e. $\boldsymbol{LA}_2(J)$ means the set of all above described differential operators. For $r \in \boldsymbol{R}$ let us denote $\chi_r: J \to \boldsymbol{R}$ as a constant function with a value r.

Proposition 1. *Let $J \subset \boldsymbol{R}$ be an open interval, $\boldsymbol{LA}_2(J) = \{L(p, q); p, q \in \boldsymbol{C}(J), p(x) > 0, x \in J\}$. For any pair of differential operators $L(p_1, q_1)$, $L(p_2, q_2) \in \boldsymbol{LA}_2(J)$ let us set*

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1\, p_2 \,, p_1\, q_2 + q_1)$$

and $L(p_1, q_1) \le L(p_2, q_2)$, *if $p_1(x) = p_2(x)$, $q_1(x) \le q_2(x)$ for an arbitrary element $x \in J$. Then $(\boldsymbol{LA}_2(J), \cdot, \le)$ is a noncommutative ordered group with a unit element $L(\chi_1 ,\chi_0)$.*

Proof: See [8]. Let us remark that for the noncommutative group $\boldsymbol{LA}_2(J)$ constructed in this way the unit element is an operator $L(\chi_1 ,\chi_0)$ and the inverse element to operator $L(p, q)$ is an operator $L(\dfrac{1}{p}, -\dfrac{q}{p})$.

In the group theory, the term normal or invariant subgroup of the given group plays also an important role. Let us note that a nonempty subset H of a group G (on the set H we consider the restriction of the operation defined on G) is called a subgroup of the group G, if $e \in H$ (a unit or a neutral element of group G); further if for every pair of elements $a, b \in H$ there applies $a.b \in H$ and also for every element $a \in H$ there exists $a^{-1} \in H$ (where a^{-1} is an inverse element to the element $a \in H \subset G$). The subgroup H of the group G is called normal (also an invariant subgroup of group G or a normal divisor of group G), if for every element $a \in G$ there applies $a^{-1} \cdot H \cdot a \subset H$ (condition P1). Let us remark that this condition is equivalent to each of the following conditions:

P2: For every element $a \in G$ there applies $a^{-1} \cdot H \cdot a = H$,
P3: For every element $a \in G$ there applies $a \cdot H \cdot a^{-1} = H$,
P4: For every element $a \in G$ there applies $a \cdot H \cdot a^{-1} \subset H$.

The proof of the equivalence of the given conditions is not complicated and could be found in any more systematic paper (possibly textbook) on the fundamentals of the group theory.

Let us now denote $\boldsymbol{L}_1\boldsymbol{A}_2(J)$ as a subset of the group $\boldsymbol{LA}_2(J)$ defined by equation

$$\boldsymbol{L_1A_2}(J) = \{ L(p, q) ; L(p, q) \in \boldsymbol{LA_2}(J), p(x) \equiv 1\}, \quad \text{so}$$
$$\boldsymbol{L_1A_2}(J) = \{L(\chi_1, q); q \in \boldsymbol{C}(J)\} .$$

Proposition 2. *($\boldsymbol{L_1A_2}(J), \cdot$) is a normal (commutative) subgroup of the group ($\boldsymbol{LA_2}(J), \cdot$) isomorphic with ($\boldsymbol{C}(J), +$).*

Proof: It is evident that a neutral element (a unit) $L(\chi_1 ,\chi_0)$ of the group $(\boldsymbol{LA_2}(J), \cdot)$ belongs also to the subgroup $\boldsymbol{L_1A_2}(J)$ of the group $(\boldsymbol{LA_2}(J), \cdot)$. Let $L(\chi_1, q) \in \boldsymbol{L_1A_2}(J)$, $L(\chi_1, u) \in \boldsymbol{L_1A_2}(J)$. Then

$$L^{-1}(\chi_1, q) = L(\chi_1, -q) \in \boldsymbol{L_1A_2}(J)$$

(similarly $L^{-1}(\chi_1, u) \in \boldsymbol{L_1A_2}(J)$) and further

$$L(\chi_1, q) \cdot L(\chi_1, u) = L(\chi_1, q+u) \in \boldsymbol{L_1A_2}(J),$$

so $(\boldsymbol{L_1A_2}(J) , \cdot)$ is a subgroup of the group $(\boldsymbol{LA_2}(J) , \cdot)$.

There remains to prove that the group $(\boldsymbol{L_1A_2}(J) , \cdot)$ is a normal subgroup of the given group. Let then $L(\chi_1, u) \in \boldsymbol{L_1A_2}(J)$, $L(p, q) \in \boldsymbol{LA_2}(J)$ be arbitrary differential operators belonging to the given sets. Then there holds: $L^{-1}(p, q) \cdot L(\chi_1, u) . L(p, q) = L(\frac{1}{p}, -\frac{q}{p}) \cdot L(p, q+u) = L(\chi_1 , \frac{q}{p} + \frac{u}{p} - \frac{q}{p}) = L(\chi_1 , \frac{u}{p}) \in \boldsymbol{L_1A_2}(J)$, so $L^{-1}(p, q) \cdot \boldsymbol{L_1A_2}(J) \cdot L(p, q) \subset \boldsymbol{L_1A_2}(J)$ for every operator $L(p, q) \in \boldsymbol{LA_2}(J)$, therefore the group $(\boldsymbol{L_1A_2}(J) , \cdot)$ is a normal subgroup of the group $(\boldsymbol{LA_2}(J) , \cdot)$. It is evident that $(\boldsymbol{L_1A_2}(J) , \cdot) \cong (\boldsymbol{C}(J), +)$. □

Now we will show the construction of a noncommutative ordered group of linear spaces of smooth functions of $\boldsymbol{C}^2$ class defined on the interval $J \subset \boldsymbol{R}$ (which can also be equal to $\boldsymbol{R}$).

Let $\varphi_1 ,\varphi_2 \in \boldsymbol{C}^2(J)$ be linearly independent functions. More precisely expressed, functions φ_1 ,φ_2 form a fundamental system of solutions of a homogenous differential equation whose coefficients are continuous functions (which means that the Wronskian $W[\varphi_1,\varphi_2]$ does not equal zero in any point of the interval J). Let us denote $V(\varphi_1,\varphi_2)$ the linear space of dimension 2 created by all functions in the form

$$y(x)=c_1\varphi_1(x)+c_2\varphi_2(x) , \tag{A}$$

where c_1 , $c_2 \in \boldsymbol{R}$, so

$$V(\varphi_1 ,\varphi_2) = \{c_1\varphi_1 + c_2 \varphi_2 ; \varphi_1 ,\varphi_2 \in \boldsymbol{C}^2(J), c_1 , c_2 \in \boldsymbol{R}\}.$$

Let us arrange a homogenous differential equation of the second order for functions $y \in V(\varphi_1 ,\varphi_2)$. There holds

$$c_1\varphi_1'(x)+c_2\varphi_2'(x) = y'(x) \tag{B}$$
$$c_1\varphi_1''(x)+c_2\varphi_2''(x)=y''(x) \tag{C}$$

We will exclude constants c_1, c_2. Since functions φ_1 ,φ_2 are linearly independent on the interval J, their Wronskian for every number $x \in J$ equals

$$W[\varphi_1,\varphi_2] = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} \neq 0$$

Let us note that for the sake of the clear arrangement we will further use the notation of the Wronskian in the form $W[\varphi_1 ,\varphi_2]$ instead of $W\varphi_1 ,\varphi_2$, which is sometimes used in the mathematics literature. From the system of equations (A), (B) we get:

$$c_1 = \frac{\begin{vmatrix} y & \varphi_2 \\ y' & \varphi_2' \end{vmatrix}}{W[\varphi_1,\varphi_2]} = \frac{y(x)\varphi_2'(x) - y'(x)\varphi_2(x)}{W[\varphi_1,\varphi_2]} \quad ,$$

$$c_2 = \frac{\begin{vmatrix} \varphi_1 & y \\ \varphi_1' & y' \end{vmatrix}}{W[\varphi_1,\varphi_2]} = \frac{y'(x)\varphi_1(x) - y(x)\varphi_1'(x)}{W[\varphi_1,\varphi_2]} \quad .$$

After substituting to (C) we will get:

$$y''(x) = y(x)\frac{\varphi_2'(x)\varphi_1''(x)}{W[\varphi_1,\varphi_2]} - y'(x)\frac{\varphi_2(x)\varphi_1''(x)}{W[\varphi_1,\varphi_2]} + y'(x)\frac{\varphi_1(x)\varphi_2''(x)}{W[\varphi_1,\varphi_2]} - y(x)\frac{\varphi_1'(x)\varphi_2''(x)}{W[\varphi_1,\varphi_2]}$$

so

$$y''(x) + y'(x)\frac{\varphi_1''(x)\varphi_2(x) - \varphi_1(x)\varphi_2''(x)}{W[\varphi_1,\varphi_2]} + y(x)\frac{\varphi_1'(x)\varphi_2''(x) - \varphi_1''(x)\varphi_2'(x)}{W[\varphi_1,\varphi_2]} = 0 \quad .$$

If we use the notation with determinants and introduce the following representation $D[\varphi_1 ,\varphi_2] = \begin{vmatrix} \varphi_1'' & \varphi_2'' \\ \varphi_1 & \varphi_2 \end{vmatrix}$, the above given equation can be represented in the form

$$y'' + \frac{D[\varphi_1,\varphi_2]}{W[\varphi_1,\varphi_2]}\, y' + \frac{W[\varphi_1',\varphi_2']}{W[\varphi_1,\varphi_2]}\, y = 0 \quad ,$$

so with the notation introduced in part 2 we can write

$$L\left(\frac{D[\varphi_1,\varphi_2]}{W[\varphi_1,\varphi_2]}, \frac{W[\varphi_1',\varphi_2']}{W[\varphi_1,\varphi_2]}\right) y = 0 \quad , \qquad \text{(D)}$$

otherwise $L(p, q)\, y = 0$, where $p = \dfrac{D[\varphi_1,\varphi_2]}{W[\varphi_1,\varphi_2]}$, $q = \dfrac{W[\varphi_1',\varphi_2']}{W[\varphi_1,\varphi_2]}$.

Let us denote $F \subset \boldsymbol{C}^2(J) \times \boldsymbol{C}^2(J)$ as the set of all ordered pairs $[\varphi_1,\varphi_2]$ of linearly independent functions φ_1 ,φ_2 , for which $D[\varphi_1 ,\varphi_2] \neq 0$ on the interval J , thus

$$\varphi_1(x)\varphi_2''(x) - \varphi_1''(x)\varphi_2(x) \neq 0 \text{ for every number } x \in J .$$

Then $V(\varphi_1 ,\varphi_2)$ is a linear space of dimension 2 of all solutions of a homogenous differential equation of the second order

$$L(p, q)\, y = 0 \, , \quad \text{so}$$

$$y'' + p(x)\, y' + q(x)\, y = 0 \, ,$$

where $p(x) = \dfrac{\begin{vmatrix} \varphi_1''(x) & \varphi_2''(x) \\ \varphi_1(x) & \varphi_2(x) \end{vmatrix}}{W[\varphi_1,\varphi_2](x)}$, $q(x) = \dfrac{W[\varphi_1',\varphi_2'](x)}{W[\varphi_1,\varphi_2](x)}$.

Example. Let $\lambda_1 , \lambda_2 \in \boldsymbol{R}$ be real numbers for which $\lambda_1 \neq \lambda_2 \neq 0$. Let us set $\varphi_1(x) = e^{\lambda_1 x}$, $\varphi_2(x) = e^{\lambda 2x}$, $x \in \boldsymbol{R}$. We will arrange a differential equation whose space of solutions is the space of dimension 2 of functions

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda 2x}, \qquad c_1, c_2 \in \boldsymbol{R} \; .$$

Through a usual procedure we will get

$$y'' - (\lambda_1 + \lambda_2)\, y' + \lambda_1\lambda_2\, y = 0 \; . \qquad \text{(E)}$$

Let us now denote the system of all spaces of dimension 2 (planes) $V(\varphi_1, \varphi_2)$ as $\boldsymbol{G}(F)$, where $[\varphi_1, \varphi_2] \in F$, and where F denotes the set of all pairs of functions $\varphi_1, \varphi_2 \in \boldsymbol{C}^2(J)$ such that

$$\frac{\begin{vmatrix} \varphi_1'' & \varphi_2'' \\ \varphi_1 & \varphi_2 \end{vmatrix}}{W[\varphi_1, \varphi_2]} > 0$$. On the system $\boldsymbol{G}(F)$ let us define a binary operation by the following formula:

For an arbitrary pair of spaces $V(\varphi_1, \varphi_2) \in \boldsymbol{G}(F)$, $V(\psi_1, \psi_2) \in \boldsymbol{G}(F)$ let us set
$V(\varphi_1, \varphi_2) \cdot V(\psi_1, \psi_2) = V(\omega_1, \omega_2)$,
where $\{\omega_1, \omega_2\}$ is the base of the space $V(\omega_1, \omega_2)$ such that the function $y=\omega_1(x)$, $y=\omega_2(x)$, $x \in J$ form the fundamental system of solutions of the linear differential equation of the second order

$$y'' + \frac{D[\varphi_1,\varphi_2] \cdot D[\psi_1,\psi_2]}{W[\varphi_1,\varphi_2] \cdot W[\psi_1,\psi_2]} \cdot y' + \frac{D[\varphi_1,\varphi_2] \cdot W[\varphi_1',\varphi_2'] + W[\varphi_1',\varphi_2'] \cdot W[\psi_1,\psi_2]}{W[\varphi_1,\varphi_2] \cdot W[\psi_1,\psi_2]} \cdot y = 0$$

i.e. $y'' + p(x)\, y' + q(x)\, y = 0$, where

$$p(x) = \frac{D[\varphi_1,\varphi_2] \cdot D[\psi_1,\psi_2]}{W[\varphi_1,\varphi_2] \cdot W[\psi_1,\psi_2]}, \quad q(x) = \frac{D[\varphi_1,\varphi_2] \cdot W[\varphi_1',\varphi_2'] + W[\varphi_1',\varphi_2'] \cdot W[\psi_1,\psi_2]}{W[\varphi_1,\varphi_2] \cdot W[\psi_1,\psi_2]} .$$

Let us further denote $\boldsymbol{H}(F)$ as the subset of the set $\boldsymbol{G}(F)$ such that $V(\varphi_1, \varphi_2) \in \boldsymbol{H}(F)$ if and only if there applies

$$\begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) + \varphi_1''(x) & \varphi_2'(x) + \varphi_2''(x) \end{vmatrix} \equiv 0$$

on the interval J . We will prove the following Theorem:

Theorem: *Let $J \subset \boldsymbol{R}$ be an open interval. The system $(\boldsymbol{G}(F), \cdot)$ is a noncommutative group of vector spaces with a neutral element $V(\chi_1, e^{-x})$, in which $\boldsymbol{H}(F)$ is a normal subgroup.*

Proof: There is a bijective correspondence between structures (more precisely grupoids) $\boldsymbol{G}(F)$ and $\boldsymbol{LA}_2(J)$ because each operator $L(p, q) \in \boldsymbol{LA}_2(J)$ determines accurately one linear space $V(\varphi_1, \varphi_2)$ of the solution of the equation $L(p,q)(y) = 0$, where the pair of functions $\varphi_1, \varphi_2 \in \boldsymbol{C}^2(J)$ form the fundamental system of solutions of the above given solution, thus the basis $\{\varphi_1, \varphi_2\}$ of the space $V(\varphi_1, \varphi_2)$. Vice versa, to every space $V(\varphi_1, \varphi_2) \in \boldsymbol{G}(F)$ there exists an equation

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad , \tag{3}$$

whose solution space of dimension two is just the space $V(\varphi_1, \varphi_2)$. The coefficients in this given equation (3) are

$$p(x) = \frac{D[\varphi_1,\varphi_2]}{W[\varphi_1,\varphi_2]} , \quad q(x) = \frac{D[\varphi_1',\varphi_2']}{W[\varphi_1,\varphi_2]} .$$

Let us note that we can formally define the isomorphism of groups $\boldsymbol{LA}_2(J)$, $\boldsymbol{G}(F)$, in which the normal subgroup $\boldsymbol{L_1A}_2(J)$ of group $\boldsymbol{LA}_2(J)$ will map on the subgroup (let us denote it $\boldsymbol{H}(F)$) of the group $\boldsymbol{G}(F)$, formed by all vector spaces $V(\varphi_1, \varphi_2)$, whose bases are constructed according to Proposition 2 by fundamental solutions systems of equations:

$$y'' + y' + \frac{D[\varphi_1',\varphi_2']}{W[\varphi_1,\varphi_2]}\, y = 0 \quad ,$$

thus $D[\varphi_1,\varphi_2] \equiv W[\varphi_1,\varphi_2]$. The last condition can be represented in the form

$$\varphi_1\varphi_2' - \varphi_1'\varphi_2 \; - \varphi_1''\varphi_2 + \varphi_1\varphi_2'' \equiv 0 \text{ , i.e.}$$
$$\varphi_1(\varphi_2' + \varphi_2'') - \varphi_2(\varphi_1' + \varphi_1'') \equiv 0 \quad ,$$

so using the determinant we get this condition in the form

$$\begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x)+\varphi_1''(x) & \varphi_2'(x)+\varphi_2''(x) \end{vmatrix} = 0 \text{, for every number } x \in J . \qquad \square$$

The following example will be very easy, in order to illustrate all the studied concepts. We will use linear spaces generated by fundamental systems of solution of linear differential equations of the second order with constant coefficients.

Example. Let $J = R$, $\varphi_1(x) = e^x, \varphi_2(x) = e^{-2x}, \psi_1(x) = e^{2x}, \psi_2(x) = e^{3x}, x \in \boldsymbol{R}$. According to the above mentioned definition we set

$$V(\varphi_1 ,\varphi_2) \cdot V(\psi_1 ,\psi_2) = V(\omega_1 ,\omega_2) ,$$

where $\{\omega_1,\omega_2\}$ is the fundamental system of solutions of the linear differential equation which will be arranged according to the above mentioned instructions. The pairs of functions $\{\varphi_1,\varphi_2\}$, $\{\psi_1,\psi_2\}$ form fundamental systems of solutions of these differential equations

$$y'' + \; y' - 2y = 0 \;,$$
$$y'' - 5y' + 6y = 0 \;,$$

in the given order. Then $\{\omega_1 ,\omega_2\}$ is the fundamental system of solutions of the differential equation

$$y'' - 5y' + 4y = 0 \; . \tag{4}$$

The characteristic equation corresponding to this differential equation is in the form $\lambda^2 - 5\lambda + 4 = 0$, with roots $\lambda = 4, \lambda_2 = 1$. The linear space of the solutions of this differential equation is formed by all functions in the form

$$y(x) = c_1 e^{4x} + c_2 e^x, \quad x \in \boldsymbol{R} ,\; c_1, c_2 \in \boldsymbol{R},$$

so $\omega_1(x) = e^{4x}$, $\omega_2(x) = e^x$, $x \in \boldsymbol{R}$. Thus we got

$$V(e^x , e^{-2x}) \cdot V(e^{2x} , e^{3x}) = V(e^{4x} , e^x) .$$

Moreover we will demonstrate that e.g.

$$V(e^{4x} , e^x) \cdot V(1 , e^{-x}) = V(1 , e^{-x}) \cdot V(e^{4x} , e^x) .$$

Indeed, $V(e^{4x},e^x) \cdot V(1,e^{-x}) = V(\varphi_1,\varphi_2)$, where $\{\varphi_1,\varphi_2\}$ is the fundamental system of solutions of the differential equation $y'' - 5y' + 4y = 0$, which is the above given equation (4) with the fundamental solution system $\varphi_1(x) = e^{4x}$, $\varphi_2(x) = e^x$, $x \in \boldsymbol{R}$. Furthermore, if we denote $V(1 , e^{-x}) \cdot V(e^{4x} , e^x) = V(\psi_1 ,\psi_2)$, the pair of functions $\{\psi_1 ,\psi_2\}$ forms the fundamental system of solutions of the same differential equation (4). The space $V(1, e^{-x})$ is therefore the unit (neutral element) of the above constructed noncommutative group $(\boldsymbol{G}(F),\cdot)$.

REMARK ON HYPERGROUPS

Now let us briefly mention the possibility of the construction of hyperstructures on the set $\boldsymbol{LA}_2(J)$. Hypergroups were introduced by F. Marty in 1934. Since then they have been studied in connection with algebraic structures, geometric structures, the algebraic theory of automata, the theory of convexity, and certain combinatorial problems of the discrete mathematics as well. Hypergroups and their applications are dealt with in details for exemple in [5], [6], [10] and [14]. A *hypergroup* is a pair $(H, *)$, where H is a nonempty set, a binary hyperoperation $*$ is the mapping of the Cartesian product $H \times H$ into the system of all nonempty subsets of the set H (often denoted $P^{*}(H)$) which satisfies following two requirements:

1º associativity axiom: $a * (b * c) = (a * b) * c$ for every triad of elements $a, b, c \in H$ (here $a * M = \bigcup_{m \in M} a * m$ for every $a \in H$, $\phi \neq M$, $M \subset H$);

2º reproduction axiom: $a * H = H = H * a$ for every $a \in H$.

For any two nonempty subsets A, B of the set H let us define their superproduct as follows

$$A * B = \bigcup \{a * b;\, a \in A, b \in B\}.$$

A subhypergrupoid of the hypergroup $(H, *)$ is a pair $(S, *)$, where $S * S \subset S$. Let us note that the relation of incidence of nonempty sets A, B, i.e. $A \cap B \neq \phi$, is in mathematics literature concerning hyperstructures usually denoted as $A \approx B$. The hypergroup $(H, *)$ is called a *transposition hypergroup* or also a *join space* if it satisfies the transposition axiom:

For each tetrad $a, b, c, d \in H$ there follows $a * d \approx b * c$ from the relation $b\backslash a \approx c/d$, where sets

$$b\backslash a = \{x \in H;\, a \in b * x\},\quad c/d = \{x \in H;\, c \in x * d\}$$

are called the *left* and *right extensions* (sometimes also left and right fraction) respectively.

Now let us bring back from the previous text the definition of the binary operation on the set $\boldsymbol{LA}_2(J)$.

For an arbitrary pair of differential operators $L(p_1, q_1), L(p_2, q_2) \in \boldsymbol{LA}_2(J)$ let us set

$$L(p_1, q_1) \cdot L(p_2, q_2) = L(p_1 p_2,\, p_1 q_2 + q_1)$$

and $L(p_1, q_1) \leq L(p_2, q_2)$, if $p_1(x) = p_2(x)$, $q_1(x) \leq q_2(x)$ for any element $x \in J$.

There follows the Proposition about the existence of the hypergroup on the set $\boldsymbol{LA}_2(J)$.

Proposition 3. *Let $J \subset \boldsymbol{R}$ be an open interval; let*

$$\boldsymbol{LA}_2(J) = \{L(p, q): \boldsymbol{C}^2(J) \rightarrow C(J);\, [p, q] \in \boldsymbol{C}_+(J) \times \boldsymbol{C}(J)\}$$

be a set of all ordinary linear differential operators of the second order – i.e. there holds

$$L(p,q)(y) = y'' + p(x)y' + q(x)y = 0,\ y \in \boldsymbol{C}^2(J).$$

Let the binary operation $*$ on $\boldsymbol{LA}_2(J)$ be defined as follows

$$L(p_1, q_1) * L(p_2, q_2) = \{L(u_1, u_2) \in \boldsymbol{LA}_2(J);\, L(p_1, q_1) \cdot L(p_2, q_2) \leq L(u_1, u_2)\},$$

where $p_1 p_2 = u_1$, $p_1 q_2 + q_1 \leq u_2$, for an arbitrary pair of operators $L(p_1, p_2), L(q_1, q_2) \in \boldsymbol{LA}_2(J)$. Then the hypergroupoid $(\boldsymbol{LA}_2(J), *)$ is a noncommutative transposition hypergroup, i.e. a noncommutative join space.

Proof: Is given in the paper [8] on page 82.

Note: In the proof of Proposition 3 there are used mathematics theories which exceed the scope of this article (e.g. the ordered group theory). Moreover, is possible to expand the construction of hyperstructures to the set of linear spaces of smooth functions of dimension two, similarly as we did in the previous text while constructing groups. The appropriate constructions are quite tedious and will be dealt with in both authors' next articles.

CONCLUSION

This article is intended for university students as an example of possible connections among mathematics theories commonly presented to them during their university study. With this article both authors follow up the previous papers, especially the latter one (e.g. [1], [2], [7], [8], [9]). We proceeded from the theory of differential equations solutions, i.e. classical mathematics analysis. In the process of constructing the noncommutative group $(\boldsymbol{G}(F),\cdot)$ we obtained a certain binary operation on the set $\boldsymbol{G}(F)$ and specified some of its properties, which belongs to the theory of algebraic structures. Such a connection between mathematical analysis and algebra is not the only one. Based on the given procedures it is possible e.g. to form the construction of binary hyperstructures, which was shown in the closing remark/comment. The above discussed connection of the classical mathematical analysis and the theory of algebraic structures and hyperstructures can enable students to penetrate the mentioned areas of higher mathematics and thus can contribute not only to deepening their knowledge, but also to developing their mathematical thinking.

References

[1] Beránek, J., Chvalina, J. *From Groups of Linear Functions to Noncommutative Transposition Hypergroups*. Edition České Budějovice: University of South Bohemia, 1999. 10 pp. Department of Mathematics Report Series, volume 7. ISBN 80-7040-392-6.

[2] Beránek, J., Chvalina, J. *Invariantní podgrupy grup obyčejných lineárních diferenciálních operátorů druhého řádu.* In Acta Mathematica 13, Nitra: Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Slovakia, 2010. p. 43-47, 5 pp. ISBN 978-80-8094-781-1.

[3] Beran, L. *Grupy a svazy*. 1st ed. Praha: SNTL–The national publisher of technical literature, 1974. 358 pp. The Math. Sem. SNTL.

[4] Borůvka, O. *Základy teorie grupoidů a grup*. 1st ed. Praha: The publisher of Czech. Academy of Sciences, 1962. 216 pp.

[5] Corsini P. (1993). *Prolegomena of Hypergroup Theory*. Aviani Editore, USA.

[6] Hošková, S., Chvalina, J. (2009) *A survey of investigations of the Brno reseach group in the hyperstructure theory since the last AHA Congress*, In: 10th International Congress on Algebraic Hyperstructures and Applications: Brno 2008, Czech Republic, Proceedings of AHA 2008, Published 2009, ISBN: 978-80-7231-688-5, p. 71-84.

[7] Chvalina, J., Moučka, J. *Action of Join Spaces of Continuous Functions on Hypergroups of Second-Order Linear Differential Operators*. In: 6th International Math. Workshop. Brno: FAST VUT Brno, 2007. 10 pp.

[8] Chvalina, J., Chvalinová, L. *Join Spaces of Linear Ordinary Differential Operators of the Second Order*. Edition Brno: MU, 2003. 10 pp. Folia Fac. Sci. Nat. Univ. Masarykianae Brunensis, Mathematica 13, p. 77 – 86.

[9] Chvalina, J., Chvalinová, L. *Solvability of certain groups of second-order linear differential operators.* In Kováčová, Monika. Proc. of the 10-th International Conference APLIMAT. Bratislava: Faculty of Mechanical Engin., Slovak University of Technology, 2011. p. 113-119, 7 pp. ISBN 978-80-89313-51-8.

[10] Chvalina, J., Hošková-Mayerová, S. (2012). *General omega-hyperstructures and certain applications of those*, Ratio Mathematica, 23, p. 3 - 20.

[11] Kalas, J., Ráb, M. *Obyčejné diferenciální rovnice.* 2nd ed. Brno: Masaryk University in Brno, 2001. 207 pp. ISBN 80-210-2589-1.

[12] Kuroš, A. G. *Teorija grupp*. 3rd ed. Moskva: Nauka 1967. 648 pp.

[13] Neuman, F. *Global Properties of Linear Ordinary Differential Equations*. 1st ed. Praha: Academia, 1991. 320 pp. ISBN 80-200-0423-8.

[14] Novák, M. *EL–semihypergroups in which the quasi-ordering is not antisymmetric.* In: Baštinec, J., Hrubý, M. Mathematics, Information Technologies and Applied Sciences 2017 (MITAV), Post-Conference Proceedings of Extended Versions of Selected Papers. Brno: University of Defence in Brno, 2017. p. 183-192. ISBN 978-80-7582-026-6.

DEA-BASED METHOD OF STEPWISE BENCHMARKING – THE CASE OF SLOVAK ECONOMIC FACULTIES

Patrik Böhm, Gabriela Böhmová
FPEDAS, Žilinská univerzita v Žiline,
Univerzitná 8215/1, 010 26 Žilina
patrik.bohm@fpedas.uniza.sk, gabriela.bohmova@fpedas.uniza.sk

Abstract: *Assessment of higher education institutions is systemic process that uses empirical data to evaluate the study programs. The aim of the assessment is not only to evaluate the scientific outcomes and compare it with other institutions, but also to find the target values of scientific outcomes. In this paper, we use data envelopment analysis (DEA) method based on the linear programming to evaluate the efficiency of selected faculties in Slovakia. Data envelopment analysis is non-parametric, linear programming based method to assess the efficiency or performance of the homogeneous decision-making units (DMU) considering multiple inputs and outputs. It divides the set of examined DMUs into two parts: efficient and inefficient units. For each inefficient unit a reference target is calculated together with corresponding efficiency gap. However, it may not be feasible for inefficient DMU to achieve the efficiency of its reference target in one step. Various methods of stepwise benchmarking were proposed in the literature. In this paper, we evaluate target selection methods for assessment of the efficiency of selected faculties in Slovakia.*

Keywords: DEA analysis, stepwise benchmarking, efficiency.

1 INTRODUCTION

Efficiency evaluation of government spending is of great importance at a time when many states are facing the tight budgets. Because of the rising education demand and public resource constraints, politicians have focused on the performance of their education systems. Although the study of economics of education has a long tradition, performance comparison has always been a difficult task. The assessment of education systems, especially higher education institutions (HEI), has a long tradition in the UK and the US. In Slovakia, the assessment is much younger, the first assessment of HEI was issued in 2005. Since then, the Academic Ranking and Rating Agency (ARRA) each year addresses the evaluation of research and educational performance of higher education institutions [1].

Experience from Slovakia and many other countries [2] shows that the assessment of the quality of HEI is a good instrument for achieving improvements in the system efficiency [3]. However, there are many other aims of evaluation of efficiency, mainly to provide the general public with an overview of easy-to-understand criteria that will help students to choose suitable school for their study. Another ambition is to initiate the competition between institutions and thus to start changes at Slovakia's HEI.

The assessment of HEI is based on two main areas, the outcomes of research and education. ARRA selected a number of indicators and divided them into five groups: teachers and students, applications for study, publications and citations, PhD. studies and grant successes. The evaluation is based on a relative rating scale system using the highest values of indicators as the benchmark. As a result, the institution's overall score is given by the average number of score of all five groups.

2 RELATED WORK

In this paper we apply DEA-based method of stepwise benchmarking to ARRA data and show how inefficient faculties should improve their outputs.

2.1 Data Envelopment Analysis

Data Envelopment analysis (DEA) is one of the important means of economic management. It is a method for the comparative efficiency assessments in contexts where multiple homogeneous units deliver goods or services [4]. It assumes the existence of a convex production frontier, which is constructed using linear programming methods and is called the best-practice or efficient frontier. The first CCR model was published in 1978 by Charnes, Cooper and Rhodes [5]. The CCR model can be written in the form below. Both primary and dual models (called multiplication and envelopment models) are shown.

Primary model	**Dual model**
$Max \quad \theta_o = \sum_{r=1}^{s} u_r\, y_{rk}$	$Min \quad \theta$
$s.t.$	$s.t.$
$\sum_{i=1}^{m} v_i x_{ik} = 1$	$\sum_{j=1}^{m} x_{ij}\lambda_j \leq \theta \cdot x_{ik};\;\; i = 1,2,\dots m$
$\sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0;\;\; j = 1,\dots,n$	$\sum_{j=1}^{n} y_{rj}\,\lambda_j \geq y_{rk};\;\; r = 1,2,\dots s$
$u_r\,, v_i \geq \varepsilon;\;\; r = 1,\cdots,s;\; i = 1,\cdots,m$	$\lambda_j \geq 0;\;\; j = 1,2,\dots n$

Here u_r, v_i, y_{rj}, and x_{ij} are the weights given to the *r*-th output, the weights given to the *i*-th input, amount of the *r*-th output and amount of the *i*-th input, respectively. Number of DMUs is denoted by *n*, *k* is the DMU under evaluation and *s* and *m* is the number of outputs and inputs, respectively.

A great variety of other models were proposed in the literature. All models use the same idea, however. The set of evaluated units called Decision Making Units (DMU) is divided by analysis into two parts: efficient and inefficient DMUs. For inefficient units, it is necessary to identify the factors that mostly influence their ineffective behaviour. For each inefficient unit, a reference set of benchmark targets is calculated together with the efficiency gap – the degree to which the DMU's outputs should be improved in order to reach efficiency. The reference set is obtained by dual model, where λ_j are dual variables. The linear combination of DMUs, which produces more output than the evaluated DMU while utilizing the same amount of input, is called the target or benchmark unit. Dual variables are coefficients for this linear combination. The results of the analysis then lead to the elimination of sources of inefficiency and to the increase of the overall efficiency of the unit.

2.2 DEA Stratification Model

One of the issues frequently addressed in the literature is that it may not be feasible for an inefficient DMU to achieve its target's efficiency in a single step – especially if the DMU is far from the efficient frontier [6]. To resolve this problem, several methods have been proposed in the literature. The common idea behind these methods is to create intermediate, more easily achievable targets, usually called local targets. Seiford & Zhu [7] proposed an algorithm to remove DMU's that lie on the original best-practice frontier to allow the remaining inefficient DMU's to create a new, second-level best-practice frontier. If DMUs that lie on this new second-level frontier are removed, a third-level best-practice frontier can be formed, etc., until there is no DMU left.

Algorithm: Determination of the *l*th-level best-practice frontier	
Step 1:	Set l=1. Evaluate DEA CCR model to J^l to obtain first-level frontier DMUs. Set first-level best-practice frontier to E^l.
Step 2:	Exclude the frontier DMUs from future DEA runs, set $J^{l+1} = J^l - E^l$. If $J^{l+1} = \varnothing$, then stop.
Step 3:	Evaluate the new subset of inefficient DMUs, J^{l+1}, by DEA CCR model to obtain a new set of efficient DMUs E^{l+1}.
Step 4:	Let $l = l + 1$ and go to Step 2.

Source: Seiford & Zhu [7]

This paper focuses on how to choose intermediate targets for inefficient units. Alirezaee & Afsharian [8] proposed a layer measurement method that provides a strategy for moving into a better layer. It lacks the information on how to choose the target DMU on each layer, however. Hong et al. [9] developed a method for classifying new DMUs into layers. They also generated a path for improving the efficiency of inefficient units, but it doesn't guarantee that it will reach the first layer. Estrada et al. [10] proposed a method of stepwise benchmarking using proximity-based target selection. In Sharma & Yu [11], data mining and DEA are fused to provide a diagnostic tool for effectively measuring the efficiency of inefficient terminals and to prescribe the step-wise projection to reach the frontier.

These methods do not address intermediate target values for particular inefficient DMU in detail. Several additional criteria were subsequently proposed in the literature that solve this impracticality problem. For example, in 2012, a target selection based on the preference structure, direction and similarity was proposed in Park, Bae, & Lim [6]. They construct a benchmarking path network based on benchmarking candidate DMUs. The optimal benchmarking path is searched based on the benchmarking objectives measured by resource improvement and the improvement direction proximity between each benchmarking candidate DMU. Park & Sung [12] proposed a new approach that integrates cross-efficiency DEA, K-means clustering and context-dependent DEA methods to minimize the resource improvement pattern inconsistency in the selection of the intermediate benchmark targets of an inefficient DMU. Ramón, Ruiz & Sirvent [13] proposed a two-step benchmarking approach within the spirit of a context-dependant DEA that minimizes the distance to the DEA efficient frontier. Three-stage performance modeling using DEA and back propagation neural network was proposed by Kwon et al. [14] and used for evaluation of the healthcare industry. Super-efficiency DEA with stepwise improvement model was applied by Suzuki & Nijkamp [15] in the context of an efficiency-improvement plan for inefficient global cities. All of these studies used various algorithms for determination of intermediate targets for inefficient units in the stepwise DEA method.

3 PROPOSED TARGET SELECTION METHOD

In this paper, we propose a simple method of the intermediate target selection for inefficient DMU. Its main advantage over methods mentioned in the literature survey is its simplicity, as it does not require any other computational methods apart from the DEA method. As we stated in the previous chapter, for each inefficient unit DEA calculates the target values – the values the DMU has to achieve to become efficient. Target values form a new, "virtual" unit, calculated as a linear combination of efficient DMUs. This target unit will be used together with other DMUs as input for DEA analysis in the upper level.

Table 1 shows the output values used in the assessment of economic faculties in Slovakia.

	Input	Education	Attractiveness	R&D	Doctoral study	Grant success
1. **Faculty of Economics** **Technical University of Košice (EF TUKE)**	1	70	69	86	88	98
2. Faculty of National Economy **University of Economics (NHF EU)**	1	84	58	33	58	42
3. Faculty of Economics and Management **Slovak University of Agriculture (FEM SPU)**	1	64	49	58	48	25
4. Faculty of Economics **Matej Bel University (EF UMB)**	1	72	55	48	55	13
5. Faculty of Business Economics **University of Economics (PHF EU)**	1	68	31	44	80	16
6. Faculty of Business Management **University of Economics (FPM EU)**	1	76	57	28	65	10
7. FPEDAS **University of Žilina (FPEDAS)**	1	76	61	14	62	13
8. Faculty of Economic Informatics **University of Economics (FHI EU)**	1	80	57	32	45	8
9. Faculty of Management **Comenius University (FM UK)**	1	56	79	29	45	14
10. Faculty of Economics **Pan-European University (FEP PEVŠ)**	1	88	59	29	37	1
11. Faculty of Business **University of Economics (OF EU)**	1	73	47	22	44	11
12. Faculty of Management **University of Prešov (FM PU)**	1	55	50	28	31	17
13. College of International Business **ISM Slovakia in Prešov (ISM PO)**	1	64	45	1	1	8
14. Faculty of Economics **J. Selye University (EF UJS)**	1	58	31	15	1	3

Table 1. Data used for evaluation of efficiency of economic faculties
Source: ARRA ranking 2015

We apply the stratification DEA model in context-dependent DEA to stratify the faculties into four levels. Set J^1 contains all faculties. Level 1 contains four faculties that lie on the global best-practice frontier: $E^1 = \{\text{EF TUKE, NHF EU, FM UK, FEP PEVŠ}\}$. These faculties work efficiently. After removing these DMUs from J^1, the remaining units form the set $J^2 = \{\text{FEM SPU, EF UMB, PHF EU, FPM EU, FPEDAS ŽU, FHI EU, OF EU, FM PU, ISM PO, EF UJS}\}$. They are evaluated using DEA again. Six faculties lie at the local best-practice frontier, so $E^2 = \{\text{FEM SPU, EF UMB, PHF EU, FPM EU, FPEDAS ŽU, FHI EU}\}$. New subset of ineffective units is $J^3 = \{\text{OF EU, FM PU, ISM PO, EF UJS}\}$. After removing these six DMUs, the remaining four faculties are evaluated. We get $E^3 = \{\text{OF EU, FM PU}\}$ and the remaining two inefficient faculties are $J^4 = \{\text{ISM PO, EF UJS}\}$.

Results for all four levels are shown in Table 2.

	Level 1	Level 2	Level 3	Level 4
Faculty	1. EF TUKE 2. NHF EU 9. FM UK 10. FEP PEVŠ	3. FEM SPU 4. EF UMB 5. PHF EU 6. FPM EU 7. FPEDAS ŽU 8. FHI EU	11. OF EU 12. FM PU	13. ISM PO 14. EF UJS

Table 2. Stratification DEA model results
Source: own calculations

Next, we find intermediate target values for one selected inefficient DMU. We selected ISM PO faculty that lies in the Level 4. Target values were already calculated in the 3rd DEA run and are listed in Table 3 as unit "*ISM PO Target 2*". This new virtual unit is added to the set J^2 and the DEA method is applied to a set of 11 DMUs. "*ISM PO Target 2*" virtual unit is inefficient and the intermediate target values for this unit are calculated. This new virtual unit is denoted by "*ISM PO Target 1*" and is consequently added to the set $J^1$and the last DEA test is applied. Target values for "*ISM PO Target 1*" unit are denoted as "*ISM PO Target*".

The output values of all intermediate targets together with the real output values of ISM PO are displayed in Table 3 and Figure 1. For the ISM PO management, this list gives the output indicators values that can be gradually used as their target values.

	Education	Attractiveness	R&D	Doctoral study	Grant
ISM PO 2015	64,0	45,0	1,0	1,0	8,0
ISM PO Target 2	68,0	47,8	23,7	40,4	12,7
ISM PO Target 1	74,8	57,9	26,0	55,5	13,9
ISM PO Target	80,6	62,4	50,5	59,8	44,6

Table 3. Intermediate target values
Source: own calculations

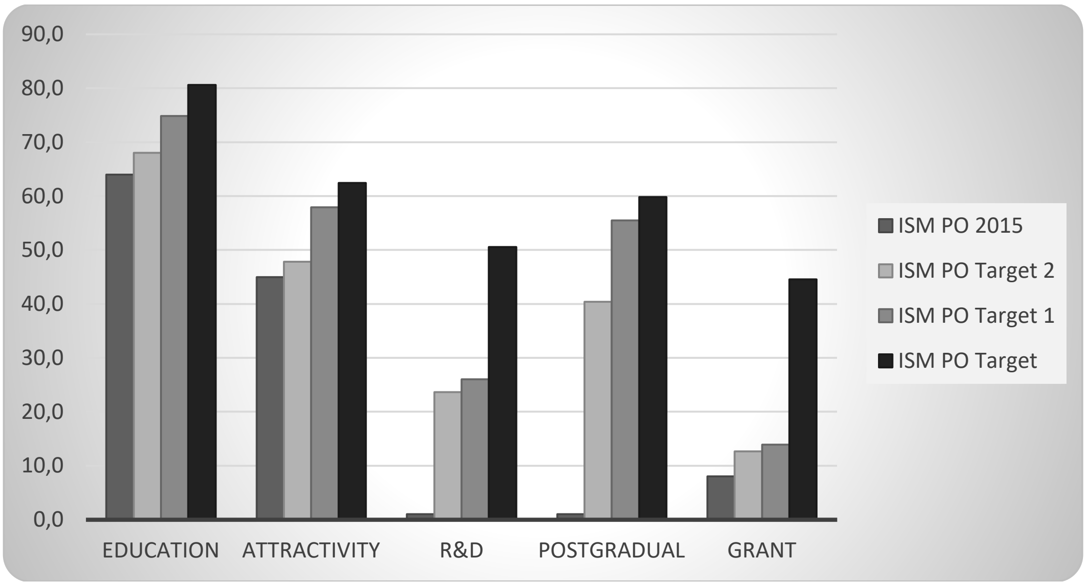


Fig. 1. Intermediate target values
Source: own

4 CONCLUSION

In this paper, we presented the stratification DEA method together with the path-finding algorithm. It computes intermediate target values for inefficient units. We evaluated 14 economic faculties in Slovakia. We divided them into 4 levels using the DEA stepwise benchmarking method. For one selected inefficient faculty, we determined intermediate target values that create a path of sequential 3-step increase of the faculty's efficiency.

Our approach differs from previously published papers in the algorithm used for searching the path for improving the efficiency of inefficient units. Unlike previous research, our method does not require the use of additional algorithms or software, but uses multiple instances of DEA analysis. In each step, the new target value is calculated using benchmark virtual units.

To evaluate the efficiency of 14 economic faculties in Slovakia, we used data collected by the Academic Ranking and Rating Agency. In order to increase the relevance of the evaluation, it would be appropriate to include other quality indicators in future research. For example, the ARRA rating does not include the use of information and communications technologies. Modern teaching tools such as e-learning or Moodle are used not only as tools for knowledge [16], but also for increasing the interest of youth in science and technology [17], which ultimately improves the quality of education.

References

[1] ARRA, Report 2015, Accessed 17 January, 2017. [Online]. [Retrieved 2018-03-10]. Available at: http://www.arra.sk/sites/arra.sk/files/file/ARRA_Sprava_2015.pdf

[2] Böhm, P, Böhmová, G. (2016). Application of stratification DEA method in efficiency evaluation of the education sector in an international perspective, *Globalization and its Socio-Economic Consequences, 16th international scientific conference proceedings, PTS I-V*, 230-237

[3] Böhm, P, Vojtekova, M. (2018). Evaluation of research and educational performance of selected faculties in Slovakia, *INTED 2018 Proceedings*, pp 6263-6270. doi: 10.21125/inted.2018.1475
[4] Thanassoulis, E. (2001). Introduction to the Theory and Application of Data Envelopment Analysis. Springer Science & Business Media New York
[5] Charnes, A., Cooper, W., Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research* 2, 429-444
[6] Park, J., Bae, H. and Lim, S. (2012). A DEA-based method of stepwise benchmark target selection with preference, direction and similarity criteria. *International Journal of Innovative Computing, Information and Control*, 8(8), 5821-5834.
[7] Seiford, L. M., Zhu, J. (2003) Context-dependant data envelopment analysis. *Measuring attractiveness and progress*. OMEGA, 31(5), 397-408
[8] Alirezaee, M., Afsharian, M. (2007). Model improvement for computational difficulties of DEA technique in the presence of special DMUs. *Applied Mathematics and Computation*, 186(2), 1600-1611. doi:10.1016/j.amc.2006.08.067
[9] Hong, H. K., Ha, S. H., Shin, C. K., Park, S. C., and Kim, S. H. (1999). Evaluating the efficiency of system integration projects using data envelopment analysis (DEA) and machine learning. *Expert Systems with Applications*, 16(3), 283-296. doi:10.1016/s0957-4174(98)00077-3
[10] Estrada, S. A., Song, H. S., Kim, Y. A., Namn, S. H., & Kang, S. C. (2009). A method of stepwise benchmarking for inefficient DMUs based on the proximity-based target selection. *Expert Systems with Applications*, 36(9), 11595-11604. doi:10.1016/j.eswa.2009.03.035
[11] Sharma, M. J., Yu, S. J. (2009). Performance based stratification and clustering for benchmarking of container terminals. *Expert Systems with Applications*, 36(3), 5016-5022. doi:10.1016/j.eswa.2008.06.010
[12] Park, J., Sung, S. (2016). Integrated Approach to Construction of Benchmarking Network in DEA-Based Stepwise Benchmark Target Selection. *Sustainability*, 8(7), 600. doi:10.3390/su8070600
[13] Ramón, N., Ruiz, J. L., & Sirvent, I. (2018). Two-step benchmarking: Setting more realistically achievable targets in DEA. *Expert Systems with Applications*, 92, 124-131. doi:10.1016/j.eswa.2017.09.044
[14] Kwon, H., Marvel, J. H., & Roh, J. J. (2017). Three-stage performance modeling using DEA–BPNN for better practice benchmarking. *Expert Systems with Applications*, 71, 429-441. doi:10.1016/j.eswa.2016.11.009
[15] Suzuki, S., & Nijkamp, P. (2017). Performance Measurement of Global Cities: Combination of a Stepwise Improvement Model with an SE Model. *New Frontiers in Regional Science: Asian Perspectives Regional Performance Measurement and Improvement*, 101-115. doi:10.1007/978-981-10-0242-7_8
[16] Kordek, D. et al. (2017). The LMS Moodle and the Moodle Mobile Application in Educational Process of Biophysics. *Mathematics, Information Technologies and Applied Sciences 2017, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2017, p. 171-176. [Online]. [Retrieved 2018-03-10]. Available at: http://mitav.unob.cz/data/MITAV%202017%20Proceedings.pdf
[17] Stuchlikova, L. et al. (2017). Engineering Education and Science & Technology Popularization Among Youngsters Supported by IT. *Mathematics, Information Technologies and Applied Sciences 2017, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2017, p. 226-234. [Online]. [Retrieved 2018-03-10]. Available at: http://mitav.unob.cz/data/MITAV%202017%20 Proceedings.pdf

COMPARISON BETWEEN GMRES AND THE METHOD OF CONJUGATE GRADIENTS FOR THE NORMAL EQUATIONS IN AN EFFICIENCY OF THEIR USE

František Bubeník, Petr Mayer
Faculty of Civil Engineering, Czech Technical University in Prague
Thákurova 7, 166 29 Praha 6, Czech Republic
Frantisek.Bubenik@cvut.cz, Petr.Mayer@cvut.cz

Abstract: *We solve $Ax = b$, where A is an arbitrary real matrix. If A is a square invertible matrix there is possible to apply the GMRES method which is considered to be the best method for solving such problems. The use of the normal equations $A^T Ax = A^T b$ is an alternative approach but it is not recommended. One of the reasons is the fact that $cond_2(A^T A) = cond_2(A)^2$. But this reason is possible to be partly eliminated by using of a better arithmetic. A priority of the normal equations is the fact that the problem is always solvable even if A is a rectangular matrix. And it is possible to apply the classical gradient methods to this problem. Some convergent properties of both approaches are compared for various classes of matrices.*

Keywords: systems of linear equations, normal equations, convergent properties, iterative methods.

INTRODUCTION

Consider a system of linear equations

$$Ax = b. \tag{1}$$

This system is intended to be solved by methods of the Krylov type. If A is a symmetric positive definite matrix then the method of the first priority to be applied is the method of conjugate gradients. If A is not such a matrix then we mostly apply the GMRES method or some of its variants.

In this paper we suggest the method of conjugate gradients applied to the system of normal equations as an alternative way to solve the problem (1). We show that in a series of cases it means a competitive alternative. The system of normal equations

$$A^T Ax = A^T b \tag{2}$$

is always solvable and A is always symmetric positive semidefinite and if moreover A is invertible then A is even definite. As a negative phenomenon there is considered the fact that

$$cond_2(A^T A) = cond_2(A)^2.$$

This fact can make, if using an insufficiently accurate arithmetic, the way through the normal equations unacceptable. There is a positive fact that $A^T A$ is always symmetric positive definite and some types of methods, for example Gauss-Seidel, the steepest descent or conjugate gradients, converge to some solution of the problem. Matrices and their properties are in [1].

1 ALGORITHMS AND THEIR FUNDAMENTAL PROPERTIES

The method of conjugate gradients was published for the first time in [3]. We introduce basic algorithms which are taken from publication [4].

Algorithm 1 *Conjugate Gradient*

1. *Compute* $r_0 := b - A\,x_0$, $p_0 := r_0$
2. *For* $j = 0, 1, \ldots,$ *until convergence Do:*
3. $\alpha_j := (r_j, r_j)/(Ap_j, p_j)$
4. $x_{j+1} := x_j + \alpha_j p_j$
5. $r_{j+1} := r_j - \alpha_j A p_j$
6. $\beta_j := (r_{j+1}, r_{j+1})/(r_j, r_j)$
7. $p_{j+1} := r_{j+1} + \beta_j p_j$
8. *EndDo*

We apply the Conjugate Gradient algorithm to the normal equations. The resulting algorithm for computation is as follows.

Algorithm 2 *Conjugate Gradient and Normal Equations*

1. *Compute* $r_0 := b - A\,x_0$, $z_0 := A^T r_0$, $p_0 := z_0$
2. *For* $j = 0, 1, \ldots,$ *until convergence Do:*
3. $w_i := Ap_i$
4. $\alpha_i := ||z_i||^2/||w_i||_2^2$
5. $x_{i+1} := x_i + \alpha_i p_i$
6. $r_{i+1} := r_i - \alpha_i w_i$
7. $z_{i+1} = A^T r_{i+1}$
8. $\beta_i := ||z_{i+1}||_2^2/||z_i||_2^2$
9. $p_{i+1} := z_{i+1} + \beta_i p_i$
10. *EndDo*

The basic GMRES algorithm can be described as follows.

Algorithm 3 *GMRES*

1. *Compute* $r_0 := b - A\,x_0$, $\beta := ||r_0||_2$, $v_1 := r_0/\beta$
2. *Define the* $(m+1) \times m$ *matrix* $\widetilde{H}_m = \{h_{ij}\}_{1 \le i \le m+1, 1 \le j \le m}$. *Set* $\widetilde{H}_m := 0$.
3. *For* $j = 1, 2, \ldots, m$, *Do:*
4.   *Compute* $w_j := Av_j$
5.   *For* $i = 1, \ldots, j$, *Do:*
6. $h_{ij} := (w_j, v_i)$
7. $w_j := w_j - h_{ij} v_i$
8.   *EndDo*
9. $h_{j+1,j} := ||w_j||_2$. *If* $h_{j+1,j} = 0$ *set* $m := j$ *and go to 12*
10. $v_{j+1} := w_j/h_{j+1,j}$
11. *EndDo*
12. *Compute* y_m *the minimizer of* $||\beta e_1 - \widetilde{H}_m y||_2$ *and* $x_m := x_0 + V_m y_m$.

Notice that the Step 2 in the Algorithm 3 GMRES generates a matrix in an upper Hessenberg form. The matrix has at least $m^2/2$ nonzero elements (m is the number of steps in GMRES). With respect to the fact that GMRES may need up to n steps (n is the size of matrix A, it is $A \in R^{n\times n}$) and the speed of convergence generally does not depend on the eigenvalues of the matrix A (se the result in [2]), then the method becomes unusable for larger matrices.

2 BASIC CONVERGENCE ANALYSIS

Krylov subspaces are defined as

$$K_k(A, v) = span\{v, Av, A^2v, \dots, A^kv\}.$$

The significance of Krylov subspaces in connection with iterative methods consists in the following property. It follows from the Cayley-Hamilton theorem that there exists a polynomial p such that

$$A^{-1} = p(A)$$

and thus a solution of the system

$$x^* = A^{-1}b = p(A)b$$

is a linear combination of vectors $b, Ab, A^2b, \dots, A^kb$ ($k = \deg\ p$). Then it is in a linear hull of these vectors and then it is an element of $K_{k+1}(A, b)$.

Consider an iterative method with an initial approximation x_0. Then

$$r_0 = b - Ax_0$$

and thus

$$x^* = A^{-1}b \in x_0 + K_{k+1}(A, r_0).$$

For the $m-$th approximation we have

$$x_m \in x_0 + K_m(A, r_0)$$

and it means that

$$x_m = x_0 + p_m(A)r_0$$

(p_m is a polynomial of degree $m - 1$ characterizing a method chosen).
Denote

$$\varepsilon_m = x_m - x^*.$$

Then

$$x_m - x^* = x_0 - x^* + p_m(A)(Ax^* - Ax_0),$$
$$\varepsilon_m = \varepsilon_0 + p_m(A)A(-\varepsilon_0).$$

In other words

$$\varepsilon_m = q_m(A)\varepsilon_0,$$

where

$$q_m(x) = 1 - xp_m(x).$$

Then

$$\deg\ q_m = m,\ q_m(0) = 1.$$

We put

$$r_m = -A\varepsilon_m = -Aq_m(A)\varepsilon_0.$$

Then

$$||r_m||^2 = (-Aq_m(A)\varepsilon_0, -Aq_m(A)\varepsilon_0) = (Aq_m(A)\varepsilon_0, Aq_m(A)\varepsilon_0) \tag{3}$$

and the GMRES method minimizes the value of $||r_m||^2$ from (3).

The method of conjugate gradients with the normal equations minimizes the following functional

$$E(x) = x^T Ax - 2x^T b,$$

usually called a functional of energy. If we express it in terms of normal equations (2), what is, in fact, just the case of the conjugate gradients method applied to the normal equations, we get the functional of energy in the form

$$E(x) = x^T A^T Ax - 2x^T A^T b. \tag{4}$$

It means that the conjugate gradients with the normal equations minimize the functional $E(x)$ from (4) in a Krylov space

$$\widetilde{K}(A^T A, \widetilde{r_0}). \tag{5}$$

As to the notation used we accept the following general convention.

Convention. Symbols with the tilde character, for example $\widetilde{r_0}$ in (5), are related to the conjugate gradients method with the normal equations and symbols without the tilde relate to the GMRES method. A Krylov space without a subscript is supposed to be with the subscript as the greatest as possible. The convention for Krylov subspaces is also applied in (5).

If we use the notation

$$r = b - Ax,$$

we get

$$||r||^2 = (r, r) = (b - Ax, b - Ax) = x^T A^T Ax - 2x^T A^T b + (b^T, b) = E(x) + const. \tag{6}$$

We can see that the minimization of the energy functional $E(x)$ for $A^T A$ from (4) is as the same as the minimization of the residuum from (6). Both the methods minimize the same value but in different spaces.

This is a distinction between the methods, the GMRES minimizes the expression in

$$K_n(A, r_0),$$

while for the conjugate gradients the minimization is in

$$\widetilde{K_n}(A^T A, \widetilde{r_0}).$$

Lemma 1 *Let A be a symmetric matrix with a symmetric spectrum (it means that if $Av = \lambda v$ then also $Aw = -\lambda w$).*
Then there exists an initial condition x_0 (corresponding to GMRES) and $\widetilde{x}_0$ (corresponding to conjugate gradients) such that

$$q_{2m} = \widetilde{q}_m,$$

where q_{2m} is the error polynomial corresponding to GMRES method and $\widetilde{q}_m$ is the same for the conjugate gradients method with the normal equations.

Proof: According to the matrix symmetry we have that $A = A^T$. There exist $\lambda_1, \ldots, \lambda_n \in R$ and $v_1, \ldots, v_n \in R^n$ such that

$$Av_i = \lambda_i\, v_i,\ ||v_i|| = 1,\ v_i^T v_j = 0 \text{ for } i \neq j.$$

Put $\varepsilon_0 = \sum_{i=1}^{n} \alpha_i\, v_i$. Then in case of GMRES we get

$$r_m = -A\, q_m(A)\, \varepsilon_0 = \sum_{i=1}^{n} -\alpha_i\, \lambda_i\, q_m(\lambda_i)\, v_i \tag{7}$$

and then

$$||r_m||^2 = \sum_{i=1}^{n} \alpha_i^2\, \lambda_i^2\, q_m^2(\lambda_i).$$

It holds that $A^T A = A^2$. Put $\widetilde{\varepsilon_0} = \sum_{i=1}^{n} \beta_i\, v_i$. Then in case of the conjugate gradients method with the normal equations we get

$$\widetilde{r_m} = -A\, \widetilde{q_m}(A^T A)\, \widetilde{\varepsilon_0} = \sum_{i=1}^{n} -\beta_i\, \lambda_i\, \widetilde{q_m}(\lambda_i^2)\, v_i.$$

Then we have

$$||\widetilde{r_m}||^2 = \sum_{i=1}^{n} \beta_i^2\, \lambda_i^2\, \widetilde{q_m}^2(\lambda_i^2).$$

We compare both methods with the same initial approximation, in particular we choose the zero approximation. So that we have $x_0 = \widetilde{x_0}$ and denoting $r_0 = b - A\, x_0$ then there is always

$$\widetilde{K_k}(A^T A, A\, r_0) = \widetilde{K_k}(A^2, A\, r_0) \subset\subset K_{2k}(A, r_0)$$

(symbol $\subset\subset$ stands for a subspace), we can see this situation in Example 1. We can write r_0 as a linear combination of eigenvectors, i. e. $r_0 = \sum_{i=1}^{n} \gamma_i\, v_i$. The coefficients are $\gamma_i = -\alpha_i\, \lambda_i$, see expression r_m in (7) for $m = 0$. If, moreover, for any pair i, k such that $\lambda_i = -\lambda_k$ is $\alpha_i = \alpha_k$ then the polynomial q_m for GMRES consists only of even powers. Then the minimum is searched for only in $\widetilde{K_k}(A^2, A\, r_0)$, because the odd powers are out of use.

Then the first and the second approximations have the same residuals, similarly the third and the fourth approximations have the same residuals and all analogous next pairs have the same residuals. Therefore the $m-$th step for the conjugate gradients method with the normal equations is equivalent to the $2m-$th step of GMRES. This means that

$$\widetilde{q_m}(\lambda_i^2) = q_{2m}(\lambda_i)$$

and the lemma is proved.

From the algorithms presented above we can see that one iteration of the GMRES method consists of one multiplication of a matrix and a vector but one iteration of the conjugate gradients method applied to the normal equations consists of two such multiplications.

Then we can state that if we do not take into account the necessary process of making orthogonality in case of GMRES, we can state that both the methods are equally laborious.

Next example shows a problem where the conjugate gradients method applied to the normal equations and GMRES coincide.

Example 1 *Consider the problem* (1) *where*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{8}$$

Then the system of normal equations in (2) *has the following matrix as the matrix of the system and the following right hand side*

$$A^T A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}.$$

The computations are carried out with the same starting approximation for both methods, i. e.

$$x^{(0)} = \widetilde{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

Then the initial residuals are

$$r^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \widetilde{r}^{(0)} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}.$$

The Krylov space in which the GMRES method looks for the approximations is

$$K_6(A, r^{(0)}) = Span\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 4 \\ 16 \\ 16 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 8 \\ -8 \\ 64 \\ -64 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 16 \\ 16 \\ 256 \\ 256 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 32 \\ -32 \\ 1024 \\ -1024 \end{pmatrix}\right)$$

while the method of conjugate gradients with the normal equations looks for approximations in the Krylov space

$$\tilde{K}_3(A^T A, \tilde{r}^{(0)}) = Span\left(\begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 8 \\ -8 \\ 64 \\ -64 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 32 \\ -32 \\ 1024 \\ -1024 \end{pmatrix}\right).$$

Individual iterations for the GMRES method follows (the iterations are calculated directly from illustrative reasons, not using the Algorithm 3)

$$x^{(1)} = x^{(0)} + \alpha_1^{(1)} r^{(0)}, \quad r^{(1)} = b - Ax^{(1)} = b - A(x^{(0)} + \alpha_1^{(1)} r^{(0)}) = b - Ax^{(0)} - \alpha_1^{(1)} A r^{(0)} =$$
$$= r^{(0)} - \alpha_1^{(1)} A r^{(0)}.$$

Then

$$\begin{aligned} \left\| r^{(1)} \right\|^2 &= \left(r^{(1)}, r^{(1)}\right) = \left(r^{(0)} - \alpha_1^{(1)} A r^{(0)}, r^{(0)} - \alpha_1^{(1)} A r^{(0)}\right) = \\ &= \left(r^{(0)}, r^{(0)}\right) - 2\alpha_1^{(1)} \left(r^{(0)}, A r^{(0)}\right) + \left(\alpha_1^{(1)}\right)^2 \left(A r^{(0)}, A r^{(0)}\right) = \\ &= 6 - 0\alpha_1^{(1)} + 42 \left(\alpha_1^{(1)}\right)^2. \end{aligned}$$

Hence we get that $\alpha_1^{(1)} = 0$ *and then*

$$x^{(1)} = x^{(0)},$$

that is

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We get similarly that

$$
\begin{aligned}
x^{(2)} &= x^{(0)} + \alpha_1^{(2)} r^{(0)} + \alpha_2^{(2)} A r^{(0)}. \\
r^{(2)} &= b - Ax^{(2)} = b - A\left(x^{(0)} + \alpha_1^{(2)} r^{(0)} + \alpha_2^{(2)} A r^{(0)}\right) = \\
&= b - Ax^{(0)} - \alpha_1^{(2)} A r^{(0)} - \alpha_2^{(2)} A^2 r^{(0)} = r^{(0)} - \alpha_1^{(2)} A r^{(0)} - \alpha_2^{(2)} A^2 r^{(0)}. \\
\left\|r^{(2)}\right\|^2 &= \left(r^{(0)} - \alpha_1^{(2)} A r^{(0)} - \alpha_2^{(2)} A^2 r^{(0)}, r^{(0)} - \alpha_1^{(2)} A r^{(0)} - \alpha_2^{(2)} A^2 r^{(0)}\right) = \\
&= \left(r^{(0)}, r^{(0)}\right) - 2\left(\alpha_1^{(2)}, \alpha_2^{(2)}\right) \left(\begin{array}{c} \left(r^{(0)}, Ar^{0)}\right) \\ \left(r^{(0)}, A^2 r^{(0)}\right) \end{array} \right) + \\
&+ \left(\alpha_1^{(2)}, \alpha_2^{(2)}\right) \left(\begin{array}{cc} \left(Ar^{(0)}, Ar^{(0)}\right) & \left(Ar^{(0)}, A^2 r^{(0)}\right) \\ \left(A^2 r^{(0)}, Ar^{(0)}\right) & \left(A^2 r^{(0)}, A^2 r^{(0)}\right) \end{array} \right) \left(\begin{array}{c} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{array} \right) = \\
&= 6 - 2\left(\alpha_1^{(2)}, \alpha_2^{(2)}\right) \left(\begin{array}{c} 0 \\ 42 \end{array} \right) + \left(\alpha_1^{(2)}, \alpha_2^{(2)}\right) \left(\begin{array}{cc} 42 & 0 \\ 0 & 546 \end{array} \right) \left(\begin{array}{c} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{array} \right).
\end{aligned}
$$

It assumes a minimum for $\alpha_1^{(2)} = 0$ *and* $\alpha_2^{(2)} = \dfrac{42}{546}$ *and we get that*

$$
x^{(2)} = \left(\begin{array}{c} 7.692307692307693e-002 \\ -7.692307692307693e-002 \\ 1.538461538461539e-001 \\ -1.538461538461539e-001 \\ 3.076923076923077e-001 \\ -3.076923076923077e-001 \end{array} \right), \quad r^{(2)} = \left(\begin{array}{c} 9.230769230769231e-001 \\ 9.230769230769231e-001 \\ 6.923076923076923e-001 \\ 6.923076923076923e-001 \\ -2.307692307692308e-001 \\ -2.307692307692308e-001 \end{array} \right).
$$

Similarly for further iterations. We get for $x^{(3)}$

$$
\alpha^{(3)} = \left(\begin{array}{c} 0 \\ 7.6923e-002 \\ 0 \end{array} \right), x^{(3)} = \left(\begin{array}{c} 7.6923e-002 \\ -7.6923e-002 \\ 1.5384-001 \\ -1.5384e-001 \\ 3.0769e-001 \\ -3.0769e-001 \end{array} \right), r^{(3)} = \left(\begin{array}{c} 9.2307e-001 \\ 9.2307e-001 \\ 6.9230e-001 \\ 6.9230e-001 \\ -2.3076e-001 \\ -2.3076e-001 \end{array} \right).
$$

If we compare the iterations $x^{(2)}$ *and* $x^{(3)}$ *(the values are displayed in a different number of decimal places but in fact the values are equal) we can see that these two successive iterations again coincide, i. e.*

$$x^{(3)} = x^{(2)}.$$

We have for $x^{(4)}$ *that*

$$
\alpha^{(4)} = \left(\begin{array}{c} 0 \\ 0.37942 \\ 0 \\ -0.01985 \end{array} \right), \quad x^{(4)} = \left(\begin{array}{c} 0.35957 \\ -0.35957 \\ 0.60007 \\ -0.60007 \\ 0.24750 \\ -0.24750 \end{array} \right), \quad r^{(4)} = \left(\begin{array}{c} 0.640427 \\ 0.640427 \\ -0.200133 \\ -0.200133 \\ 0.010007 \\ 0.010007 \end{array} \right)
$$

and for $x^{(5)}$ that

$$\alpha^{(5)} = \begin{pmatrix} 0 \\ 0.37942 \\ 0 \\ -0.01985 \\ 0 \end{pmatrix}, \quad x^{(5)} = \begin{pmatrix} 0.35957 \\ -0.35957 \\ 0.60007 \\ -0.60007 \\ 0.24750 \\ -0.24750 \end{pmatrix}, \quad r^{(5)} = \begin{pmatrix} 0.640427 \\ 0.640427 \\ -0.200133 \\ -0.200133 \\ 0.010007 \\ 0.010007 \end{pmatrix}.$$

We can see again that

$$x^{(5)} = x^{(4)}.$$

For $x^{(6)}$:

$$\alpha^{(6)} = \begin{pmatrix} 0 \\ 1.31250 \\ 0 \\ -0.32812 \\ 0 \\ 0.01562 \end{pmatrix}, \quad x^{(6)} = \begin{pmatrix} 1 \\ -1 \\ 0.5 \\ -0.5 \\ 0.25 \\ -0.25 \end{pmatrix}, \quad r^{(6)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The residual $r^{(6)}$ contains all zeros and then $x^{(6)}$ is an exact solution of the problem.

Now we introduce calculations to the method of conjugate gradients applied to the normal equations. We apply the Algorithm 2 and get

$$\tilde{x}^{(1)} = \begin{pmatrix} 0.076923 \\ -0.076923 \\ 0.153846 \\ -0.153846 \\ 0.307692 \\ -0.307692 \end{pmatrix}, \quad \tilde{r}^{(1)} = \begin{pmatrix} 0.92308 \\ 0.92308 \\ 0.69231 \\ 0.69231 \\ -0.23077 \\ -0.23077 \end{pmatrix},$$

$$\tilde{x}^{(2)} = \begin{pmatrix} 0.35957 \\ -0.35957 \\ 0.60007 \\ -0.60007 \\ 0.24750 \\ -0.24750 \end{pmatrix}, \tilde{r}^{(2)} = \begin{pmatrix} 0.640427 \\ 0.640427 \\ -0.200133 \\ -0.200133 \\ 0.010007 \\ 0.010007 \end{pmatrix}, \tilde{x}^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 0.5 \\ -0.5 \\ 0.25 \\ -0.25 \end{pmatrix}, \tilde{r}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The residual $\tilde{r}^{(3)}$ consists of all zeros and then $\tilde{x}^{(3)}$ is an exact solution which coincides with $x^{(6)}$. Let us emphasize that the following equalities hold

$$x^{(2)} = \tilde{x}^{(1)}, \;\; x^{(4)} = \tilde{x}^{(2)}, \;\; x^{(6)} = \tilde{x}^{(3)}$$

and the fact corresponds to the situation in Lemma 1.

Lemma 2 *Let A be a skew-symmetric matrix.*

Then there exists an initial condition x_0 (corresponding to GMRES) and $\widetilde{x}_0$ (corresponding to conjugate gradients) such that

$$q_{2m} = \widetilde{q}_m,$$

where q_{2m} is the error polynomial corresponding to GMRES method and $\widetilde{q}_m$ is the same for the conjugate gradients method with the normal equations.

Proof: The proof is analogous to Lemma 1, however, here we use the symmetry of spectrum but with respect to the imaginary axis what follows from the skew-symmetry of the matrix.

Now we present a more general situation when the eigenvalues of the matrix A are neither real nor purely imaginary.

Theorem 1 *Let A be a matrix in the form*

$$A = Q^T D Q, \quad Q^T Q = I,$$

where D is a block diagonal matrix

$$D = \left(\begin{array}{ccc} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ & \dots & \\ 0 & 0 & D_{\frac{n}{2}} \end{array} \right). \tag{10}$$

The blocks are 2×2 matrices

$$D_j = \left(\begin{array}{cc} a_j & b_j \\ -b_j & a_j \end{array} \right).$$

Then the conjugate gradients method and the normal equations needs $n/2$ iterations to reach the exact solution and GMRES needs n iterations for the same. It means that both the processes consist of the same number of operations of the kind that a matrix is multiplied by a vector.

Proof: Let us determine the eigenvalues of the matrix A. The characteristic polynomial of a block D_j is

$$(a_j - \lambda_j)^2 + b_j^2 = 0.$$

Then the eigenvalues of the matrix D_J are

$$\lambda_{j,12} = a_j \pm i\, b_j.$$

The eigenvalues of D_j are also eigenvalues of A. Then we can write

$$A^T A = Q^T D^T Q\, Q^T D\, Q = Q^T D^T D\, Q.$$

The diagonal blocks of the matrix $D^T D$ are

$$D_j^T D_j = \left(\begin{array}{cc} a_j & -b_j \\ b_j & a_j \end{array} \right) \left(\begin{array}{cc} a_j & b_j \\ -b_j & a_j \end{array} \right) = \left(\begin{array}{cc} a_j^2 + b_j^2 & 0 \\ 0 & a_j^2 + b_j^2 \end{array} \right).$$

Thus the matrix $A^T A$ has all eigenvalues double and the Theorem is proved.

Now we introduce an example which illustrates the results of Lemma 2 and Theorem 1.

Example 2 *Consider the problem* (1) *with the same right hand side* b *as in Example 1, see* (8)*, but with the following matrices* A_1*,* A_2*,* A_3 *and* A_4 *as the matrices of the system. Let*

$$A1 = \begin{pmatrix} 1.00000 & 0.89443 & 2.92119 & -2.82843 & 3.00000 & -0.57735 \\ 2.68328 & -2.20000 & -0.65320 & -1.26491 & 1.34164 & -0.25820 \\ 1.46059 & 2.61279 & -1.80000 & -1.54919 & 1.64317 & -0.31623 \\ 3.77124 & 1.68655 & 2.06559 & 3.66667 & 4.47834 & 0.40825 \\ -1.66667 & -0.74536 & -0.91287 & -5.42115 & 3.33333 & 0.57735 \\ 0.57735 & 0.25820 & 0.31623 & -0.40825 & -0.57735 & 2.00000 \end{pmatrix}, \tag{11}$$

$$A2 = \begin{pmatrix} 0.33333 & 0.59628 & 2.55604 & 0.47140 & -0.33333 & -0.57735 \\ 2.38514 & -2.33333 & -0.81650 & 0.21082 & -0.14907 & -0.25820 \\ 1.09545 & 2.44949 & -2.00000 & 0.25820 & -0.18257 & -0.31623 \\ -0.47140 & -0.21082 & -0.25820 & 2.00000 & -0.70711 & 0.40825 \\ 0.33333 & 0.14907 & 0.18257 & 0.70711 & 2.00000 & 0.57735 \\ 0.57735 & 0.25820 & 0.31623 & -0.40825 & -0.57735 & 2.00000 \end{pmatrix}, \tag{12}$$

$$A3 = \begin{pmatrix} 0.00000 & -1.78885 & 1.46059 & 0.94281 & -0.66667 & -0.57735 \\ 1.78885 & 0.00000 & -3.26599 & 0.42164 & -0.29814 & -0.25820 \\ -1.46059 & 3.26599 & 0.00000 & 0.51640 & -0.36515 & -0.31623 \\ -0.94281 & -0.42164 & -0.51640 & 0.00000 & -1.41421 & 0.40825 \\ 0.66667 & 0.29814 & 0.36515 & 1.41421 & 0.00000 & 0.57735 \\ 0.57735 & 0.25820 & 0.31623 & -0.40825 & -0.57735 & 0.00000 \end{pmatrix}, \tag{13}$$

$$A4 = \begin{pmatrix} 1.00000 & -1.78885 & -0.36515 & -2.35702 & -0.33333 & 1.73205 \\ 0.00000 & 2.60000 & -2.12289 & -1.05409 & -0.14907 & 0.77460 \\ -1.82574 & 1.14310 & 2.40000 & -1.29099 & -0.18257 & 0.94868 \\ 0.47140 & 0.21082 & 0.25820 & -1.33333 & 3.06413 & -1.22474 \\ -2.33333 & -1.04350 & -1.27802 & -1.17851 & -0.66667 & -1.73205 \\ -1.73205 & -0.77460 & -0.94868 & 1.22474 & 1.73205 & 2.00000 \end{pmatrix}. \tag{14}$$

These matrices have different properties with respect to eigenvalues. Let us describe them.

Matrix A_1 *possesses three distinct pairs of complex conjugate eigenvalues.*

Matrix A_2 *possesses a pair of double eigenvalues and another different pair of eigenvalues.*

Matrix A_3 *is a skew-symmetric and possesses three pairs of purely imaginary eigenvalues.*

Matrix A_4 *possesses three pairs of complex conjugate eigenvalues, generally different but all are of the same absolute value.*

All the matrices A_1 *to* A_4 *are orthogonally similar to a block diagonal matrix* D *as in* (10).

Suppose again, as in Example 1, the same zero initial approximation, see (9). *We compare the residuals corresponding to six iterations of both methods individually for each of the matrices* A_1 *in* (11)*,* A_2 *in* (12)*,* A_3 *in* (13) *and* A_4 *in* (14). *It is seen that the sixth iteration reaches an exact solution.*

The resulting residuals are collected in the following Tab. 1. The symbol GMR stands fo the GMRES method and CGN for the method of conjugate gradients applied to the normal equations.

it	A1-GMR	A1-CGN	A2-GMR	A2-CGN	A3-GMR	A3-CGN	A4-GMR	A4-CGN
0	2.4495	2.4495	2.4495	2.4495	2.4495	2.4495	2.4495	2.4495
1	2.1846	1.0460	1.6687	0.79220	2.4495	1.4699	2.3482	0
2	1.2031	0.45759	1.6107	0	1.4699	0.70491	2.0872	0
3	1.1700	0	1.5178	0	1.4699	0	1.8930	0
4	1.1025	0	0	0	0.70491	0	0.71818	0
5	0.8003	0	0	0	0,70491	0	0.45197	0
6	0	0	0	0	0	0	0	0

Tab. 1. *The process of convergence*
Source: own

CONCLUSION

In this paper an alternative way for solving the systems of linear equations is proposed. The main contribution of the paper is that there are shown situations where the method of conjugate gradients applied to the system of normal equations is comparable to GMRES. Moreover, there are introduced classes of problems where the application of normal equations is even better then GMRES. The normal equations method is significantly easier then GMRES. It belongs to the significant added values of the authors.

References

[1] Golub, Gene H., Van Loan, Charles F. Matrix Computations. The Johns Hopkins University Press, Baltimor, Maryland, 2013.

[2] Greenbaum, A., Pták, V., Strakoš, Z. Any nonincreasing convergence curve is possible for GMRES. *SIAM J. Matrix Anal. Appl.*, 17 (1996).

[3] Hestenes, M. R., Stiefel, E. L. Methods of conjugate gradients for solving linear systems. *Journal of Research of the National Bureau of Standards*, Section B, 1952.

[4] Saad, J. Iterative Methods for Sparse Linear Systems. PWS Publishing Company, Boston, 1996.

GLOBAL SOLUTIONS TO MIXED-TYPE NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Josef Diblík, Gabriela Vážanová
The Faculty of Electrical Engineering and Communication Brno University of Technology,
Technická 3058/10, 61600, Brno, Czech Republic.
Email: diblik@feec.vutbr.cz, xvincu00@stud.feec.vutbr.cz

Abstract: *In this paper the criteria for existence of global solutions to nonlinear mixed-type functional differential equations are formulated. To prove the criteria, Schauder-Tychonoff fixed point theorem is used. A linear variant of derived results are given. Illustrative examples are considered as well.*

Keywords: global solution, mixed-type functional differential equations, nonlinear and linear equations, Schauder-Tychonoff fixed point theorem.

INTRODUCTION

A lot of papers deal either with delayed differential equations or advanced differential equations. We can refer, for example, to papers by Diblík [1], Diblík and Kúdelčíková [2], Pituk and Röst [6] or Diblík and Koksch [3].

The topic of mixed-type functional differential equations has been discussed in the paper by Pinelas [5]. She studies the asymptotic behavior of semi-global solutions to the linear scalar differential equation. Her work is by its topic very close to our investigations.

In the paper the following notation will be used: For $r > 0$ let $\mathrm{C}_r := \mathrm{C}\left([0, r], \mathbb{R}^n\right)$ be the Banach space of continuous functions from the interval $[0, r]$ to $\mathbb{R}^n$ equipped with the supremum norm

$$\|\psi\|_r = \sup_{\alpha\in[0,r]} |\psi\left(\alpha\right)|, \qquad \psi \in \mathrm{C}\left([0, r], \mathbb{R}^n\right),$$

where $|\cdot|$ is the maximum norm in $\mathbb{R}^n$.

For a function $y = y\left(t\right)$, continuous on an interval $[t - D, t]$, $t \in \mathbb{R}$, $D > 0$ we define a delayed-type function $y_t \in \mathrm{C}_D$ by formula $y_t\left(\tau\right) = y\left(t - \tau\right)$ where $\tau \in (0, D]$. Similarly, for a function $y = y\left(t\right)$, continuous on an interval $[t, t + A]$, $t \in \mathbb{R}$, $A > 0$, we define an advanced-type function $y^t \in \mathrm{C}_A$ by formula $y^t\left(\sigma\right) = y\left(t + \sigma\right)$ where $\sigma \in (0, A]$. Throughout the rest of the paper we assume that $D > 0$ and $A > 0$ are fixed.

The following definitions of continuity and quasi-boundedness are motivated by definitions in [4]. We say that the functional $f(t, y_t, y^t)$ is continuous if it is continuous with respect to t on $\mathbb{R}$ for each given continuous function $y \colon \mathbb{R} \to \mathbb{R}^n$. The functional $f \colon \mathbb{R} \times \mathrm{C}_D \times \mathrm{C}_A \to \mathbb{R}^n$ is said to be quasi-bounded if f is bounded on every set of the form $[a, b] \times \mathrm{C}\left([0, D], \Omega_1\right) \times \mathrm{C}\left([0, A], \Omega_2\right)$

where $a < b$ and Ω_1, Ω_2 are closed bounded sets of $\mathbb{R}^n$.

By $\mathbb{R}^n_{\geqslant 0}$ ($\mathbb{R}^n_{>0}$) we denote the set of all componentwise nonnegative (positive) vectors v in $\mathbb{R}^n$, i.e., $v = (v^1, \dots, v^n)$ with $v^i \geqslant 0$ ($v^i > 0$) for $i = 1, \dots, n$. For $u, v \in \mathbb{R}^n$, we denote $u \leqslant v$ if $v - u \in \mathbb{R}^n_{\geqslant 0}$, $u \ll v$ if $v - u \in \mathbb{R}^n_{>0}$, and $u < v$ if $u \leqslant v$ and $u \neq v$. In order to avoid unnecessary additional definitions, we use, whenever the meaning is not ambiguous, the same symbols $\mathbb{R}^n_{\geqslant 0}$ ($\mathbb{R}^n_{>0}$) to denote relevant subsets of the set $\mathbb{R}^n$.

In the paper we will consider a system of nonlinear mixed-type functional differential equations

$$\dot{y}(t) = f\left(t, y_t, y^t\right), \tag{1}$$

where $f\colon \mathbb{R} \times \mathrm{C}_D \times \mathrm{C}_A \to \mathbb{R}^n$ is a continuous and quasi-bounded functional. A solution we understand in the following meaning: A continuous function $y\colon \mathbb{R} \to \mathbb{R}^n$ is a global solution of (1) if it is continuously differentiable on $\mathbb{R}$ and satisfies (1) on $\mathbb{R}$. The existence of global solutions to mixed-type functional differential equations (1) will be proved. We provide an application of the results on linear equations. Main results are Theorem 1 and Theorem 3.

1 EXISTENCE OF GLOBAL SOLUTIONS

To prove the existence of global solutions to differential equation (1), we assume that there exist continuously differentiable functions $\beta, \gamma\colon \mathbb{R} \to \mathbb{R}^n$ and a constant vector $k \in \mathbb{R}^n$ such that

$$\beta(-\infty) = \gamma(-\infty) = k, \tag{2}$$

and for every $t \in \mathbb{R}$

$$\beta(t) \leqslant \gamma(t), \tag{3}$$
$$\beta'(t) \leqslant f(t, \beta_t, \beta^t), \tag{4}$$
$$\gamma'(t) \geqslant f(t, \gamma_t, \gamma^t). \tag{5}$$

By $\mathcal{S}$ we denote the set of the functions $y \in \mathrm{C}(\mathbb{R}, \mathbb{R}^n)$ such that $\beta(t) \leqslant y(t) \leqslant \gamma(t)$, that is,

$$\mathcal{S} = \{y \in \mathrm{C}(\mathbb{R}, \mathbb{R}^n)\colon \beta(t) \leqslant y(t) \leqslant \gamma(t), t \in \mathbb{R}\}. \tag{6}$$

In the following theorem, we will state sufficient conditions for the existence of global solutions to equation (1).

Theorem 1. *Let $\beta, \gamma\colon \mathbb{R} \to \mathbb{R}^n$ be continuously differentiable functions, satisfying* (2)-(5) *and the conditions*

$$f(t, \beta_t, \beta^t) \leqslant f(t, y_t, y^t), \tag{7}$$
$$f(t, \gamma_t, \gamma^t) \geqslant f(t, y_t, y^t), \tag{8}$$

for $t \in \mathbb{R}$ and $y \in \mathcal{S}$.

Then, there exists a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ satisfying $y(-\infty) = k$ and the inequality

$$\beta(t) \leqslant y(t) \leqslant \gamma(t) \tag{9}$$

for $t \in \mathbb{R}$.

Proof. Obviously, $\mathcal{S}$ is nonempty, closed, convex subset of $\mathrm{C}(\mathbb{R}, \mathbb{R}^n)$. We define an operator $T\colon \mathcal{S} \to \mathrm{C}(\mathbb{R}, \mathbb{R}^n)$ by

$$(Ty)(t) = k + \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s, \quad t \in \mathbb{R}. \tag{10}$$

The operator T is well-defined, the existence of the integral for $y \in \mathcal{S}$ is guaranteed by conditions (2)-(8), since

$$\begin{aligned} \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \geqslant \int_{-\infty}^{t} f(s, \beta_s, \beta^s) \mathrm{d}s \geqslant \int_{-\infty}^{t} \beta'(s) \mathrm{d}s \\ \geqslant \beta(t) - \beta(-\infty) = \beta(t) - k \end{aligned} \tag{11}$$

and

$$\begin{aligned} \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \leqslant \int_{-\infty}^{t} f(s, \gamma_s, \gamma^s) \mathrm{d}s \leqslant \int_{-\infty}^{t} \gamma'(s) \mathrm{d}s \\ \leqslant \gamma(t) - \gamma(-\infty) = \gamma(t) - k. \end{aligned} \tag{12}$$

Moreover, the operator T is continuous. Now, we are going to prove that $T(\mathcal{S}) \subset \mathcal{S}$. Let $y \in \mathcal{S}$. The inequality (11) implies that

$$(Ty)(t) = k + \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \geqslant k + \beta(t) - k = \beta(t) \tag{13}$$

and inequality (12) implies

$$(Ty)(t) = k + \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \leqslant k + \gamma(t) - k = \gamma(t). \tag{14}$$

Further, we show that functions from $T(\mathcal{S})$ are uniformly bounded and equicontinuous on each compact subinterval of $\mathbb{R}$. The equicontinuity follows from the relation

$$|(Ty)(t) - (Ty)(t')| = \left| \int_{t'}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \right|$$

and the fact that f is quasi-bounded. The uniform boundedness on a compact subinterval follows from inequalities (13) and (14). Therefore, by the Arzela-Ascoli theorem, the closure of $T(\mathcal{S})$ is compact in $\mathrm{C}(\mathbb{R}, \mathbb{R}^n)$ and according to Schauder-Tychonoff fixed-point theorem there exists an $y \in \mathcal{S}$ such that $Ty = y$. This fixed point y is a global solution of (1). Moreover, it satisfies $y(-\infty) = k$ and the inequality (9), because y is a function from $\mathcal{S}$. $\square$

Let us remark that our technique of the proof is similar to the one used, for example, in [2], [6].

1.1 A linear system

In this section, we will show the applicability of Theorem 1. Let us consider a linear system

$$\begin{aligned} \dot{y_1}(t) &= (\sin t + 1) y_2(t-1), \\ \dot{y_2}(t) &= 0.1\mathrm{e}^{\cos t} y_1(t+0.25) + e^{-t^2-10} \end{aligned} \tag{15}$$

for $t \in \mathbb{R}$. This is a variant of equation (1) with

$$n = 2,$$

$$f(t, y_t, y^t) = \left(f_1(t, y_t, y^t), f_2(t, y_t, y^t)\right) = \left((\sin t + 1) y_2(t-1),\ 0.1\mathrm{e}^{\cos t} y_1(t+0.25) + e^{-t^2-10}\right).$$

We will prove the existence of a global solution by Theorem 1. We need to show, that for suitable functions β, γ will hold conditions (2)-(5) and (7), (8). Conditions (2) and (3) will be fulfilled for $k = 0$ and

$$\begin{aligned} \beta &= (\beta_1,\ \beta_2) = \left(-\mathrm{e}^t,\ -\mathrm{e}^t\right), \\ \gamma &= (\gamma_1,\ \gamma_2) = \left(\mathrm{e}^{2t},\ \mathrm{e}^{2t}\right). \end{aligned}$$

Conditions (7), (8) hold, because of the linearity and the fact that the right-hand sides of (15) are non-decreasing with respect to $y_2(t-1)$ and $y_1(t+0.25)$. Now, let us estimate the expressions $f(t, \beta_t, \beta^t)$ and $f(t, \gamma_t, \gamma^t)$ and verify that (4), (5) hold:

$$\begin{aligned}
& f_1(t, \beta_{1t}, \beta_{2t}, \beta_1^t, \beta_2^t) \\
&\qquad = (\sin t + 1)\left(-\mathrm{e}^{(t-1)}\right) \geqslant -2\mathrm{e}^{-1}\mathrm{e}^t = -0.735759\mathrm{e}^t \geqslant -\mathrm{e}^t = \beta_1', \\
& f_2(t, \beta_{1t}, \beta_{2t}, \beta_1^t, \beta_2^t) \\
&\qquad = 0.1\mathrm{e}^{\cos t}\left(-\mathrm{e}^{(t+0.25)}\right) + e^{-t^2-10} \geqslant -0.3\mathrm{e}^{0.25}\mathrm{e}^t = -0.385208\mathrm{e}^t \geqslant -\mathrm{e}^t = \beta_2', \\
& f_1(t, \gamma_{1t}, \gamma_{2t}, \gamma_1^t, \gamma_2^t) \\
&\qquad = (\sin t + 1)\left(\mathrm{e}^{2(t-1)}\right) \leqslant 2\mathrm{e}^{-2}\mathrm{e}^{2t} = 0.270671\mathrm{e}^{2t} \leqslant 2\mathrm{e}^{2t} = \gamma_1', \\
& f_2(t, \gamma_{1t}, \gamma_{2t}, \gamma_1^t, \gamma_2^t) \\
&\qquad = 0.1\mathrm{e}^{\cos t}\left(\mathrm{e}^{2(t+0.25)}\right) + e^{-t^2-10} \leqslant 0.3\mathrm{e}^{0.5}\mathrm{e}^{2t} + \mathrm{e}^{2t} = 1.494616\mathrm{e}^{2t} \leqslant 2\mathrm{e}^{2t} = \gamma_2'
\end{aligned}$$

for $t \in \mathbb{R}$. We have verified all the assumptions from Theorem 1, therefore there exists a global solution of system (15) $y\colon \mathbb{R} \to \mathbb{R}^2$ satisfying $y(-\infty) = 0$ and the inequality

$$-\mathrm{e}^t \leqslant y_i(t) \leqslant \mathrm{e}^{2t}$$

for $t \in \mathbb{R}$, $i = 1, 2$.

1.2 Global solutions for a linear equation

In the following part we will consider a linear equation

$$\dot{y}(t) = c(t) y\left(t - \tau(t)\right) + d(t) y\left(t + \sigma(t)\right) + \omega(t), \quad t \in \mathbb{R}, \tag{16}$$

where $c\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, $d\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ are bounded continuous functions and $\tau\colon \mathbb{R} \to (0, D]$, $\sigma\colon \mathbb{R} \to (0, A]$, $\omega\colon \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 2. *Suppose that there exist continuously differentiable functions $\beta, \gamma\colon \mathbb{R} \to \mathbb{R}$ satisfying* (2) *and* (3) *and*

$$\beta'(t) \leqslant c(t)\beta(t-\tau(t)) + d(t)\beta(t+\sigma(t)) + \omega(t), \tag{17}$$

$$\gamma'(t) \geqslant c(t)\gamma(t-\tau(t)) + d(t)\gamma(t+\sigma(t)) + \omega(t) \tag{18}$$

on $\mathbb{R}$.

Then, there exists a global solution $y(t)$ of (16) *on* $\mathbb{R}$ *such that $y(-\infty) = k$ and*

$$\beta(t) \leqslant y(t) \leqslant \gamma(t) \tag{19}$$

for $t \in \mathbb{R}$.

Proof. Because equation (16) is a special case of (1), we will apply Theorem 1 to prove the existence of global solution of (16). We assume that the conditions (2) and (3) hold, so the remaining task is to verify conditions (4), (5), (7) and (8).

The inequalities (4) and (5) are equivalent to inequalities (17) and (18) if we put

$$f(t, y_t, y^t) := c(t)y\left(t-\tau(t)\right) + d(t)y\left(t+\sigma(t)\right) + \omega(t)$$

and consequently,

$$f(t, \beta_t, \beta^t) = c(t)\beta\left(t-\tau(t)\right) + d(t)\beta\left(t+\sigma(t)\right) + \omega(t),$$

$$f(t, \gamma_t, \gamma^t) = c(t)\gamma\left(t-\tau(t)\right) + d(t)\gamma\left(t+\sigma(t)\right) + \omega(t).$$

Conditions (7) and (8) hold, because of the linearity of $f(t, y_t, y^t)$ and the fact that $c(t)$ and $d(t)$ are nonnegative functions. Therefore, by Theorem 1, there exists a global solution of (16) satisfying $y(-\infty) = k$ and the inequality (19). $\square$

1.2.1 A linear example

In this part, we consider a linear scalar equation

$$\dot{y}(t) = (20 + \sin t) \cdot y(t-1) + (1 + \cos t)\mathrm{e}^{-1} \cdot y(t + 0.25) + 0.3\mathrm{e}^{3t}, \quad t \in \mathbb{R}. \tag{20}$$

This is an equation of the type (16), where

$$c(t) = 20 + \sin t,\ \ d(t) = (1 + \cos t)e^{-1},\ \ \omega(t) = 0.3\mathrm{e}^{3t},$$
$$\tau(t) = 1,\ \ \sigma(t) = 0.25.$$

If we find a continuously differentiable functions $\beta(t)$ and $\gamma(t)$ such that the conditions (2) and (3), (17) and (18) hold on $\mathbb{R}$, the existence of a global solution will be guaranteed by Theorem 2. Let us set

$$\beta(t) = -e^{9t}, \quad \gamma(t) = e^{3t} \quad \text{and} \quad k = 0.$$

It is easy to see that the conditions (2) and (3) are fullfilled. Now, we will verify the inequalities (17) and (18). For the calculations we used Wolfram Alpha software [7], available online. The inequality (17) becomes

$$\begin{aligned} c(t)\beta\left(t-\tau(t)\right) + & d(t)\beta\left(t+\sigma(t)\right) + \omega(t) \\ & = -(20+\sin t)\mathrm{e}^{9(t-1)} - (1+\cos t)\mathrm{e}^{-1}\mathrm{e}^{9(t+0.25)} + 0.3\mathrm{e}^{3t} \\ & \geqslant -21\mathrm{e}^{-9}\mathrm{e}^{9t} - 2\mathrm{e}^{-1}\mathrm{e}^{2.25}\mathrm{e}^{9t} \doteq -6.98328\mathrm{e}^{9t} \geqslant -9\mathrm{e}^{9t} = \beta'(t) \end{aligned}$$

for every $t \in \mathbb{R}$ and the inequality (18) may be estimated as

$$\begin{aligned} c(t)\gamma\left(t-\tau(t)\right) + & d(t)\gamma\left(t+\sigma(t)\right) + \omega(t) \\ & = (20+\sin t)\mathrm{e}^{3(t-1)} + (1+\cos t)\mathrm{e}^{-1}\mathrm{e}^{3(t+0.25)} + 0.3\mathrm{e}^{3t} \\ & \leqslant 21\mathrm{e}^{-3}\mathrm{e}^{3t} + 2\mathrm{e}^{-1}\mathrm{e}^{0.75}\mathrm{e}^{3t} + 0.3\mathrm{e}^{3t} \doteq 2.9\mathrm{e}^{3t} \leqslant 3\mathrm{e}^{3t} = \gamma'(t) \end{aligned}$$

for every $t \in \mathbb{R}$ as well.

Therefore, according to Theorem 2, the equation (20) has a global solution $y(t)$ such that $y(-\infty) = 0$ and

$$-\mathrm{e}^{9t} \leqslant \; y(t) \leqslant \mathrm{e}^{3t}$$

for $t \in \mathbb{R}$.

2 A GENERALIZATION

In the first section we assumed that the functions β, γ are continuously differentiable functions on $\mathbb{R}$. This assumption restricts the applicability of Theorem 1. Below we will assume that the functions β and γ are continuous on $\mathbb{R}$ and continuously differentiable almost everywhere.

Theorem 3. *Let $\beta, \gamma \colon \mathbb{R} \to \mathbb{R}^n$ be continuous functions on $\mathbb{R}$ and continuously differentiable on $\mathbb{R} \setminus \mathcal{M}$, where $\mathcal{M} = \{t_i \in \mathbb{R}, i = 1, \dots, n,\; n \in \mathbb{N} \cup \{\infty\},\; t_1 < t_2 < \cdots < t_n\}$. Let there be a constant $k \in \mathbb{R}^n$ such that*

$$\beta(-\infty) = \gamma(-\infty) = k. \tag{21}$$

Moreover, for every $t \in \mathbb{R}$, $y \in \mathcal{S}$ it holds that

$$\beta(t) \leqslant \gamma(t), \tag{22}$$

$$f(t, \beta_t, \beta^t) \leqslant f(t, y_t, y^t), \tag{23}$$

$$f(t, \gamma_t, \gamma^t) \geqslant f(t, y_t, y^t), \tag{24}$$

and for $t \in \mathbb{R} \setminus \mathcal{M}$ hold

$$\beta'(t) \leqslant f(t, \beta_t, \beta^t), \tag{25}$$

$$\gamma'(t) \geqslant f(t, \gamma_t, \gamma^t). \tag{26}$$

Then, there exists a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ satisfying $y(-\infty) = k$ and the inequality

$$\beta(t) \leqslant y(t) \leqslant \gamma(t)$$

for $t \in \mathbb{R}$.

Proof. The scheme of the proof is similar to the proof of Theorem 1. As above, the set $\mathcal{S}$ given by formula (6) is nonempty, closed, convex subset of $\mathrm{C}(\mathbb{R}, \mathbb{R}^n)$. We use the operator $T\colon \mathcal{S} \to \mathrm{C}(\mathbb{R}, \mathbb{R}^n)$ defined by (10). The operator T is well-defined, the existence of the integral for $y \in \mathcal{S}$ is guaranteed by conditions (21), (23)-(26). We show it: There are three cases which may occur: $t < t_1, t \in [t_i, t_i + 1)$ for some index i or $t \geqslant t_n$. Let $t < t_1$, then the inequalities (11) and (12) hold. In the case $t \in [t_i, t_{i+1})$ for some index i the integral may be estimated as

$$\begin{aligned}
\int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s &\geqslant \int_{-\infty}^{t_1} f(s, \beta_s, \beta^s)\mathrm{d}s + \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} f(s, \beta_s, \beta^s)\mathrm{d}s + \int_{t_i}^{t} f(s, \beta_s, \beta^s)\mathrm{d}s \\
&\geqslant \int_{-\infty}^{t_1} \beta'(s)\mathrm{d}s + \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} \beta'(s)\mathrm{d}s + \int_{t_i}^{t} \beta'(s)\mathrm{d}s \\
&= \beta(t_1) - \beta(-\infty) + \sum_{j=1}^{i-1} [\beta(t_{j+1}) - \beta(t_j)] + \beta(t) - \beta(t_i) \\
&= \beta(t) - k
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s &\leqslant \int_{-\infty}^{t_1} f(s, \gamma_s, \gamma^s)\mathrm{d}s + \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} f(s, \gamma_s, \gamma^s)\mathrm{d}s + \int_{t_i}^{t} f(s, \gamma_s, \gamma^s)\mathrm{d}s \\
&\leqslant \int_{-\infty}^{t_2} \gamma'(s)\mathrm{d}s + \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} \gamma'(s)\mathrm{d}s + \int_{t_i}^{t} \gamma'(s)\mathrm{d}s \\
&= \gamma(t_1) - \gamma(-\infty) + \sum_{j=1}^{i-1} [\gamma(t_{j+1}) - \gamma(t_j)] + \gamma(t) - \gamma(t_i) \\
&= \gamma(t) - k.
\end{aligned} \tag{28}$$

If $t \geqslant t_n$ the process of estimating the integral will be similar to (27) and (28). The only change will be replacing index i by index n and the result remains the same:

$$\int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \geqslant \beta(t) - k \tag{29}$$

and

$$\int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \leqslant \gamma(t) - k. \tag{30}$$

Moreover, the operator T is continuous. Now, we are going to prove that $T(\mathcal{S}) \subset \mathcal{S}$. According to inequalities (11), (12), (27)-(30) for the operator T holds

$$(Ty)(t) = k + \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \geqslant k + \beta(t) - k \geqslant \beta(t)$$

and also

$$(Ty)(t) = k + \int_{-\infty}^{t} f\left(s, y_s, y^s\right) \mathrm{d}s \leqslant k + \gamma(t) - k \leqslant \gamma(t).$$

From this point the proof continues in the same way as the proof of Theorem 1. □

2.0.2 A nonlinear example

In this part, we consider a nonlinear scalar equation

$$\dot{y}(t) = (20 + 0.001 \sin t) \cdot y(t-1) + (1 + \cos t)\mathrm{e}^{-1} \cdot y(t + 0.25) + 0.001\mathrm{e}^{3t + (2/\pi) \arctan y(t)} \quad (31)$$

where $t \in \mathbb{R}$. Let $k = 0$ and

$$\beta(t) = \begin{cases} -\mathrm{e}^{9t} & \text{for} \quad t \leq 0, \\ -\mathrm{e}^{4t} & \text{for} \quad t > 0, \end{cases} \qquad \gamma(t) = \begin{cases} \mathrm{e}^{3t} & \text{for} \quad t \leq 0, \\ \mathrm{e}^{4t} & \text{for} \quad t > 0. \end{cases}$$

These chosen functions β, γ are continuous on $\mathbb{R}$ and continuously differentiable except the point 0. They also satisfy (21) and (22).

Functions $\arctan$ and e are increasing, therefore

$$\mathrm{e}^{(2/\pi) \arctan \beta(t)} \leqslant \mathrm{e}^{(2/\pi) \arctan y(t)} \leqslant \mathrm{e}^{(2/\pi) \arctan \gamma(t)}$$

for $y \in \mathcal{S}$ and $t \in \mathbb{R}$. The remaining part of the right-hand side of the equation (31) is linear, therefore inequalities (23), (24) hold for $y \in \mathcal{S}$ and $t \in \mathbb{R}$.

Inequalities (25) and (26) have to be checked on four intervals:

$$t \in (-\infty, -0.25], \qquad t \in (-0.25, 0], \qquad t \in (0, 1], \qquad t \in (1, \infty).$$

In the first case when $t \in (-\infty, -0.25]$ the right-hand side of the inequality (25) may be estimated in the following way:

$$\begin{aligned} f(t, \beta_t, \beta^t) &= (20 + 0.001 \sin t) \cdot (-\mathrm{e}^{9(t-1)}) + (1 + \cos t)\mathrm{e}^{-1} \cdot (-\mathrm{e}^{9(t+0.25)}) + 0.001\mathrm{e}^{3t + (2/\pi) \arctan y(t)} \\ &\geqslant -20.001\mathrm{e}^{9t-9} - 2\mathrm{e}^{-1+9t+2.25} \\ &= \mathrm{e}^{9t}(-20.001\mathrm{e}^{-9} - 2\mathrm{e}^{1.25}) \doteq -6.98\mathrm{e}^{9t} \\ &\geqslant -9\mathrm{e}^{9t} = \beta'(t) \end{aligned}$$

and the right-hand side of the inequality (26) as:

$$\begin{aligned} f(t, \gamma_t, \gamma^t) &= (20 + 0.001 \sin t) \cdot \mathrm{e}^{3(t-1)} + (1 + \cos t)\mathrm{e}^{-1} \cdot \mathrm{e}^{3(t+0.25)} + 0.001\mathrm{e}^{3t + (2/\pi) \arctan y(t)} \\ &\leqslant 20.001\mathrm{e}^{3t-3} + 2\mathrm{e}^{-1+3t+0.75} + 0.001\mathrm{e}^{3t+1} \\ &\leqslant \mathrm{e}^{3t}(20.001\mathrm{e}^{-3} + 2\mathrm{e}^{-0.25} + 0.001\mathrm{e}) \doteq 2.56\mathrm{e}^{3t} \\ &\leqslant 3\mathrm{e}^{3t} = \gamma'(t). \end{aligned}$$

Now, we will consider t from interval $(-0.25, 0]$ and the estimations will be the following:

$$\begin{aligned}
f(t,\beta_t,\beta^t) &= (20+0.001\sin t)\cdot\left(-\mathrm{e}^{9(t-1)}\right) + (1+\cos t)\mathrm{e}^{-1}\cdot\left(-\mathrm{e}^{4(t+0.25)}\right) + 0.001\mathrm{e}^{3t+(2/\pi)\arctan y(t)}\\
&\geqslant -20.001\mathrm{e}^{9t-9} - 2\mathrm{e}^{-1+4t+1}\\
&= \mathrm{e}^{9t}\left(-20.001\mathrm{e}^{-9} - 2\mathrm{e}^{-5t}\right)\\
&\geqslant \mathrm{e}^{9t}\left(-20.001\mathrm{e}^{-9} - 2\mathrm{e}^{-5\cdot(-0.25)t}\right) \doteq -6.98\mathrm{e}^{9t}\\
&\geqslant -9\mathrm{e}^{9t} = \beta'(t)
\end{aligned}$$

and

$$\begin{aligned}
f(t,\gamma_t,\gamma^t) &= (20+0.001\sin t)\cdot \mathrm{e}^{3(t-1)} + (1+\cos t)\mathrm{e}^{-1}\cdot \mathrm{e}^{4(t+0.25)} + 0.001\mathrm{e}^{3t+(2/\pi)\arctan y(t)}\\
&\leqslant 20.001\mathrm{e}^{3t-3} + 2\mathrm{e}^{-1+4t+1} + 0.001\mathrm{e}^{3t+1}\\
&= \mathrm{e}^{3t}(20.001\mathrm{e}^{-3} + 2\mathrm{e}^{t} + 0.001\mathrm{e})\\
&\leqslant \mathrm{e}^{3t}(20.001\mathrm{e}^{-3} + 2 + 0.001\mathrm{e}) \doteq 2.998\mathrm{e}^{3t}\\
&\leqslant 3\mathrm{e}^{3t} = \gamma'(t).
\end{aligned}$$

The third case considers $t \in (0, 1]$, then:

$$\begin{aligned}
f(t,\beta_t,\beta^t) &= (20+0.001\sin t)\cdot\left(-\mathrm{e}^{9(t-1)}\right) + (1+\cos t)\mathrm{e}^{-1}\cdot\left(-\mathrm{e}^{4(t+0.25)}\right) + 0.001\mathrm{e}^{3t+(2/\pi)\arctan y(t)}\\
&\geqslant -20.001\mathrm{e}^{9t-9} - 2\mathrm{e}^{-1+4t+1}\\
&= \mathrm{e}^{4t}\left(-20.001\mathrm{e}^{5t-9} - 2\right)\\
&\geqslant \mathrm{e}^{4t}\left(-20.001\mathrm{e}^{-4} - 2\right) \doteq -2.37\mathrm{e}^{4t}\\
&\geqslant -4\mathrm{e}^{4t} = \beta'(t)
\end{aligned}$$

and

$$\begin{aligned}
f(t,\gamma_t,\gamma^t) &= (20+0.001\sin t)\cdot \mathrm{e}^{3(t-1)} + (1+\cos t)\mathrm{e}^{-1}\cdot \mathrm{e}^{4(t+0.25)} + 0.001\mathrm{e}^{3t+(2/\pi)\arctan y(t)}\\
&\leqslant 20.001\mathrm{e}^{3t-3} + 2\mathrm{e}^{-1+4t+1} + 0.001\mathrm{e}^{3t+1}\\
&= \mathrm{e}^{4t}\left(20.001\mathrm{e}^{-t-3} + 2 + 0.001\mathrm{e}^{-t+1}\right)\\
&\leqslant \mathrm{e}^{4t}(20.001\mathrm{e}^{-3} + 2 + 0.003) \doteq 3\mathrm{e}^{4t}\\
&\leqslant 4\mathrm{e}^{4t} = \gamma'(t).
\end{aligned}$$

And finally, for $t \in (1, \infty)$ the following holds:

$$\begin{aligned}
f(t,\beta_t,\beta^t) &= (20+0.001\sin t)\cdot\left(-\mathrm{e}^{4(t-1)}\right) + (1+\cos t)\mathrm{e}^{-1}\cdot\left(-\mathrm{e}^{4(t+0.25)}\right) + 0.001\mathrm{e}^{3t+(2/\pi)\arctan y(t)}\\
&\geqslant \mathrm{e}^{4t}(-20.001\mathrm{e}^{-4} - 2) \doteq -2.37\mathrm{e}^{4t}\\
&\geqslant -4\mathrm{e}^{4t} = \beta'(t)
\end{aligned}$$

and

$$\begin{aligned}
f(t,\gamma_t,\gamma^t) &= (20+0.001\sin t)\cdot \mathrm{e}^{4(t-1)} + (1+\cos t)\mathrm{e}^{-1}\cdot \mathrm{e}^{4(t+0.25)} + 0.001\mathrm{e}^{3t+(2/\pi)\arctan y(t)}\\
&\leqslant \mathrm{e}^{4t}\left(20.001\mathrm{e}^{-4} + 2 + 0.001\mathrm{e}^{-t+1}\right)\\
&\leqslant \mathrm{e}^{4t}\left(20.001\mathrm{e}^{-4} + 2 + 0.003\right) \doteq 2.37\mathrm{e}^{4t}\\
&\leqslant 4\mathrm{e}^{4t} = \gamma'(t).
\end{aligned}$$

According to Theorem 3, there exists a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ satisfying $y(-\infty) = 0$ and the inequalities

$$\begin{aligned} -\mathrm{e}^{9t} &\leqslant y(t) \leqslant \mathrm{e}^{3t}, \qquad t \leqslant 0 \\ -\mathrm{e}^{4t} &\leqslant y(t) \leqslant \mathrm{e}^{4t}, \qquad t > 0. \end{aligned}$$

2.1 A linear variant

Below we will formulate a consequence for linear equations. Let us consider the equation (16) with coefficients mentioned in section 1.2, i.e. $c\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, $d\colon \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ are bounded continuous functions and $\tau\colon \mathbb{R} \to (0, D]$, $\sigma\colon \mathbb{R} \to (0, A]$, $\omega\colon \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 4. *Let $\beta, \gamma\colon \mathbb{R} \to \mathbb{R}$ be continuous functions on $\mathbb{R}$ and continuously differentiable on $\mathbb{R} \setminus \mathcal{M}$. Let there be a constant vector $k \in \mathbb{R}$ such that* (21) *holds. Moreover, suppose that for every $t \in \mathbb{R}$ holds the inequality* (22) *and for $\mathbb{R} \setminus \mathcal{M}$ hold the inequalities* (25) *and* (26)*.*

Then, there exists a global solution $y(t)$ of (16) *such that $y(-\infty) = k$ and*

$$\beta(t) \leqslant y(t) \leqslant \gamma(t) \tag{32}$$

for $t \in \mathbb{R}$.

Proof. The idea of the proof is similar to the proof of Theorem 2. We assume that conditions (21), (22), (25) and (26) are fulfilled. The conditions (23) and (24) hold for a linear equation with positive coefficients. Therefore, according to Theorem 3 there exists a global solution of (16) such that $y(-\infty) = k$ and (32) hold.

□

CONCLUSION

In the paper, the existence of global solutions of advance-delay system (1) is proved. Simultaneously, upper and lower estimates of global solutions are derived. As a method of proof, Schauder-Tychonoff fixed point theorem is applied. The obtained results are adapted to linear equations and illustrated by examples.

To compare our result with other papers, we refer to [2], [5] and [6]. In [2] was proved the existence of global solutions for advanced differential system. Pituk and Röst in [6] have proved the existence of the solution on half-axis. We have used similar methods and proved the existence of solutions to advance-delay systems on whole real line. In [5] Pinelas considers linear scalar equations.

References

[1] Diblík, J. Integral criteria for the existence of positive solutions of first-order linear differential advanced-argument equations. Appl. Math. Lett. (2016), 1–7. http://dx.doi.org/10.1016/j.aml.2016.07.016

[2] Diblík, J., Kúdelčíková, M. Two classes of asymptotically different positive solutions to advanced differential equations via two different fixed-point principles. Math. Meth. Appl. Sci. Vol. 40, No. 5, 2017, 1422–1437. DOI: 10.1002/mma.4064.

[3] Diblík, J., Koksch, N. Sufficient conditions for the existence of global solutions of delayed differential equations. *J. Math. Anal. Appl.*, Vol. 318, No. 1, 2006, 611–625.

[4] Driver, R. D. Ordinary and Delay Differential Equations. Spriger Verlag, New York, 1977. ISBN 978-0-387-90231-9.

[5] Pinelas, S. Asymptotic behavior of solutions to mixed type differential equation. *Electronic Journal of Differential Equations*, No. 210, 2014, 1-9. ISSN 1072-6691.

[6] Pituk, M., Röst, G. Large time behavior of a linear delay differential equation with asymptotically small coefficient. *Boundary Value Problems*, No. 1, 2014, 1–9. ISSN 1687-2770.

[7] Wolfram Alpha [online software]. [Cit. 2018-09-25]. Available at: `<https://wolframalpha.com>`.

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MATHEMATICAL TOOLS FOR CREATING MODELS OF INFORMATION AND COMMUNICATION NETWORK SECURITY

Irada Dzhalladova[1] **and Miroslava Růžičková**[2]

[1] National University of Economics, Department of Computer Mathematics and Information Security,
Kyiv 03068, Peremogy 54/1, Ukraine
Email: idzhalladova@gmail.com

[2] University of Białystok, Faculty of Mathematics and Informatics,
K. Ciołkowskiego 1M, 15-245 Białystok, Poland
Email: miroslava.ruzickova@gmail.com, m.ruzickova@math.uwb.edu.pl

Abstract: *The paper deals with a system of difference equations with a small parameter, where the coefficients depend on a random process. A method of asymptotic expansions for constructing the mathematical expectation of solutions is proposed and used in the investigation the stability of solutions. An application of the results to a scalar difference equation of considered type is shown.*

Keywords: system of difference equations, small parameter, stability of solutions, asymptotic expansions, mathematical expectation.

AMS Subject Classifications: 34F05

INTRODUCTION

The development and stability of systems depend on an understanding of the system, on the rich experience and the interaction of system with the environment. Timely and current information can allow the system to stabilize, adapt and restore the structure of the system or subsystem in the event of possible violations.

Obviously, in developing solutions, management of complex systems always has to take into account the uncertainty and risk, while allowing some regularities of the probabilistic nature in accordance with the role of individual or mass of random phenomena. In our opinion, based on the stochastic approach, we can study a number of aspects concerning the operation of the system in different areas of activity. Our aim is to develop models of these activities using elements of random processes.

Markov models are widely used in management. They form the basis of modern arsenal of probabilistic methods in relation to the description of the state of the managed object and the transition from one state to another at time with an acceptable degree of accuracy and reliability.

Thus, investigating stability of solutions of difference equations with random coefficients depending on Markov or non-Markov, in particular semi-Markov, process represents a current problem.

The basis for most authors in the development of the stability of stochastic systems was the theory of stability of a deterministic system developed by Lyapunov. However, the method of Lyapunov functions is often difficult to apply to the study of the stability of non-stationary dynamic systems. This can be explained by the fact that it is inconvenient to use the Lyapunov functions in the sense of the Lyapunov stability concept for this type of systems. The investigation of the Lyapunov stability of differential systems with random parameters becomes even more complicated.

Our task is to obtain a reliable and simple method for studying stability of stochastic systems of difference equations. In this paper, we propose the method of asymptotic expansions for constructing the mathematical expectation of solutions to stochastic system of difference equations with a small parameter. The method of matched asymptotic expansions and the representation of a solution in a convergent series were used, for example, in [5] and [7] respectively, to study deterministic difference equations. The dynamic system and methods discussed in this paper are very well suited for use as models for protecting information in cyberspace.

1 STATEMENT OF THE PROBLEM

Let $(\Omega,\, \mathcal{F},\, P)$ be a probability space. The space Ω is called the sample space, $\mathcal{F}$ is the set of all possible events (the σ-algebra), and P is some probability measure on Ω. On such a probability space, we consider the initial value problem formulated for stochastic dynamic system with a small parameter

$$X_{n+1} = X_n + \mu\, A(n,\, \xi_n) X_n, \quad n = 1, 2, \ldots \tag{1}$$

$$X_0 = \varphi(\omega), \tag{2}$$

where A is an $m \times m$ matrix with random elements, $\varphi : \Omega \to \mathbb{R}^m$. A sequence $\xi = \{\xi_i\}_{i=1}^{\infty}$ of random variables $\xi_i \colon \Omega \to S$, $i = 0, 1, 2, \ldots$ is called a discrete-time stochastic chain on the state space S. In our considerations, ξ is first a random chain with an infinite number of states, then a random Markov chain taking a finite number of states $\theta_1,\, \theta_2,\, \ldots,\, \theta_q$ with probabilities $p_k(n) = P\,\{\, \xi_n = \theta_k\}$, $k = 1, 2, \ldots, q$, $n = 1, 2, \ldots$ that satisfy the system of difference equations

$$p_k(n+1) = \sum_{s=1}^{q} \pi_{ks} p_s(n), \quad k = 1, 2, \ldots, q \tag{3}$$

where $\pi_{jk}(s, n) = P\left(\xi_n = \theta_k \,\middle|\, \xi_s = \theta_j\right), \quad k, j = 1, 2, \ldots, q.$
Denoting the transition matrix as $\Pi = \left(\pi_{ks}(n)\right)_{k,s=1}^{q}$, (3) can be rewritten into the matrix form

$$P(n+1) = \Pi(n)\, P(n) \tag{4}$$

where $P(n) = (p_1(n), p_2(n), \ldots, p_q(n))^T$.

Definition 1 *The m-dimensional random vector X_n, $n = 1, 2, \ldots$ is called a solution of the initial value problem* (1), (2) *if X_n satisfies* (1) *and initial condition* (2) *in the sense of strong solution of the initial Cauchy problem.*

It is known that in general case, stochastic difference equations can not be solved in a closed form with the exception of several classes of these equations. In a broader sense, by solving a stochastic equation we mean finding the statistical characteristics of the solution. The main method for solving these equations is via finding the probability distribution function as a function of time using the equivalent Fokker-Planck equation, which tells us how the probability distribution function evolves in time. Moreover, solving these differential equations is not an easy task. Therefore, finding an explicit solution to a stochastic difference equation is possible only in some special cases. There is, thus, a great need for a simple and straightforward presentation of the methods for obtaining a solution to stochastic models. Another method, namely the method of moment equations for the study of dynamical systems with random structure is used in [2, 3, 4, 6]. The stability of the zero solution to stochastic differential systems with four-dimensional Brownian motion is studied, for example in [1].

In this paper, a solution to (1) is assumed as a power series in small parameter μ, containing the mathematical expectation of solution. Then we construct difference equations for the mathematical expectation of a random solution to system (1) and we apply the solution of such a system in the investigation of the stability.

2 MAIN RESULTS

2.1 Random chain with infinite number of states

First let ξ be some random chain with an infinite number of states. We find the random solution in the form of an asymptotic series in powers of μ,

$$X_n = Y_n + \sum_{k=1}^{\infty} \mu^k \Phi_k(n,\, \xi_n) Y_n, \quad n = 1, 2, \dots \tag{5}$$

where $Y_n = E^{(1)}\{X_n\}$ is the mathematical expectation of X_n, and $\Phi_k(n,\, \xi_n)$ are unknown matrices satisfying assumption:

$$E^{(1)}\{\Phi_k(n,\, \xi_n)\} = \Theta, \quad k = 1, 2, \dots, \tag{6}$$

which means that their mathematical expectation is a zero matrix.

Next, we find the vector Y_n also in the form of an asymptotic series in powers of μ,

$$Y_{n+1} = Y_n + \sum_{k=1}^{\infty} \mu^k B_k(n) Y_n, \quad n = 1, 2, \dots \tag{7}$$

with unknown matrices $B_k(n)$.
Substituting (5) and (7) into equation (1) we get the following system of equations

$$Y_n + \sum_{k=1}^{\infty} \mu^k B_k(n) Y_n + \sum_{k=1}^{\infty} \mu^k \Phi_k(n+1,\, \xi_{n+1}) \left[Y_n + \sum_{k=1}^{\infty} \mu^k B_k(n) Y_n \right]$$

$$= Y_n + \sum_{k=1}^{\infty} \mu^k \Phi_k(n,\, \xi_n) Y_n + \mu A(n,\, \xi_n) \left[Y_n + \sum_{k=1}^{\infty} \mu^k \Phi_k(n,\, \xi_n) Y_n \right],$$

from where

$$\left(I + \sum_{k=1}^{\infty} \mu^k B_k(n) \right) \left(I + \sum_{k=1}^{\infty} \mu^k \Phi_k(n+1,\, \xi_{n+1}) \right) = \left(I + \sum_{k=1}^{\infty} \mu^k \Phi_k(n,\, \xi_n) \right) \Big(I + \mu A(n,\, \xi_n) \Big). \tag{8}$$

Equating the coefficients at the same powers of μ in (8), we obtain system of linear matrix difference equations,

$$\Phi_1(n+1,\, \xi_{n+1}) - \Phi_1(n,\, \xi_n) + B_1(n) = A(n,\, \xi_n), \tag{9}$$

$$\begin{aligned} &\Phi_k(n+1,\, \xi_{n+1}) - \Phi_k(n,\, \xi_n) + B_k(n) \\ &\qquad = A(n,\, \xi_n)\Phi_{k-1}(n,\, \xi_n) - \sum_{s=1}^{k-1} \Phi_s(n+1,\, \xi_{n+1}) B_{k-s}(n), \quad k = 2, 3, \ldots \end{aligned} \tag{10}$$

Application of mathematical expectation on equation (9), and taking into account (6), we can determine the first unknown matrix,

$$B_1(n) = E^{(1)}\{A(n,\, \xi_n)\}. \tag{11}$$

In the same way, from equations (10) we obtain relations

$$B_k(n) = E^{(1)}\{A(n,\, \xi_n)\Phi_{k-1}(n,\, \xi_n)\}, \quad k = 2, 3, \ldots . \tag{12}$$

On the other hand, if we take into account $\Phi_1(0,\, \xi_0) = \Theta$, then from (9) follows

$$\Phi_1(n,\, \xi_n) = \sum_{s=1}^{n-1} \big(A(s,\, \xi_s) - B_1(s)\big).$$

From where, and from (12) if $k = 2$, we get an expression for $B_2(n)$,

$$B_2(n) = E^{(1)} \left\{ A(n,\, \xi_n) \sum_{s=1}^{n-1} \big(A(s,\, \xi_s) - B_1(s)\big) \right\}. \tag{13}$$

Continuing similar calculations, we can obtain an expression for any matrix $B_k(n), k = 3, 4, \ldots$. In conclusion, we formulate the lemma.

Lemma 1 *Let ξ_n be a random chain with an infinite number of states. Then the random solution to stochastic dynamical system* (1) *with a small parameter μ can be found in the form of asymptotic series* (5), (7) *where the matrices $\Phi_k(n,\, \xi_n), B_k(n), k = 1, 2, \ldots$ are defined by* (9), (10).

Remark 1 *The previous considerations concerned any random process with an infinite number of states. If we assume that the random chain ξ_n can take only the finite number of states $\theta_1, \theta_2, \ldots, \theta_q$, and the probabilities of the distribution of values $\theta_1, \theta_2, \ldots, \theta_q$ are known:*

$$p_k(n),\, p_{kk_1}(n, n_1),\, \ldots\, , p_{kk_1 \ldots k_q}(n, n_1,\, \ldots\, , n_q),$$

then the matrices $B_k(n), k = 1, 2$ are defined as follows

$$\begin{aligned} B_1(n) &= \sum_{k=1}^{q} A(n,\, \theta_k) p_k(n), \\ B_2(n) &= \sum_{k=1}^{q} \sum_{k=1}^{q} A(n,\, \theta_k)\big(A(s,\, \theta_1) - B_1(s)\big) p_{kk_1}(n, s). \end{aligned}$$

2.2 Markov chain with finite number of states

The calculations above can be simplified if we assume that the process ξ is a Markov chain for which (3) and (4) are satisfied. It is easy to see, that from (4) the following is true:

$$\begin{aligned} P(s) = P\big((s-1)+1\big) &= \Pi(s-1)P(s-1) \\ &= \Pi(s-1)\Pi(s-2)P(s-2), \\ &\vdots \\ &= \Pi(s-1)\Pi(s-2)\cdots\Pi\big((s-(s-n)\big)P(n), \\ &\qquad\qquad n,s=1,2,\ldots,\quad s>n. \end{aligned}$$

Denote

$$\begin{aligned} &M(s,n) = \Pi(s-1)\Pi(s-2),\ldots,\Pi(n), \quad n,s=1,2,\ldots,\ s>n, \\ &M(n,n) = I \end{aligned}$$

where I is the identity matrix.
It should be noted, if the transition matrix Π does not depend on n, then $M(s,n)=\Pi^{s-n}$.
It is obvious that $M(n+1,n)=\Pi(n)$, and $P(s)=M(s,n)\,P(n)$. Thus, for the elemets of the matrix $M(s,n)=\big(m_{jk}(s,n)\big)_{j,k=1}^{q}$, $n,s=1,2,\ldots$ we have

$$m_{jk}(s,n) = P\left(\xi_n=\theta_k \,\middle|\, \xi_s=\theta_j\right), \quad k,j=1,2,\ldots,q.$$

Under assumpion $n\geq n_1\geq n_2\cdots\geq n_s$, the joint distribution can be calculated as follows

$$p_{kk_1\ldots k_s}(n,n_1,\,\ldots\,,n_s) = m_{kk_1}(n,n_1)\,m_{k_1k_2}(n_1,n_2)\,\cdots\,m_{k_{s-1}s}(s-1,n_s)\,p_{ks}(n,s)$$

As a result, the mathematical expectation of the functions $a(n,\xi_n)$, $n=1,2,\ldots$ that are dependent on the random chain ξ_n has the form

$$\begin{aligned} &E^{(1)}\left\{a(n,\xi_n)a(n_1,\xi_{n_1})\,\cdots\,a(n_s,\xi_{n_s})\right\} \\ &\qquad = \sum_{kk_1\ldots k_s=1}^{q} p_{kk_1\ldots k_s}(n,n_1,\,\ldots\,,n_s)\,a_k(n)\,a_{k_1}(n_1)\,\cdots\,a_{k_s}(n_s) \\ &\qquad = \sum_{kk_1\ldots k_s=1}^{q} a_k(n)\,m_{kk_1}(n,n_1)\,\cdots\,a_{k_s}(n_s)\,m_{k_{s-1}k_s}(n_{s-1},n_s)p_{k_s}(n_s) \end{aligned} \tag{14}$$

where $a_k(n)=a(n,\theta_k)$, $k=1,2,\ldots,q$.
Denoting

$$\begin{aligned} &A(n) = \operatorname{diag}\big(a_1(n),a_2(n),\,\ldots,\,a_q(n)\big) \\ &C = (1,1,\,\ldots,\,1), \qquad \dim C = q, \end{aligned}$$

formula (14) can be rewritten into the matrix form

$$\begin{aligned} &E^{(1)}\left\{a(n,\xi_n)a(n_1,\xi_{n_1})\,\cdots\,a(n_s,\xi_{n_s})\right\} \\ &\qquad = CA(n)\,M(n,n_1)\,A(n_1)\,\cdots\,M(n_{s-1},n_s)\,A(n_s)\,P(s). \end{aligned} \tag{15}$$

The obtained result we formulate in the following Theorem

Theorem 1 *Let ξ_n be a random Markov chain with the finite number of states $\theta_1, \theta_2, \ldots, \theta_q$. Then the mean solution to stochastic dynamical system* (1) *with a small parameter μ can be found in the form of asymptotic series* (5), (7) *where the matrices $\Phi_k(n, \xi_n), B_k(n), k = 1, 2, \ldots$ are defined by* (9), (10) *and* (15).

3 INVESTIGATION OF THE STABILITY OF THE MEAN SOLUTION OF THE LINEAR FIRST-ORDER DIFFERENCE EQUATION

Consider the linear difference equation of the first order,

$$X_{n+1} = X_n + \mu\, a(\xi_n) X_n, \quad n = 1, 2, \ldots \tag{16}$$

where the Markov chain ξ can take three possible states θ_1, θ_2, and θ_3 with probabilities $p_1(n)$, $p_2(n)$, and $p_3(n)$ that satisfy the system of difference equations (4) in the form

$$\begin{aligned} p_1(n+1) &= \big(1 - (\alpha+\beta)\big)p_1(n) + \delta p_2(n) + \mu p_3(n), \\ p_2(n+1) &= \alpha p_1(n) + \big(1 - (\delta+\gamma)\big)p_2(n) + \nu p_3(n), \\ p_3(n+1) &= \beta p_1(n) + \gamma p_2(n) + \big(1 - (\mu+\nu)\big)p_3(n) \end{aligned} \tag{17}$$

where $\alpha, \beta, \gamma, \delta, \mu, \nu \in (0, 1]$. Suppose that the initial probabilities

$$p_1 = \frac{\alpha+\beta}{w}, \quad p_2 = \frac{\gamma+\delta}{w}, \quad p_3 = \frac{\nu+\mu}{w}, \qquad w = \alpha+\beta+\gamma+\delta+\mu+\nu.$$

The matrix of this system, this is the transition matrix

$$\Pi = \begin{pmatrix} 1-\alpha-\beta & \delta & \mu \\ \alpha & 1-\delta-\gamma & \nu \\ \beta & \gamma & 1-\mu-\nu \end{pmatrix},$$

does not depend on n, and has eigenvalues

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{1-w-\sqrt{D}}{2}, \qquad \lambda_3 = \frac{1-w+\sqrt{D}}{2}$$

where

$$\begin{aligned} D &= w^2 - 4g, \\ g &= \alpha\gamma + \alpha\mu + \alpha\nu + \beta\delta + \beta\gamma + \beta\nu + \delta\mu + \delta\nu + \gamma\mu. \end{aligned}$$

Thus, the general solution to (17) can be written in the form

$$P(n) = c_1\kappa_1 + c_2\kappa_2 \left(\frac{1-w-\sqrt{D}}{2}\right)^n + c_3\kappa_3 \left(\frac{1-w+\sqrt{D}}{2}\right)^n$$

where $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}^3$. The n-th power of the matrix Π can be calculated ([37]) as:

$$\Pi^n = z_1 + \frac{\left(1-w-\sqrt{D}\right)^n}{2^n} z_2 + \frac{\left(1-w+\sqrt{D}\right)^n}{2^n} z_3$$

where

$$
\begin{aligned}
z_1 &= \frac{4}{1+2w+4g}\big(A^2-(1-w)A\big)+\frac{1-2w+4g}{4}I,\\
z_2 &= \frac{2}{\big(1+w+\sqrt{D}\big)\sqrt{D}}\big(A^2-A(\lambda_1+\lambda_2)I+\lambda_1\lambda_3 I\big)\\
z_3 &= \frac{2}{\big(-1-w+\sqrt{D}\big)\sqrt{D}}\big(A^2-A(\lambda_1+\lambda_2)I+\lambda_1\lambda_2 I\big),\\
A &= \mathrm{diag}(a_1,a_2,a_3).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Pi^{n-1}+\Pi^{n-2}+\cdots+\Pi &= (n-1)z_1+\frac{1-w-\sqrt{D}}{1+w+\sqrt{D}}z_2+\frac{1-w+\sqrt{D}}{1+w-\sqrt{D}}z_3\\
&= z_1+z_2+z_3.
\end{aligned}
$$

The solution to (16) we find in the form (5) where Y_n is in the form (7), taking into account only two terms of the power series. By Remark 1, using (15), and the fact that the matrix Π does not depend on n, for $b_i(n), i=1,2$, we obtain

$$
\begin{aligned}
b_1(n) &= a_1p_1+a_2p_2+a_3p_3=\frac{a_1(\alpha+\beta)+a_2(\gamma+\delta)+a_3(\mu+\nu)}{w},\\
b_2(n) &= CA\big(\Pi^{n-1}+\Pi^{n-2}+\cdots+\Pi\big)(A-b_1I)P.
\end{aligned}
$$

In addition that $CAz_1(A-b_1I)P=\Theta$, for b_2 we have

$$
b_2=\lim_{n\to\infty}b_2(n)=A(z_2+z_3)(A-b_1I)P.
$$

Thus,

$$
\begin{aligned}
b_2 = (a_1,a_2,a_3)\frac{1}{w^2\big[(1+w)^2-D\big]^2}\Big(&4A^2(w-1)^2-16-4D\\
&+A\ \frac{(1+w)(3-4w+w^2-D)+(1+w)^2+D}{0.25}\\
&-4I(1+w)\big((1-w)^2-D\big)\Big)\begin{pmatrix} r_1 & 0 & 0\\ 0 & r_2 & 0\\ 0 & 0 & r_3\end{pmatrix}\begin{pmatrix}\alpha+\beta\\ \gamma+\delta\\ \mu+\nu\end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
r_1 &= a_1(\gamma+\delta+\mu+\nu)-a_2(\gamma+\delta)-a_3(\mu+\nu),\\
r_2 &= a_2(\alpha+\beta+\mu+\nu)-a_1(\alpha+\beta)-a_3(\mu+\nu),\\
r_3 &= a_3(\alpha+\beta+\gamma+\delta)-a_1(\alpha+\beta)-a_2(\gamma+\delta).
\end{aligned}
$$

The stability of the equation (16) can now be determined by the stability of the equation (7) which has the form

$$
y_{n+1}=\big(1+\mu b_1+\mu^2 b_2+o(\mu^3)\big)y_n
$$

where $o(\cdot)$ is the little Laudau symbol.

Is is obvious, that the trivial solution to (16) is stable in the mean, if the inequality

$$1 + \mu b_1 + \mu^2 b_2 + o(\mu^3) < 0$$

with sufficiently small μ.

4 CONCLUSION

Our approach to the modelling of information and communication network security differs from the approach of many researches, which, as a rule, take into account of the maximum possible number of factors affecting information about safety, that is they make basically a classification architecture that helps organizations to implement their information security strategies.

As a mathematical model of the problems under discussion we use dynamical system of difference equations with a small parameter, where the coefficients depend on a random process. The input parameters of the model can be the stochastic parameters of the threats, which are obtained from the statistics of their occurrence and elimination. We can determine the domain of stability of the information system, this is, if the system is ready to work in conditions of its security.

References

[1] Baštinec, J., Klimešová, M. Stability of the Zero Solution of Stochastic Differential Systems with Four-Dimensional Brownian Motion. In Mathematics, Information Technologies and Applied Sciences 2016, (post-conference proceedings of extended versions of selected papers.) *Brno: University of Defence.* (2016), 7 – 30.

[2] Diblík, J., Dzhalladova, I., Michalková, M., Růžičková, M. Modeling of applied problems by stochastic systems and their analysis using the moment equations. *Adv. Difference Equ.*, Vol. 2013:152, (2013), 12 pp.

[3] Diblík, J., Dzhalladova, I., Michalková, M., Růžičková, M. Moment equations in modeling a stable foreign currency exchange market in conditions of uncertainty. *Abstr. Appl. Anal.*, Vol. 2013 (2013), Art. ID 172847, 11 pp.

[4] Dzhalladova, I., Růžičková, M., Štoudková Růžičková, V., Stability of the zero solution of nonlinear differential equations under the influence of white noise, *Adv. Difference Equ.*, Vol. 2015:143, (2015), 11 pp.

[5] Hall, C.L., Lustri, C.J., Multiple scales and matched asymptotic expansions for the discrete logistic equation, *Nonlinear Dyn.*, (2016) 85: 1345. https://doi.org/10.1007/s11071-016-2764-7.

[6] Růžičková, M., Dzhalladova, I., The optimization of solutions of the dynamic systems with random structure, *Abstr. Appl. Anal.*, Vol. 2011, (2011), Art. ID 486714, 18 pp.

[7] Zerizer, T., Perturbation method for linear difference equations with small parameters, *Adv. Difference Equ.*, Vol. 2006, (2006), Art. ID 19214, 1-12 pp.

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SPACIAL COMBINED BENDING-GYRATORY VIBRATION - EQUATIONS OF MOTION

Petr Hrubý, Dana Smetanová, Tomáš Náhlík
Institute of Technology and Business in České Budějovice
Okružní 10, 370 01 České Budějovice, e-mails: dochruby@mail.vstecb.cz,
smetanova@mail.vstecb.cz, nahlik@mail.vstecb.cz

Abstract: *The paper deals with the construction of the motion equation of the section of the one-dimensional linear continuum in the state of the spatial combined-gyratory vibration. The basic decomposition of the general spatial motion of the rigid body in the center is used to define the condition of the continuum element. The general spatial motion is replaced by an irresistible shifting and relative spherical motion. The transverse oscillation of the continuum in the field of centrifugal forces conveys a significant dependence of the numbers on the angular velocity of the rotation of the system.*

Keywords: one-dimensional continuum, equation of motion, D'Alembert principle, combined-gyratory vibration.

INTRODUCTION

The presented equations of motion in state of spacial combined bending-gyratory vibration are useful tool for description of all mechanisms with shafts (e.g. shafts in cars, gear pumps, ...). In engineering constructions, the most endangered parts are rotating components.
Reliability of shaft endangers in particular two limit states. In the vicinity of resonance there is an enormous increase in the amplitudes of the state variables and the achievement of the yield strength of the material. These conditions often occur with the coupling shafts of Cardan mechanisms. The torque is transmitted here over long distances. Shafts are long and slender and are prone to transverse bending. The gearbox shafts are compact and operate at a sufficient distance from the resonant area. In this case, they are threatened by fatigue fractures; they need to be checked for safety to fatigue. A similar situation to gearboxes is with gear pump shafts.
The separated rotation of shaft is studied by authors of [1]. Lanzutti et al.[6] presents a failure analysis of transmission gearbox (and its components) used in motor of a food centrifugal dryer tested with a life test procedure developed by Electrolux Professional. Sinitsin and Shestakov [7] present comprehensive analysis of the angular and linear accelerations of moving elements (shafts, gears) by wireless acceleration sensor of moving elements.
The coupling problems between shafting torsional vibration and speed control system of diesel engine is studied by Yibin et al. [8]. The torque is transmitted to relatively long distances by shafts in engines. The combined motions are presented in papers [4] and [5]. Moreover the both paper are devoted of spectral properties of vibrations. The presented paper focuses on analytical construction of equations of motion of one-dimensional continuum in state of combined motions, especially spacial combined bending-gyratory vibration. It is generalization of the paper [5] to spacial case.
The paper is structured as follows. Firstly, we consider an element of one-dimensional continuum (see Fig. 1) with the case of homogeneous field with constant annular cross-section and we describe the force systems being in equilibrium.

Secondly, we present the equations of motion in state of spacial combined bending-gyratory vibration. The next part of our paper analyses results of the aforementioned research.

1 EQUILIBRIUM CONDITIONS AND INERTIAL EFFECTS ACTING ON ELEMENT OF CONTINUUM

Consider an element of one-dimensional continuum (see Fig. 1, [5]) with the case of homogeneous field with constant annular cross-section. The element modeling a shaft has an inner radius r_1 and outer radius r_2. The length is l.
The element (as homogeneous field) is further characterized by the stiffness parameter E (module of elasticity in tension or compression) and the weight parameter (density) ϱ. The theoretical background of elasticity and strength is possible see to [3].

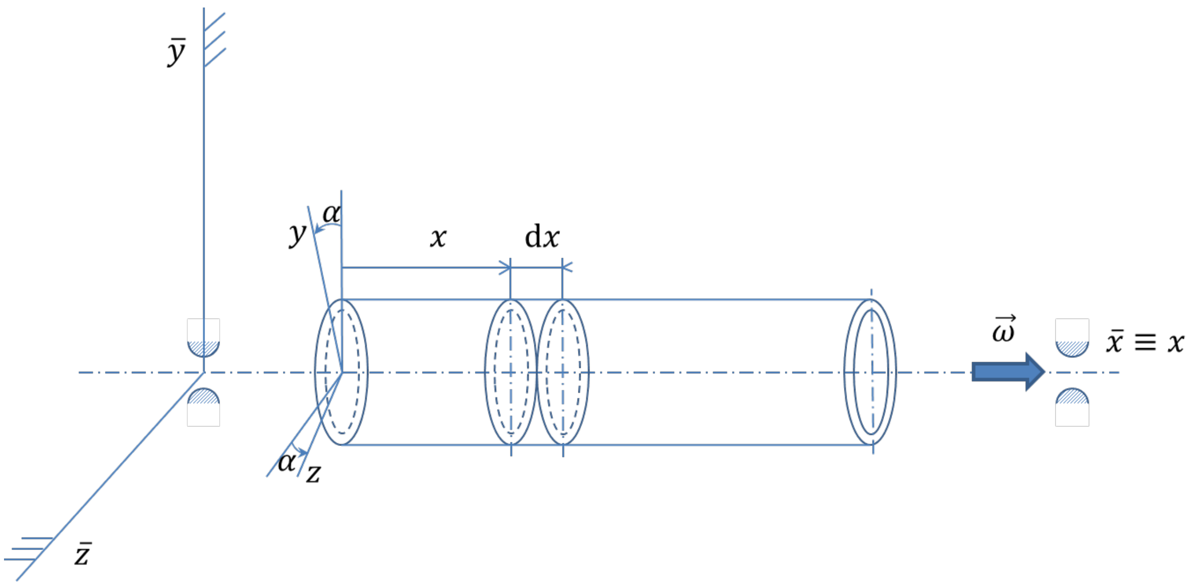

Fig. 1. The element of continuum
Source: own

The external forces acting on the element make state combined bending-gyratory vibration. In that case, the continuum element making general spatial motion which is composed of three simple movements - namely rotation, shift and spherical motion. Let's obtain the equations of motion from the equilibrium of the acting forces.
The force system acts on the length element $\mathrm{d}x$ of continuum (Fig. 1) which is imaginary removed and released from the system. The inertial effects acting on the element can be in the center of the element generally replaced by the inertial force having components $D_y,\ D_z$, the moment of inertia couple having components $M_{D_y},\ M_{D_z}$ and shear force $Q_y,\ Q_z$.
The force systems being in equilibrium is expressed by formulas

$$\begin{aligned} &D_y + \frac{\partial Q_y}{\partial x}\mathrm{d}x = 0, \\ &M_{D_z} + Q_y\mathrm{d}x - \frac{\partial M_z}{\partial x}\mathrm{d}x = 0, \\ &D_z + \frac{\partial Q_z}{\partial x}\mathrm{d}x = 0, \\ &M_{D_y} + \frac{\partial M_y}{\partial x}\mathrm{d}x - Q_z\mathrm{d}x = 0. \end{aligned} \tag{1}$$

The inertial force of the sliding motion is obtained from the following formula

$$\mathrm{d}D = -a\,\mathrm{d}m, \tag{2}$$

where $\mathrm{d}m = \mu\,\mathrm{d}x$ and μ is weight of the continuum length.
Dynamic models are formed by parts of the annular cross-section. The weight of the unit of length is expressed directly from the area of annular cross-section in the following form

$$\mu = \pi\varrho\left(r_2^2 - r_1^2\right). \tag{3}$$

It should be emphasized that system (with respect coordinates x, y, z) rotates with constant angular velocity ($\omega = const.$). Assume that the constant angular velocity of system rotation is respect with solving all the problem mentioned in this paper. If the results are given of some quantities depending on the angular velocity ω then the angular velocity is understood as a parameter. The acceleration of the center being achieved by time derivative of the position vector

$$r(t) = (x(t), y(t), z(t)). \tag{4}$$

For simplification we use the notation $r = (x, y, z)$.
The position vector is expressed in rotating system, however, it is necessary to differentiate it in the "non-rotating" coordinates $\bar{x}, \bar{y}, \bar{z}$ (see Fig. 1). Once the implementation of relevant derivatives the acceleration vector is obtained in the form

$$a = \left(0, \ddot{y} - \omega^2 y - 2\omega\dot{z}, \ddot{z} - \omega^2 z - 2\omega\dot{y}\right). \tag{5}$$

Substituting the (3) and (5) to (2) we obtain

$$\mathrm{d}D = -\mu\,\mathrm{d}x\left(0, \ddot{y} - \omega^2 y - 2\omega\dot{z}, \ddot{z} - \omega^2 z - 2\omega\dot{y}\right). \tag{6}$$

Rotation of the element about the angles β, γ correspond to the continuum deformation in the planes xz, xy. The angles β and γ are very small angles. In next, this fact allows to receive strong simplification. For small angles the following equations are possible aplied

$$\beta = -\frac{\partial z}{\partial x},\ \gamma = -\frac{\partial y}{\partial x}. \tag{7}$$

The vectors of the inertial force $D_{x,z,y}$ and of the moment of inertia couple $M_{D_{x,z,y}}$ can be rewritten into the form suitable for further solutions

$$\mathrm{d}D_{x,z,y} = -\mu\,\mathrm{d}x\left(0,\ \frac{\partial^2 y}{\partial t^2} - \omega^2 y - 2\omega\frac{\partial z}{\partial t},\ \frac{\partial^2 z}{\partial t^2} - \omega^2 z - 2\omega\frac{\partial y}{\partial t}\right) \tag{8}$$

and

$$\mathrm{d}M_{D_{x,z,y}} = -\bar{\mu}\,\mathrm{d}x\left(0,\ \frac{\partial^3 y}{\partial x\partial t^2} - \omega^2\frac{\partial z}{\partial x},\ \frac{\partial^3 z}{\partial x\partial t^2} - \omega^2\frac{\partial y}{\partial x}\right), \tag{9}$$

where μ is (3) and

$$\bar{\mu} = \pi\varrho\left(r_2^4 - r_1^4\right) \tag{10}$$

is a constant of annular-cross section of the continuum element.

2 EQUATIONS OF MOTION IN STATE OF SPACIAL COMBINED BENDING-GYRATORY VIBRATIONS

The equations of motion are obtained using the D'Alembert principle to forces. The bending moment vector of the elastic forces is given by relation

$$M = -E\ J\ \left(0,\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 y}{\partial x^2}\right), \tag{11}$$

where E is stiffness parameter (the module of elasticity in tension or compression) and

$$J = \frac{\pi}{4}\left(r_2^4 - r_1^4\right) = \frac{1}{4\varrho}\bar{\mu} \tag{12}$$

is a moment of an inertia - of an annular cross-section. As mentioned in the introduction to this chapter, equations of motion are obtained from the equations (1) force systems equilibrium by D'Alembert principle.
The equations of motion of the spacial combined bending-gyratory vibrations of the annular cross-sections one-dimensional continuum are expressed by formulas

$$EJ\frac{\partial^4 y}{\partial x^4} - \bar{\mu}\frac{\partial^4 y}{\partial x^2 \partial t^2} - \bar{\mu}\omega^2\frac{\partial^2 y}{\partial x^2} + \mu\frac{\partial^2 y}{\partial t^2} - 2\mu\omega\frac{\partial z}{\partial t} - \mu\omega^2 y = 0, \tag{13}$$

and

$$EJ\frac{\partial^4 z}{\partial x^4} - \bar{\mu}\frac{\partial^4 z}{\partial x^2 \partial t^2} - \bar{\mu}\omega^2\frac{\partial^2 z}{\partial x^2} + \mu\frac{\partial^2 z}{\partial t^2} + 2\mu\omega\frac{\partial y}{\partial t} - \mu\omega^2 z = 0. \tag{14}$$

Substituting the equalities (3), (10) and (12) to above equations of motion (13), (14) and modifying it we obtain the final form of the equations of motion

$$\frac{\partial^4 y}{\partial x^4} - \frac{\varrho}{E}\left(\frac{\partial^4 y}{\partial x^2 \partial t^2} + \omega^2\frac{\partial^2 y}{\partial x^2}\right) + \frac{4\varrho}{E\left(r_1^2 + r_2^2\right)}\left(\frac{\partial^2 y}{\partial t^2} - 2\omega\frac{\partial z}{\partial t} - \omega^2 y\right) = 0, \tag{15}$$

and

$$\frac{\partial^4 z}{\partial x^4} - \frac{\varrho}{E}\left(\frac{\partial^4 z}{\partial x^2 \partial t^2} + \omega^2\frac{\partial^2 z}{\partial x^2}\right) + \frac{4\varrho}{E\left(r_1^2 + r_2^2\right)}\left(\frac{\partial^2 z}{\partial t^2} - 2\omega\frac{\partial y}{\partial t} - \omega^2 z\right) = 0. \tag{16}$$

Note that the analytical solutions of the equations (15) and (16) are real functions y, z of real variables t, x.
With respect to the deflection of the continuum cross-section we can rewrite the formulas (15), (16) in following way.
We calculate $(15) + i(16)$ and we obtain the formula

$$\begin{aligned}\frac{\partial^4 y}{\partial x^4} + i\frac{\partial^4 z}{\partial x^4} - \frac{\varrho}{E}\left[\left(\frac{\partial^4 y}{\partial x^2 \partial t^2} + i\frac{\partial^4 z}{\partial x^2 \partial t^2}\right) + \omega^2\left(\frac{\partial^2 y}{\partial x^2} + i\frac{\partial^2 z}{\partial x^2}\right)\right] + \\ + \frac{4\varrho}{E\left(r_1^2 + r_2^2\right)}\left[\left(\frac{\partial^2 y}{\partial t^2} + i\frac{\partial^2 z}{\partial t^2}\right) + 2\omega\left(-\frac{\partial z}{\partial t} + i\frac{\partial y}{\partial t}\right) - \omega^2\left(y + iz\right)\right] = 0.\end{aligned} \tag{17}$$

The above formula (17) rewritten with notation: $v = y + iz$ has form:

$$\frac{\partial^4 v}{\partial x^4} - \frac{\varrho}{E}\left(\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\omega^2 \partial^2 v}{\partial x^2}\right) + \frac{4\rho}{E\left(r_1^2 + r_2^2\right)}\left(\frac{\partial^2 v}{\partial t^2} - \omega^2 v + 2i\omega\frac{\partial v}{\partial t}\right) = 0 \tag{18}$$

and the formula represents equation of motion in the state of spacial combined bending-gyratory vibrations.
Note that $v = y + iz$ describes the deflection of the continuum cross-section at the coordinate x at the general time t.

CONCLUSION

The shafts have a lot of interesting technical properties and they are studied intensively from different point of view. An interesting view at the behavior of shafts is situation when they move. The movement of the shafts is described by equations of motion (i.e. partial differential equations). It is possible study different motions of shafts (e.g. rotation, bending, shift, ...) and their combinations.

More frequently separated movements of shaft are studied (see [1], [8]). The combination of movements is presented less often because in the situation there is more complicated calculation [5]. A procedure for vibration analysis of the device based on measured data in simulated operating modes in mechanisms is studied in [2]. In [9], these new trends in torsional vibration calculation for various vehicles are briefly described, with attention paid not only to practical use, but above all to how and to what extent these themes should be presented to students.

In the paper [5] there is presented mathematical model of combined bending-gyratory vibration. Especially the paper is devoted a finite element for 1-dimensional linear continuum in the state of combined bending-gyratory vibration. An application of the finite element method is designed and tuned a method for calculating eigenvalues and vectors of a stepped shaft in the state of combined bending-gyratory vibration.

In the paper the equations of motion of spacial combined bending-gyratory vibration of one-dimensional linear continuum are presented. The model can be used both for the calculation of natural frequencies and shape oscillations and for calculation of steady response in case of oscillation enforced by discrete excitation in any cut of the continuum. The presented paper is generalization of [5] to spacial vibrations. The problem is solved analytically. In [5] there is used finite element method.

References

[1] Ben Arab, S., Dias Rodriges, J., Bouaziz, S., Haddar, M. A finite element based on Equivalent Single Layer Theory for rotating composite shafts dynamic analysis. *Composite structures*, No.178, 2017, p. 135-144.

[2] Dekys, V., Kriska, P., Lietava, L., Kopas, P., Stalmach, O. Simplified estimate of fatigue damage based on dynamic analysis.*Scientific Journal of Silesian University of Technology. Series Transport.* No. 99, 2018, p. 15-23.

[3] Hájek, E., Reif, P., Valenta, F. *Elasticity and Strength I.*, SNTL/ALFA. Prague. 1988. ISBN 04-224-88

[4] Hrubý, P., Hlaváč, Z. *Aplikace MKP při řešení spektrálních a modálních vlastností* Výzkumná zpráva č. 102 07 91 (Research Report), ZČU Plzeň, 1991, p. 36

[5] Hrubý, P., Smetanová, D., Náhlík, T. Proposal Mathematical Model for Calculation of Modal and Spectral Properties. In: *Mathematics, Information Technologies and Applied Sciences 2017, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2017, p. 131-140. [Online]. [Cit. 2018-04-11]. Available at: <`http://mitav.unob.cz/data/MITAV%202017%20Proceedings.pdf`>. ISBN 978-80-7582-026-6 .

[6] Lanzutti, A., Gagliardi, A., Raffaelli, A., Simonato, M., Furlanetto, R., Mgnan, M., Andreatta, F., Fedrizzi, L. Failure analysis of gear, shafts and keys of centrigual washers failed during life test. *Engineering Failure Analysis*, No. 79, 2017, p. 634-641.

[7] Sinitsin, V. V., Shestakov, A. L.Wireless acceleration sensor of moving elements for condition monitoring of mechanism.*Measurement Science and Technology*, No. 28, 2017, p. 1-8.

[8] Yibin, G., Wanyou, L., Shuwen, Y., Xiao, H., Yunbo, Y., Zhipeng, W., Xiuzhen, M. Diesel engine torsional vibration control coupling with speed control system. *Mechanical Systems and Signal Processing*, No. 94, 2017, p. 1-13.
[9] Zoul, V., P. Kováč. A brief overview about the development of torsional vibration calculation and education of methods for their calculation. *Scientific Journal of Silesian University of Technology. Series Transport.* No. 99, 2018, p. 205-211.

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DIGITAL EDUCATION AND INFORMATIONAL THINKING OF ELEMENTARY SCHOOL PUPIL

Miroslava Huclová
31. základní škola Plzeň
Elišky Krásnohorské 10, 323 00 Plzeň, huclovami@zs31.plzen-edu.cz

Abstract: *The article deals with teaching in the educational field of Information and Communication Technologies (ICT) at elementary school. It gives an insight into the important part of the knowledge and skills that the new generation will need in further education and training future profession.*
In the first part of the paper presents the educational concept of basic Czech education republics valid curricular documents and strategies of the Czech Republic's educational policy in the digital world with by formulating cross-sectional priorities.
The second part of the article brings action research is basic research questions. To examine with the use of computer technology was attended by pupils from the 5th to 9th grade of elementary school, giving a view from the position of the target group. This is followed by a detailed evaluation of the research questions and an analysis of respondents' answers to the given issue. It deals with the results of the stated goals of the basic education and its effectiveness. It provides a preview and feedback on the teaching of Informatics at elementary school. Here is a view of another possible shift to the development of information thinking of pupils and the pitfalls associated with this phenomenon. It responds to the vision and strategy of digital education in the upcoming years in the context of this research.

Keywords: computer science, information and communication technologies (ICT), qualitative research, elementary school, informational thinking, learning objectives, digital education.

INTRODUCTION

The environment in which future generations will live and educate fundamentally changes, particularly as a result of digital technology. With this is related the requirement for education of elementary school students. Information technology would thus have to break through the whole process of teaching at elementary school. It is therefore necessary to raise the information thinking of students and to extend their digital literacy and skills to a new dimension. The paper analyses the current state of the curriculum and provides an excursion of the current knowledge and skills of elementary school pupils in this field.

1 THE BASIC EDUCATION CONCEPT

1.1 Curricular documents

Educational documents are obligatory document, according to which each school is governed by teaching. The state level in the curriculum system is the National Education Program and Framework Educational Programs (hereinafter RVP). The National Education Program defines initial education as a whole. RVP define obligatory education frameworks for its individual stages – pre-primary, primary and secondary education. The school level consists of school educational programs (ŠVP) that provide education at each school (RVP_ZV, 2017 p. 5) [1]. ŠVP is created by each school according to the principles set out in relevant RVP. In

the system of basic education, according to the ŠVP, it is taught from the school year 2007/2008. There has been no revision in the information and communication technology education field since the RVP ZV, although the professional public has repeatedly reminded this. There has been a situation where educational documents are obsolete; they do not reflect the level of development and the available digital technologies.
In November 2014, the government supported the Digital Education Strategy by 2020 [2]. It was created on the proposal of the Ministry of Education, Youth and Sports. It responds to the continuous development of digital technologies and anticipates the gradual involvement of modern technologies in teaching. Currently comments are being made on the working papers for the variation proposals for updating the Framework Educational Programs in the field of ICT and (new) informatics.

1.2 Strategy of Czech educational policy

The Czech Republic's educational policy strategy in the digital world formulates cross-sectional priorities:

- to reduce inequalities in education;
- to promote quality teaching and teachers as a key prerequisite;
- to responsibly and effectively manage the education system.

Within the strategy, measures are grouped into seven main lines of intervention to address the priorities set:

1. Ensure non-discriminatory access to digital educational resources.
2. Ensure conditions for the development of digital literacy and information literacy of students.

Formulation of measures to achieve the objectives of the given intervention: ensure a system of regular innovations of the framework educational programs. Emphasis on digital technologies in the curriculum and their modernization. Modernize the ICT education area in the framework of educational programs, emphasize information thinking. Linking formal and non-formal learning to informative learning.

3. Ensure conditions for the development of digital literacy and informative thinking of teachers.
4. Ensure the building and renewal of the educational infrastructure.
5. Promote innovative practices, monitoring, evaluating and disseminating their results.
6. Ensure a system that encourages the development of schools in the area of digital technology integration into teaching and school life.
7. Increase public understanding of the objectives and processes of integrating technology into education [2].

For the described action research and the identification of research issues is important the priority described in Point 2 – Ensuring the conditions for the development of digital literacy and information literacy of pupils.

2 ACTION RESEARCH

Terms of research are targeted to obtain the necessary information on the achieved digital education of students of elementary school with regard to current curricular documents. The research was carried out by students from 5th to 9th year. Research goal was to determine the achieved information profile of the students and his shift in digital literacy and information thinking. There was observed a glance with regard to individual years and the results were analysed with taking into account established research criteria.

2.1 Research category

This action research can be classified into the following categories:
Description: research describes phenomena and focuses on the questions: who, how, and pin.
Research techniques are: statistical surveys, field observations, and case study that gives an image of specific similarities of situation, phenomenon or relationships.
Research features: Applied research: answers questions that are of direct relevance to practice. It seeks solutions to practical problems.
Research strategy used: Qualitative research strategy: a weakly structured research strategy focused on smaller samples; researcher has got close relationship to the subject´s, the attitude of the researcher is within the situation [3].
Basic approaches to qualitative research: Case study – Study of social groups (group of pupils).

2.2 Practical part of the research

For the above research, the basic steps are specified:
Research area: achieved information profile of the students and its shift in digital literacy and information thinking in of each year.
Research problem: What are the student's informational profile and its digital education and information thinking?
Purpose of the research: To ensure the effectiveness of ICT teaching at primary school with regard to existing curricular documents.
Importance for science: Research points to the feedback of teaching with using ICT at elementary school, not only in information and education areas communication technology. Important information will provide the result of research for a further digital education strategy at elementary school. Research is also important for creators and producers of modern teaching resources and aids suitable for teaching s using ICT.

2.3 Basic research questions

What is student´s relationship to subject Informatics due to each year?
How does a student's evaluate the use and application of acquired knowledge and skills from the subject Informatics in other subjects?
How students assess their digital education and information thinking?
How do students assess the current educational content of the subject Informatics for their future use?
What is the connection between choosing a future profession and teaching s using ICT?

2.4 Restriction of research

The limitation and delimitation of the qualitative part of the research is given by subject of interest, subject of Informatics and other subjects, which are used in ICT at the 31st elementary school in Pilsen. All subjects taught correspond to ŠVP "School for the 21st Century", which is based on RVP ZV. Conclusions therefore concern the information profile of the students of this school.
This results in the following research limitations:
The study and its conclusions must be taken as local because the research is conducted on the students of a particular elementary school. This is also an advantage (the group of respondents is homogeneous and has the same conditions of education at the same school all the time) and a disadvantage (for a wider generalization it is a smaller research sample). The author is aware of all these restrictions they are taken into account when interpreting the results.

2.5 Implementation of research

The research was carried out as follows:

- analysis of the situation at selected elementary schools, analysis of the educational field Information and communication technology according to the RVP ZV a ŠVP 31. elementary schools in Pilsen "School for the 21st Century" with regard to the content definition of the subject, the forms and methods used in teaching and the student's outcomes; analysis of other teaching subjects where ICT is used in teaching;
- selecting appropriate research groups ;
- compiling a questionnaire with questions that are in context with research problem;
- addressing students (respondents), explaining the purpose of the research, obtaining consent research conditions, access to research questions on network drives;
- collection, processing, completion and analysis of acquired data;
- evaluation of research.

Time research: research was carried out during the school year 2017/2018.

2.6 Role of researcher in relation to terrain

Researcher in terrain has due to terrain of the role of a dedicated researcher (a person who meets with his / her respondents outside research); freely according to [4].

3 THE SITUATION OF THE SCHOOL CONDUCTING RESEARCH

3.1 Teaching with using ICT

School Education Program 31st Elementary School v Pilsen is the subject of Informatics (educational area of Information and Communication Technologies) in the 5th and 6th grade. The subsidy is one hour a week. The subject is compulsory for all students. The subject Informatics enables students to achieve the basic level of information literacy. Students will acquire basic computer skills and learn to orient themselves in the world of information.

From the 7th grade, one group of students is educated in a compulsory elective course of Informatics in a time subsidy of 3 hours a week. Other groups in this year have other optional subjects at the same time. Subject chooses from wide range of optional subjects. Teacher in these subjects try to make the most of ICT in teaching. In the 8th and 9th grades students use ICT in the subject of Art Education. This is mainly about vector and raster graphics processing, working with data communication etc. The subject is included in the Arts and Culture educational field. The thematic plans were adapted to correspond to the Art Education and corresponded with the RVP ZV. The subsidy is one hour a week in each grade.

3.2 Material and technical equipment of the school

The school is a full primary school with a capacity of 850 students. The teaching staff consists of a director, two representatives and 52 teachers, including all components of school counseling. For teaching with using ICT there are two large classroom equipped. Other two classrooms are fully equipped with iPads for each pupil. The school has modern measurement systems that students can use to teach natural science subjects. Processing systems use ICT. Classrooms have at their disposal (30, 30) computers from 2018. There is an interactive touch panel in a number of classes. The complete ICT services for the school are provided by the statutory city of Pilsen, which is the founder of the school.

3.3 Professional qualifications of teachers schools from the use of ICT

Teacher's professional competence can be defined as an "open and development-capable system of professional qualities that cover the full range of components of knowledge, skills,

experience, attitudes and personal prerequisites that are interdependent, and understood holistically." [5]
From the point of view of professional competence, the teachers of the 31st elementary school in Plzeň are fully qualified in their educational field. In recent years, other educators have gone through a number of educational programs to use ICT in teaching.

3.4 Digital literacy framework for school pupils

Student knows security and ergonomics when working with PC and multimedia, is familiar with the history of computer technology, names hardware and software, works with operating systems, text editors, knows text typography, processes and creates vector files, creates animation, works with raster graphics and digital photos, orientated in printing technology, 3D printing and creation of files for 3D printing, knows the principle of recording devices, can work with cloud storage, can get and work with digital image recording, working and organizes data in spreadsheets, works with Internet, digitized parts of the world cultural heritage, its conclusions can be processed and defended. Works with network drives, communicates via computer network, email. It uses school information systems. It respects the Copyright Act, knows the dangers and risks of social communication and can solve any pitfalls. It knows GDPR, it can identify which data it can publish. It is orientated in training programs suitable for individual subjects, can search in the databases of educational resources, can use the programs for their cognitive and educational process. It uses antivirus programs, knows the principles of computer virus prevention, and works securely with email communication and programs [6].

4 RESEARCH

4.1 Research sample

The sample was students from the 5th to the 9th grade of that school. Two classes took part in each year. In the 7th grade only one class. Teachers in Informatics were educated in these years, teaching ICT courses took place with teachers of the school.
The survey was attended by 264 respondents.

5th grade	64 pupils
6th grade	65 pupils
7th grade	22 pupils
8th grade	61 pupils
9th grade	52 pupils

4.2 Questionnaire

Research was created as a data source with an electronic questionnaire using the Google Forms application. In the questionnaire respondents responded electronically to the research questions. Questions in the questionnaire were chosen with respect to all ages. After the electronic questioning was ended, the pupils conducted a group discussion on the issue. Observations were recorded in the prepared form.
The first round of questions deals with the student's personal relationship to the subject. The second round of questions deals with the use of acquired knowledge and skills in the subject of Informatics in other subjects from the student's point of view, the third round of questions records the digital education and the student's informal thinking, and the last round deals with the educational content of the subject. The conclusion is devoted to the future profession of the students and his perspective on the use and use of ICT

5 EVALUATION OF QUESTIONS

5.1 Relation of the subject of Informatics

Question: *What is your relationship to the subject of Informatics?*
Students have a positive relationship to the subject of Informatics. Evaluations were done as a mark in school. Of the 264 respondents, 46.6% of the students opted for a good grade, 39% of the students scored, 8.3% of the students, 4% of the students voted enough, 2% of the pupils voted poorly. No relationship has been recorded here in the evaluation of the question among the students of each year.
Group conversation: during group conversation students positively evaluated the use of computer technology, valued other forms and methods of teaching. They highlighted particular electronic document processing, information gathering and communication with ICT. During a group interview, some people found that they answered inaccurately. Although the form was explained to them, they rated the answer, not as a mark in school but as a score. By doing so, the answers with the mark are devalued enough and insufficiently.
Question: *How many hours a week would you like to have Informatics?*
Students would increase lessons in 48, 4 % or retained in 43, 2 %. For the reduce were 8.4 % of students. No relationship has been recorded here in the evaluation of the question among the students of each year.
Group conversation: during a group interview, students agreed on a higher hourly grant. They wondered at the expense of what subject they would classify. They declined increase of the increase the total weekly number of hours. They were thinking that the subject could be offered only to those interested after leaving the class. Students who have decided to reduce the number of hours did not engage in a discussion.

5.2 Using acquired knowledge and skills from of the subject Informatics v other subjects

Question: *In which subject do you use computers or iPads?*
Students choose among subjects that are taught at school. Here is a direct link in the form of teachers in individual subjects and in each class. In the 5th grade, one teacher predominates. In particular, they use teaching programs in mathematics and Czech. Other uses pupils do not know. Higher grades use ICT by teachers. Statistically, most pupils report use in art education, mathematics and history. Other subject is behind.
Group conversation: during a group interview, pupils reported cases of using computers and iPads. There was an obvious share in the teacher's personality and his erudition in the use of teaching programs and electronic resources. Teachers who lead students to use ICT in class are assessed as experienced and professionally trained. Students in the 8th grade and in the 9th grade were able to differentiate between the complexities of the work done in the individual subjects using the ICT. From getting information and working with individual information, through work with tutorials to creating individual files using acquired knowledge. Most mentioned subjects were mathematics (GeoGebra, Wolfram Alpha) and art education (creation and modification of vector and raster files).
Question: *Which ability from ICT do you use in other subjects?*
This question had a free answer in the form of a text. Pupils in the 5th grade responded in general. They were unable to write a specific use. In the 6th and 7th year, they mainly chose to obtain and process information for papers in individual subjects. In the higher grades students were able to describe the specific use of programs. Repeatedly mentioned the ability to create and present their digital works.
Group conversation: in the group interview, the answers were in line with the previous question. Once again, they know the usage depending on the teacher´s usage. The ability to

use presentational programs and present their work to classmates was a breakthrough in higher years. In the 9th year they mentioned the possibility of cloud storage and their use for their work.

5.3 Digital education and the student's computer thinking

Question: *Do you use digital materials obtained on the Internet during your home preparation?*

Of the 264 respondents, 49.2% replied that they use digital materials for home preparation and can search for them. 13.2% said they did not use digital materials and could not find them. 25.4% say they use digital materials that the teacher sends them. 12.2% use digital materials made available to the parent.

Group conversation: in the group interview students provided digital materials they used for their home preparation. Again, there was a difference in the grades. In general the students appreciated the teachers who provide them with educational resources in electronic form. A special section was the electronic textbooks available to students in some subjects. Here, again, the influence of the pedagogue, which uses them and taught the pupils with the electronic textbook, was again influenced. They also mentioned learning materials that they use with some teachers and which were created under an open Creative Commons license.

Question in questionnaire: *Can you create computer-based learning materials?*

Of the 264 respondents, 51.9% of the respondents answered that they were able to create learning materials, 43.2% said they never tried to create learning materials, and 4.9% did not know whether they could create learning materials.

Group conversation: during the group conversation, the age of the pupils and their experience were again evident. Deeper inquiries have shown that learning materials can create students in higher grades. These are in particular digital materials created in presentation programs or materials using programs that have been taught by the teacher. In the lower grades, the interview revealed that some of the answers are distorted by over-confident pupils.

Question in questionnaire: *Can you present digital materials in front of a group?*

Of the 264 respondents, 47.3% of the respondents answered that they were able to present digital materials in front of the group, 41.6% said they never tried to present their work to the group and 10.8% of students could not present their work in front of the group.

Group conversation: during the group conversation, the age of the students and their experience were again evident. The conclusions of the students were identical with the previous group interview.

Question: *Can you write in which area are computers dangerous?*

This question had a free answer in the form of a text. Pupils in the 5th grade were able to describe the risks they were familiar with during the course, but also within the Preventive School Program. Students from higher grades were able to describe specific threats, particularly cyberbullying, social networking threats, communications, passwords. They used the specific knowledge they gained in teaching in the subject of Informatics, but also in the Preventive School Schedule programs.

Group conversation: during the group conversation, the age of the pupils and their experience were again evident. Students from higher talked about specific threats. They were interested in news in this area. The conversation became an exchange of practical information and experience.

Question: *What social networks do you use?*

161 pupils from 264 pupils use Facebook, 52 pupils from Twitter, 12 pupils Lidé.cz, Instagram 186 pupils, Google+ 115 pupils, 45 pupils no social network.

Group conversation: during the group conversation, the age of the pupils and their experience were again evident. The conversation once again revealed the knowledge of specific threats in communication, depending on the pupils' age.
Question: *Do you use cloud storage?*
Of the 264 respondents, 34.1% of pupils answered yes, 23.9% did not, and 42% did not know what cloud storage was.
Group conversation: during the group conversation, the age of the pupils and their experience were again evident. Here, full-time tuition in the subject of Informatics was fully demonstrated. Pupils of higher age can use this technology; pupils of lower grades are not acquainted with this.

5.4 Educational content of the subject

Question in the questionnaire: *From the following lesson, choose what you will need in the future.*
Students assessed that most would need internet and electronic communication, raster graphics and digital photography, vector graphics, text editor, typography of text writing, hardware and software knowledge. Students from higher have broadened their response to knowledge of information presentation and security, and working with data. The less important the students attached to the history of computer technology and the basics of programming.
Group conversation: Again, the conversation reveals a connection between the students' age and the number of hours they have been taught. In lower grades there were majority of answers about working with educational programs. Higher grades demonstrate deeper knowledge and better communicate about the meaning and use of curriculum.

5.5 Selection of the pupil's future profession

Question in the questionnaire: *What would you like to be in the future?*
In the free answer, the students used to designate different professions (from technician, doctor, lawyer, car mechanic, confectioner, etc.). Their future career orientation was largely unrelated to areas of ICT.
Group conversation: in conversations, students, regardless of their age, are fully aware that their future profession will be closely related to using computer technology. They have found few professions where there is no representation of computer technology. They are fully aware of the sense of education as a preparation for a future profession.

6 RESEARCH REPORT

From the conclusions presented in the previous chapter, the following answers to the research questions can be identified:
Students have a positive relationship with the subject of Informatics, regardless of the year they are studying. It evaluates positively other methods of teaching and using digital technologies.
The use and application of the knowledge gained from the subject of Informatics in other subjects is directly dependent on the personality of the teacher of the subject. Research has clearly demonstrated that in subjects where the educator implements digital technology in school education, students are an active learner with the use of ICT. In this research question, the need to ensure the conditions for the development of digital literacy and the informational thinking of teachers was manifested the most. At the same time, it is necessary to regularly update the Framework Educational Programs (RVP) so that they are in line with the latest

scientific knowledge and developments in digital technology (not only in the subject of Informatics).
The rating of digital education and information thinking depends on the age of the students and the number of hours worked by Informatics. Again, it corresponds directly to the previous research question and its conclusions.
The current educational content of the subject Informatics for its future use can be seen in the practical application of vector and raster graphics, text editors and spreadsheets and in the presentation of their work and the possibilities of communication in the digital environment.
The future profession of students does not affect the subject of Informatics in any way. Students perceive digital technology as a common part of its environment and are fully aware that their control is part of their education and a prerequisite for their future life.

CONCLUSION

The research report provides a unique view of the subject of Informatics at a primary school in the Czech Republic. It clearly defines students' view of teaching and their knowledge and skills. It establishes the role of the pedagogue in the educational process and the need for its development to achieve the quality of teaching at elementary school. The mechanisms and conditions for quality digital education of primary school students in the Czech Republic are clearly described. Therefore, it can be stated that the digital education strategy, which follows the Czech Republic's Education Policy Strategy by 2020, is a necessary support for digital education, which is becoming increasingly necessary. Especially in elementary schools, students need to develop in information thinking across all educational disciplines. A key player to this goal is not only technical equipment, but also a pedagogue with full interdependence across the curriculum.

References

[1] RVP_ZV. *Rámcový vzdělávací program pro základní vzdělávání (se změnami provedenými k 1. 9. 2017*). MŠMT ČR. [Online] 2018. http://www.msmt.cz/file/43792/.

[2] *Strategie digitálního vzdělávání do roku 2020*. Ministerstvo školství, mládeže a tělovýchovy: Strategické a koncepční dokumenty [online]. Praha, 2018, 2018 [cit. 2018-05-22]. Dostupné z: http://www.msmt.cz/vzdelavani/skolstvi-v-cr/strategie-digitalniho-vzdelavani-do-roku-2020.

[3] Švaříček, R., Šeďová, K. *Kvalitativní výzkum v pedagogických vědách*. Praha: Portál, 2007. ISBN 978-80-7367-313-0.

[4] Heindl, J. *Kvalitativní výzkum: základní teorie, metody a aplikace*. Praha: Portál, 2008. ISBN 978-80-7367-485-4.

[5] Vašutová, J. *Profese učitele v českém vzdělávacím kontextu*. Brno: Paido, 2004. ISBN 80-7315-082-4.

[6] Škola pro 21. století – ŠVP 31. základní školy. *31. základní škola Plzeň* [online]. Plzeň [cit. 2018-05-22]. Dostupné z: http://www.zs31.plzen-edu.cz/dokumenty/svp/skola-pro-21-stoleti-svp-31-zakladni-skoly.aspx.

SOME PROPERTIES OF WEAK ISOMETRIES IN DIRECTED GROUPS

Milan Jasem
Faculty of Chemical and Food Technology
Slovak University of Technology, Radlinského 9, 812 37 Bratislava, Slovak Republic
milan.jasem@stuba.sk

Abstract: *In the paper weak isometries in directed groups are investigated. A weak isometry f in a directed group G is a mapping from G into G such that $|f(x) - f(y)| = |x - y|$ for all $x, y \in G$. It is shown that set of all weak isometries in a directed group G forms a group under the composition of mappings and some results of K. L. N. Swamy concerning isometries in abelian lattice ordered groups are extended to all directed groups. The question whether a directed group is determined by its group of weak isometries is also considered. Further, it is proved that in any directed group G such that if $g' \in G^+$, then $g' = 2g$ for some $g \in G^+$, there exists a one-to-one correspondence between stable weak isometries in G and direct decompositions of G with abelian second factor.*

Keywords: directed group, intrinsic metric, weak isometry, direct decomposition.

INTRODUCTION

An isometry in an abelian lattice ordered group G was introduced by K. L. N. Swamy in [13] as a bijection $f : G \to G$ preserving the intrinsic metric $d(x,\, y) = |x - y|$, i. e. $|x - y| = |f(x) - f(y)|$ for each $x,\, y \in G$. W. C. Holland [1] considered whether other intrinsic metrics might be naturally defined on a lattice ordered group. Isometries in non-abelian lattice ordered groups were studied by J. Jakubík [2, 3]. The mappings preserving the intrinsic metric $d(x,\, y)$ in a representable lattice ordered group were dealt with by J. Jakubík in [4]. J. Rachůnek [12] generalized the notions of an intrinsic metric and of an isometry to any partially ordered group and investigated isometries in a 2-isolated abelian Riesz group. Isometries in Riesz groups and distributive multilattice ordered groups were examined by the author in [5, 6, 8, 9]. Weak isometries in directed groups were studied in [7]. Partially ordered semigroups were used for constructions of EL-semihypergroups in [11].

1 PRELIMINARIES

We review some notions and notations used in the paper. Let G be a partially ordered group (notation po-group). The group operation will be written additively.

If M is a subset of G, then we denote by $U(M)$ and $L(M)$ the set of all upper bounds and the set of all lower bounds of the set M in G, respectively. If $a_1, \ldots, a_n \in G$, then we write $U(a_1, \ldots, a_n)$ and $L(a_1, \ldots, a_n)$ instead of $U(\{a_1, \ldots, a_n\})$ and $L(\{a_1, \ldots, a_n\})$, respectively.

If for $x, y \in G$ there exists the least upper bound (greatest lower bound) of the set $\{x, y\}$ in G, then it will be denoted by $x \vee y$ $(x \wedge y)$.

If for elements x and y in G there exists $x \vee y$ in G, then there also exist $x \wedge y$, $(-x) \vee (-y)$, $(-x) \wedge (-y)$ in G and $(x \vee y) + c = (x + c) \vee (y + c)$, $c + (x \vee y) = (c + x) \vee (c + y)$ for each $c \in G$. Moreover, $-(x \vee y) = (-x) \wedge (-y)$.

The dual assertions are valid, too.

The absolute value $|x|$ of an element x of a po-group G is defined by $|x| = U(x, -x)$.

If A is a subset of G, then $A^+ = \{x \in A; x \geq 0\}$.

The set of all subsets of G will be denoted by $exp\, G$.

A po-group G is called directed if $U(x, y) \neq \emptyset$ and $L(x, y) \neq \emptyset$ for every $x, y \in G$.

A po-group G is called 2-isolated if $2a \geq 0$ implies $a \geq 0$ for every $a \in G$.

K. L. N. Swamy [13] showed that in any abelian lattice ordered group H the mapping $d : H \times H \to H$ defined by $d(x, y) = |x - y|$ is an intrinsic metric (or an autometrization) in H, i. e. it satisfies the formal properties of a distance function:

(M_1) $d(x, y) \geq 0$ with equality if and only if $x = y$ (positive definiteness),
(M_2) $d(x, y) = d(y, x)$ (symmetry),
(M_3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A simple example of an intrinsic metric is well known metric $d(x, y) = |x - y|$ in the additive group R of all real numbers with the natural order.

J. Rachůnek [12] defined an intrinsic metric on G as a mapping $d : G \times G \to exp\, G$ satisfying the following conditions for every $x, y, z \in G$:

$(\overline{M}_1)$ $d(x, y) \subseteq U(0)$ and $d(x, y) = U(0)$ if and only if $x = y$,
$(\overline{M}_2)$ $d(x, y) = d(y, x)$,
$(\overline{M}_3)$ $d(x, y) \supseteq d(x, z) + d(z, y)$

and showed that in any 2-isolated abelian Riesz group H the mapping $d : H \times H \to exp\, H$ defined by $d(x, y) = |x - y|$ is an intrinsic metric in H.

The conditions $(\overline{M}_1)$ - $(\overline{M}_3)$ concerning a po-group G are analogous to the above mentioned conditions (M_1) - (M_3) for an abelian lattice ordered group H.

In [10] it was shown that $d(x, y) = |x - y|$ is an intrinsic metric in any 2-isolated abelian po-group.

A mapping $f : G \to G$ is called a weak isometry in G if $|x - y| = |f(x) - f(y)|$ for every $x, y \in G$.

A weak isometry f is called a stable weak isometry, if $f(0) = 0$.

A weak isometry f is called an isometry, if f is a bijection.

If $f : G \to G$ and $g : G \to G$, then the mapping $f \circ g : G \to G$ defined by $(f \circ g)(x) = g(f(x))$ for each $x \in G$ is called a composition of mappings f and g.

A mapping $t : G \to G$ defined by $t(x) = x + c$ for each $x \in G$, where $c \in G$, is called a right translation by c in G and will be denoted by t_c. The set of all right translations in G will be denoted by $T(G)$. If $t_b, t_c \in T(G)$, then $t_b \circ t_c = t_{b+c}$.

If f is a weak isometry in G, then the mapping g defined by $g(x) = f(x) - f(0)$ for each $x \in G$ is a stable weak isometry in G. The mapping g is called a stable weak isometry associated with the weak isometry f and will be denoted by $\bar{f}$. Thus $f(x) = \bar{f}(x) + f(0)$ for each $x \in G$. If we put $t(x) = x + f(0)$ for each $x \in G$, then $f(x) = t(\bar{f}(x))$ for each $x \in G$.

Hence each weak isometry in G can be represented as a composition of a stable weak isometry and a right translation. Thus for finding of all weak isometries it suffices to determine all stable weak isometries. Every right translation is an isometry.

A mapping g of a set M into M is called an involutory mapping (or an involution) if $g(g(x)) = x$ for every $x \in M$. We will write $g^2(x)$ instead of $g(g(x))$.

Let H_1 and H_2 be groups and partially ordered sets. Let the mapping $\varphi : H_1 \to H_2$ be a bijection.
(i) If $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$ for each $x, y \in H_1$, then φ is called an order isomorphism (shortly o-isomorphism) of H_1 and H_2.
(ii) If $\varphi(x + y) = \varphi(x) + \varphi(y)$ for each $x, y \in H_1$, then φ is called a group isomorphism (shortly g-isomorphism) of H_1 and H_2.
(iii) If φ is an o-isomorphism and also a g-isomorphism of H_1 and H_2, then φ is called an og-isomorphism.

We will need the following assertions and we will often apply them without special references.

A_1. Any weak isometry in a directed group is a bijection [7, Corollary 5].
(Hence the notions of an isometry and of a weak isometry coincide in any directed group.)

A_2. Any stable weak isometry f in a directed group H is an involutory group homomorphism and $f(x) + x = x + f(x)$ for each $x \in H$ [7, Theorems 3 and 4].

2 PROPERTIES OF WEAK ISOMETRIES IN DIRECTED GROUPS

Theorem 1 *The set $I(G)$ of all weak isometries in a directed group G is a group under the composition of mappings. The set $SI(G)$ of all stable weak isometries in G and the set $T(G)$ of all right translations in G are subgroups of $I(G)$.*

Proof. Let f be a weak isometry in G. Since f is a bijection, there exists an inverse mapping f^{-1} of the mapping f in G. Let $x, y \in G$. Then $|f^{-1}(x) - f^{-1}(y)| = |f(f^{-1}(x)) - f(f^{-1}(y))| = |x - y|$. Thus f^{-1} is a weak isometry in G. Let h and g be weak isometries in G, z, $t \in G$. Then $|(g \circ h)(z) - (g \circ h)(t)| = |h(g(z)) - h(g(t))| = |g(z) - g(t)| = |z - t|$. Hence the composition of weak isometries h and g is also a weak isometry in G. Therefore the set $I(G)$ of all weak isometries in a directed group G is a group under the composition of mappings. Clearly, $SI(G)$ and $T(G)$ are subgroups of $I(G)$.

Remark. The set $I(G)$ of all weak isometries in a directed group G can be partially ordered in a natural manner by letting for f, $g \in I(G)$, $f \leq g$ to mean $f(x) \leq g(x)$ for each $x \in G$.

Theorem 2 *Let G be a directed group and f a weak isometry in G. Then the following conditions are equivalent:*
(i) f is order preserving (i. e. $x \leq y \Rightarrow f(x) \leq f(y)$),
(ii) $\bar{f}$ is order preserving,
(iii) $\bar{f}(x) = x$ for each $x \in G$.

Proof. (i) $\Rightarrow$ (ii). Let f is order preserving. Let $y, z \in G, y \leq z$. Then $f(y) \leq f(z)$ and hence $\bar{f}(y) = f(y) - f(0) \leq f(z) - f(0) = \bar{f}(z)$. Thus $\bar{f}$ is order preserving.
(ii) $\Rightarrow$ (i). We assume now that $\bar{f}$ is order preserving, $y, z \in G, y \leq z$. Thus $\bar{f}(y) \leq \bar{f}(z)$. This implies $f(y) = \bar{f}(y) + f(0) \leq \bar{f}(z) + f(0) = f(z)$. Hence f is order preserving.
(ii) $\Rightarrow$ (iii). Let $\bar{f}$ is order preserving. Let $t \in G^+$. Then from $|t - 0| = |\bar{f}(t) - \bar{f}(0)| = |\bar{f}(t)|$ it

follows that $\bar{f}(t) \leq t$. Thus $0 \leq -\bar{f}(t) + t$. This yields $0 = \bar{f}(0) \leq \bar{f}(-\bar{f}(t) + t) = -(\bar{f})^2(t) + \bar{f}(t) = -t + \bar{f}(t) = -(-\bar{f}(t) + t)$. Therefore $-\bar{f}(t) + t = 0$. Hence $\bar{f}(t) = t$.

Now, let $x \in G$. Since G is a directed group, $x = x_1 - x_2$ for some $x_1, x_2 \in G^+$. Then $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = x_1 - x_2 = x$.
(iii) $\Rightarrow$ (ii). It is obvious.

Corollary 1 *A weak isometry f in a directed group G is order preserving if and only if $f(x) = x + f(0)$ for each $x \in G$ (i. e. f is a translation).*

Theorem 3 *Let G be a directed group and f a weak isometry in G. Then the following conditions are equivalent:*
(i) f is order reversing (i. e. $x \leq y \Rightarrow f(x) \geq f(y)$),
(ii) $\bar{f}$ is order reversing,
(iii) $\bar{f}(x) = -x$ for each $x \in G$.

Proof. (i) $\Rightarrow$ (ii). Let $y, z \in G$, $y \leq z$. Let f is order reversing. Then $f(y) \geq f(z)$ and hence $\bar{f}(y) = f(y) - f(0) \geq f(z) - f(0) = \bar{f}(z)$. Thus $\bar{f}$ is order reversing.
(ii) $\Rightarrow$ (i). We assume now that $\bar{f}$ is order reversing, $y, z \in G$, $y \leq z$. Thus $\bar{f}(y) \geq \bar{f}(z)$. This implies $f(y) = \bar{f}(y) + f(0) \geq \bar{f}(z) + f(0) = f(z)$. Hence f is order reversing.
(ii) $\Rightarrow$ (iii). Let $\bar{f}$ is order reversing. Let $t \in G^+$. Then from $|t - 0| = |\bar{f}(t) - \bar{f}(0)| = |\bar{f}(t)|$ it follows that $-\bar{f}(t) \leq t$. Thus $0 \leq \bar{f}(t) + t$. This yields $0 = \bar{f}(0) \geq \bar{f}(\bar{f}(t) + t) = (\bar{f})^2(t) + \bar{f}(t) = t + \bar{f}(t) = \bar{f}(t) + t$. Hence $\bar{f}(t) + t = 0$. Therefore $\bar{f}(t) = -t$.

Now, let $x \in G$. Since G is a directed group, $x = x_1 - x_2$ for some $x_1, x_2 \in G^+$. Thus $\bar{f}(x_1 + x_2) = \bar{f}(x_1) + \bar{f}(x_2) = -x_1 - x_2 \leq 0$. Then from $|(x_1 + x_2) - 0| = |\bar{f}(x_1 + x_2) - \bar{f}(0)|$ we obtain $\bar{f}(x_1 + x_2) = -x_2 - x_1$. Thus $-x_1 - x_2 = -x_2 - x_1$ and hence $x_1 + x_2 = x_2 + x_1$. Then we have $\bar{f}(x) = \bar{f}(x_1 - x_2) = \bar{f}(x_1) - \bar{f}(x_2) = -x_1 + x_2 = -(-x_2 + x_1) = -(x_1 - x_2) = -x$.
(iii) $\Rightarrow$ (ii). It is obvious.

Corollary 2 *A weak isometry f in a directed group G is order reversing if and only if $f(x) = -x + f(0)$ for each $x \in G$.*

Corollaries 1 and 2 extend results of K. L. N. Swamy concerning isometries in abelian lattice ordered groups [13, Theorems 2 and 3] to all directed groups.

Theorem 4 *Let G be a directed group and f a weak isometry in G. Then f^2 is a translation such that $f^2(x) = x + f^2(0)$ for each $x \in G$.*

Proof. Let f be a weak isometry in G, $x \in G$. Then $f(x) = t(\bar{f}(x))$, where t is a translation in G defined by $t(x) = x + f(0)$ for each $x \in G$. Since $\bar{f}$ is an involutory group homomorphism, we have $f^2(x) = f(t(\bar{f}(x))) = f(\bar{f}(x) + f(0)) = t(\bar{f}(\bar{f}(x) + f(0))) = t((\bar{f})^2(x) + \bar{f}(f(0))) = (\bar{f})^2(x) + \bar{f}(f(0)) + f(0) = x + f(f(0)) - f(0) + f(0) = x + f^2(0)$.

Theorem 4 generalizes Corollary 1 of K. L. N. Swamy [13].

Theorem 5 *Let H be a non-void subset of a directed group G and f a stable weak isometry in G. Then H is a subgroup of G if and only if $f(H)$ is a subgroup of G.*

Proof. Let H be a subgroup of G. Since f is a group homomorphism, $f(H)$ is also a subgroup of G. Assume now that $f(H)$ is a subgroup of G. Since f is an involution, we have $f(f(H)) = H$. Hence H is a subgroup of G.

Theorem 6 *Let G and H be directed groups. Let $\varphi : G \to H$ be an og-isomorphism of G and H.*
(i) Let f be a weak isometry in G. Then the mapping $f^{\varphi} : H \to H$ such that $f^{\varphi}(\varphi(x)) = \varphi(f(x))$ for each $x \in G$ is a weak isometry in H.
(ii) The mapping $\theta : I(G) \to I(H)$ defined by $\theta(f) = f^{\varphi}$ for each $f \in I(G)$, where f^{φ} is as in (i), is an og-isomorphism of $I(G)$ and $I(G)$ such that $\theta(T(G)) = T(H)$.

Proof. (i) First we prove that $\varphi(|a-b|) = |\varphi(a) - \varphi(b)|$ for each $a, b \in G$.
Let $c \in |a-b|$. Thus $c \geq a-b$, $c \geq b-a$ and hence $\varphi(c) \geq \varphi(a-b) = \varphi(a) - \varphi(b)$, $\varphi(c) \geq \varphi(b-a) = \varphi(b) - \varphi(a)$. This implies $\varphi(c) \in U(\varphi(a)-\varphi(b), \varphi(b)-\varphi(a)) = |\varphi(a)-\varphi(b)|$. Therefore $\varphi(|a-b|) \subseteq |\varphi(a) - \varphi(b)|$.

Let $d' \in |\varphi(a) - \varphi(b)|$. Then $d' = \varphi(d)$ for some $d \in G$. Thus $\varphi(d) \geq \varphi(a) - \varphi(b) = \varphi(a-b)$, $\varphi(d) \geq \varphi(b) - \varphi(a) = \varphi(b-a)$. From this follows that $d \geq a-b$, $d \geq b-a$. Hence $d \in |a-b|$. Therefore $|\varphi(a) - \varphi(b)| \subseteq \varphi(|a-b|)$.

Let $x', y' \in H$. Thus $\varphi(x) = x'$, $\varphi(y) = y'$ for some $x, y \in G$. Then $|f^{\varphi}(x') - f^{\varphi}(y')| = |\varphi(f(x)) - \varphi(f(y))| = \varphi(|f(x) - f(y)|) = \varphi(|x-y|) = |\varphi(x) - \varphi(y)| = |x'-y'|$. Therefore f^{φ} is a weak isometry in H.
(ii) First we prove that θ is a bijection. Let g, $h \in I(G)$, $\theta(g) = \theta(h)$. Thus $g^{\varphi}(z) = h^{\varphi}(z)$ for any $z \in H$.

Let $x \in G$. Then $\varphi(x) \in H$ and hence $g^{\varphi}(\varphi(x)) = h^{\varphi}(\varphi(x))$. This yields $\varphi(g(x)) = \varphi(h(x))$. From this follows that $g(x) = h(x)$. Therefore $g = h$.

It is easy to see that the mapping φ^{-1} is an og-isomorphism of H and G.

Let $t \in I(H)$. In view of (i) we have that the mapping $u = t^{\varphi^{-1}}$ is a weak isometry in G. Let $x \in G$. Then $u^{\varphi}(\varphi(x)) = \varphi(u(x)) = \varphi(t^{\varphi^{-1}}(x)) = \varphi(t^{\varphi^{-1}}(\varphi^{-1}(\varphi(x)))) = \varphi(\varphi^{-1}(t(\varphi(x)))) = t(\varphi(x))$. Thus $t = u^{\varphi} = \theta(u)$. Hence θ is a bijection.

Let $g_1, g_2 \in I(G), g_1 \leq g_2$. Let $d \in H$. Then $g_1(\varphi^{-1}(d) \leq g_2(\varphi^{-1}(d)$. Hence $g_1^{\varphi}(d) = g_1^{\varphi}(\varphi(\varphi^{-1}(d))) = \varphi(g_1(\varphi^{-1}(d))) \leq \varphi(g_2(\varphi^{-1}(d))) = g_2^{\varphi}(\varphi(\varphi^{-1}(d))) = g_2^{\varphi}(d)$. Therefore $\theta(g_1) = g_1^{\varphi} \leq g_2^{\varphi} = \theta(g_2)$.

Analogously we can prove that if h_1, $h_2 \in I(G)$ and $\theta(h_1) \leq \theta(h_2)$, then $h_1 \leq h_2$.

Now assume that $g, h \in I(G)$, $x \in G$. Then we have $\theta(g \circ h)(\varphi(x)) = (g \circ h)^{\varphi}(\varphi(x)) = \varphi((g \circ h)(x)) = \varphi((h(g(x))) = h^{\varphi}(\varphi(g(x))) = h^{\varphi}(g^{\varphi}(\varphi(x))) = (g^{\varphi} \circ h^{\varphi})(\varphi(x)) = (\theta(g) \circ \theta h)(\varphi(x))$. Hence $\theta(g \circ h) = \theta(g) \circ \theta(h)$.

Let $t \in T(G)$ be the translation by the element $c \in G$. Thus $t(x) = x + c$ for each $x \in G$. Let $y \in H$. Then $\theta(t)(y) = t^{\varphi}(y) = t^{\varphi}(\varphi(\varphi^{-1}(y))) = \varphi(t(\varphi^{-1}(y))) = \varphi(\varphi^{-1}(y) + c) = \varphi(\varphi^{-1}(y)) + \varphi(c) = y + \varphi(c)$. Therefore $t^{\varphi} \in T(H)$. Hence $\theta(T(G)) \subseteq T(H)$.

Assume now that $t \in T(H)$. Since θ^{-1} is an og-isomorphism of H and G, we get that $\theta^{-1}(t)$ is a translation in G. Thus $T(H) \subseteq \theta(T(G))$.

Theorem 6 generalizes Theorem 1 of K. L. M. Swamy [14]

Theorem 7 *Let G and H be directed groups. If there exists an og-isomorphism θ of $I(G)$ and $I(H)$ such that $\theta(T(G)) = T(H)$, then G is og-isomorphic with H.*

Proof. Let θ be an og-isomorphism of $I(G)$ and $I(H)$. Let $a \in G$. Thus $t_a \in I(G)$ and $\theta(t_a)$ is a translation by a' in H for some $a' \in H$. Hence $\theta(t_a) = t_{a'}$.

Put $\varphi(a) = a'$ for each $a \in G$. Clearly φ is a bijection.

Let $b, c \in G$. Then $t_{(b+c)'} = \theta(t_{b+c}) = \theta(t_b \circ t_c) = \theta(t_b) \circ \theta(t_c) = t_{b'} \circ t_{c'} = t_{b'+c'}$. This implies $(b+c)' = b' + c'$. Thus $\varphi(b+c) = \varphi(b) + \varphi(c)$.

If $b, c \in G$ and $b \leq c$, then $t_b \leq t_c$. This yields $t_{b'} = \theta(t_b) \leq \theta(t_c) = t_{c'}$. Thus $t_{b'}(x) \leq t_{c'}(x)$ for each $x \in H$. Hence $x + b' \leq x + c'$ for each $x \in H$. From this follows that $b' \leq c'$. Therefore $\varphi(b) \leq \varphi(c)$.

Analogously we can prove that if $b, c \in G$ and $\varphi(b) \leq \varphi(c)$, then $b \leq c$. Thus φ is an og-isomorphism.

Theorem 7 generalizes Theorem 2 of K. L. M. Swamy [14]

A po-group H is called a direct product of its subgroups P and Q (notation $H = P \times Q$) if the following conditions are satisfied:

(i) Each element $x \in H$ can be uniquely represented in the form $x = p + q$, where $p \in P, q \in Q$ (elements p and q we denote by x_P and x_Q and call components of x in the direct factors P and Q, respectively),

(ii) $p + q = q + p$ for each $p \in P, q \in Q$,

(iii) $x \leq y$ if and only if $x_P \leq y_P$ and $x_Q \leq y_Q$ for each $x, y \in H$.

In the case that $H = P \times Q$ it is also spoken about the direct decomposition of H.
If $H = P \times Q$, then clearly $(x+y)_P = x_P + y_P$, $(x+y)_Q = x_Q + y_Q$, $(x-y)_P = x_P - y_P$, $(x-y)_Q = x_Q - y_Q$ for all $x,\, y \in H$.

Analogously we can define direct decomposition of a partially ordered monoid.

We recall that a monoid is a non-empty set which is closed under an associative binary operation $+$ and has a zero element 0.

A partially ordered monoid H is called a direct product of its submonoids P and Q (notation $H = P \times Q$) if the conditions (i) - (iii) from the definition of a direct decomposition of a po-group above are satisfied.

Theorem 8 *Let G be a directed group and f a stable weak isometry in G. Let $A_1 = \{x \in G^+; f(x) = x\}$, $B_1 = \{x \in G^+; f(x) = -x\}$, $A = A_1 - A_1$, $B = B_1 - B_1$. Then*

(i) A_1 is a monoid and a convex subset of G,

(ii) A is a convex subgroup of G and $f(a) = a$ for each $a \in A$,

(iii) B_1 is a commutative monoid and a convex subset of G,

(iv) B is an abelian convex subgroup of G and $f(b) = -b$ for each $b \in B$,

(v) $a_1 \wedge b_1 = 0$, $a_1 \vee b_1 = a_1 + b_1 = b_1 + a_1, a_1 = 0 \vee (a_1 - b_1), b_1 = 0 \vee (b_1 - a_1)$ for each $a_1 \in A_1, b_1 \in B_1$,

(vi) $a + b = b + a$ for each $a \in A$, $b \in B$,

(vii) $A_1 \cap B_1 = \{0\}$, $A \cap B = \{0\}$.

Proof. (i) Let $a_1,\, a_2 \in A_1$. Then $f(a_1 + a_2) = f(a_1) + f(a_2) = a_1 + a_2$. Hence A_1 is a monoid.

Let $a,\, b \in A_1,\, c \in G,\, a \leq c \leq b$. Since $c \geq 0$, from $|c - 0| = |f(c) - f(0)| = |f(c) - 0|$ we get $c \geq f(c)$. Further, from $|b - c| = |f(b) - f(c)| = |b - f(c)|$ we obtain $b - c \geq b - f(c)$. This yields $f(c) \geq c$. Thus $f(c) = c$ and hence $c \in A_1$. Therefore A_1 is a convex subset of G.

(ii) Let $x, y \in A$. Thus $x = x_1 - x_2$, $y = y_1 - y_2$, where $x_1, x_2, y_1, y_2 \in A_1$. Then $x - y = x_1 - x_2 - (y_1 - y_2) = x_1 - x_2 + y_2 - y_1 = x_1 - x_2 + y_2 + x_2 - x_2 - y_1 = (x_1 - x_2 + y_2 + x_2) - (y_1 + x_2)$. Since $x_1 - x_2 + y_2 + x_2 \geq 0$, $y_1 + x_2 \geq 0$ and $f(x_1 - x_2 + y_2 + x_2) = f(x_1) - f(x_2) + f(y_2) + f(x_2) = x_1 - x_2 + y_2 + x_2$, $f(y_1 + x_2) = f(y_1) + f(x_2) = y_1 + x_2$, we have $x - y \in A$. Hence A is a subgroup of G.

Further, $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = x_1 - x_2 = x$.

Let a, $b \in A$, $c \in G$, $a \leq c \leq b$. Then $0 \leq c - a \leq b - a$. Since $0, b - a \in A_1$, in view of (i) we have $c - a \in A_1$. Then $c - a + a = c \in A$. Thus A is a convex subset of G.

(iii) Let b_1, $b_2 \in B_1$. Then $f(b_1 + b_2) = f(b_1) + f(b_2) = -b_1 - b_2 \leq 0$. Then from $|(b_1 + b_2) - 0| = |f(b_1 + b_2) - f(0)| = |-f(b_1 + b_2)|$ we get $f(b_1 + b_2) = -(b_1 + b_2)$. Thus $-b_1 - b_2 = -b_2 - b_1$. This yields $b_1 + b_2 = b_2 + b_1$. Therefore B_1 is a commutative monoid.

Let a, $b \in B_1$, $c \in G$, $a \leq c \leq b$. From $|b - c| = |f(b) - f(c)| = |-b - f(c)| = |f(c) + b|$ it follows that $b - c \geq f(c) + b$. This yields $0 \geq b - c - b \geq f(c)$. Since $|c - 0| = |f(c) - f(0)| = |-f(c)|$, we have $c = -f(c)$. Thus $f(c) = -c$ and hence $c \in B_1$. Therefore B_1 is a convex subset of G.

(iv) Let $x, y \in B$. Thus $x = x_1 - x_2$, $y = y_1 - y_2$ where $x_1, x_2, y_1, y_2 \in B_1$. Then $x - y = x_1 - x_2 - (y_1 - y_2) = x_1 - x_2 + y_2 - y_1 = x_1 - x_2 + y_2 + x_2 - x_2 - y_1 = (x_1 - x_2 + y_2 + x_2) - (y_1 + x_2)$. Since $x_1 - x_2 + y_2 + x_2 \geq 0$, $y_1 + x_2 \geq 0$ and $f(x_1 - x_2 + y_2 + x_2) = f(x_1) - f(x_2) + f(y_2) + f(x_2) = -x_1 + x_2 - y_2 - x_2 = -(x_1 - x_2 + y_2 + x_2)$, $f(y_1 + x_2) = f(y_1) + f(x_2) = -y_1 - x_2 = -(y_1 + x_2)$, we have $x_1 - x_2 + y_2 + x_2$, $y_1 + x_2 \in B_1$ and hence $x - y \in B$. Thus B is a subgroup of G.

In view of (iii) we have $x + y = x_1 - x_2 + y_1 - y_2 = y_1 - y_2 + x_1 - x_2 = y + x$.

Further, $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = -x_1 + x_2 = -(-x_2 + x_1) = -(x_1 - x_2) = -x$.

Let a, $b \in B$, $c \in G$, $a \leq c \leq b$. Then $0 \leq c - a \leq b - a$. Since $0, b - a \in B_1$, in view of (iii) we have $c - a \in B_1$. Then $c - a + a = c \in B$. Thus B is a convex subset of G.

(v) Let $a_1 \in A_1, b_1 \in B_1$. From $|(a_1 + b_1) - 0| = |f(a_1 + b_1) - f(0)| = |f(a_1) + f(b_1)| = |a_1 - b_1|$ we obtain $a_1 + b_1 = (a_1 - b_1) \vee (b_1 - a_1)$. Since $a_1 - b_1 \leq a_1 \leq a_1 + b_1$, $b_1 - a_1 \leq b_1 \leq a_1 + b_1$, we have $a_1 \vee b_1 = a_1 + b_1$ and hence $a_1 \wedge b_1 = 0$. Thus $a_1 + b_1 = b_1 + a_1$.

Further from $a_1 \vee b_1 = a_1 + b_1$ it follows that $0 \vee (a_1 - b_1) = a_1$, $0 \vee (b_1 - a_1) = b_1$.

(vi) Let $a \in A, b \in B$. Thus $a = a_1 - a_2, b = b_1 - b_2$ where a_1, $a_2 \in A_1, b_1$, $b_2 \in B_1$. In view of (v) we get $a + b = a_1 - a_2 + b_1 - b_2 = b_1 - b_2 + a_1 - a_2 = b + a$.

(vii) Let $x \in A_1 \cap B_1$. Then $f(x) = x$, $f(x) = -x$. Hence $-x = x \geq 0$. This yields $x = 0$.

Let $y \in A \cap B$. Thus $y = a_1 - a_2 = b_1 - b_2$, where a_1, $a_2 \in A_1$, b_1, $b_2 \in B_1$. In view of (v) we obtain $a_1 + b_2 = a_2 + b_1$. Then $a_1 - b_2 = f(a_1 + b_2) = f(a_2 + b_1) = a_2 - b_1$. By (v), $a_1 = 0 \wedge (a_1 - b_2)$, $a_2 = 0 \wedge (a_2 - b_1)$. This implies $a_1 = a_2$ and hence $y = 0$.

Theorem 9 *Let G be a directed group such that for each $g' \in G^+$ there exists $g \in G^+$ such that $g' = 2g$. Let f be a stable weak isometry in G. Let A_1, B_1, A, B be as in Theorem 8. Then $G = A \times B$ and $f(x) = x_A - x_B$ for each $x \in G$.*

Proof. Let $x' \in G^+$. Thus $x' = 2x$ for some $x \in G^+$. Then $|x - 0| = |f(x) - f(0)| = |f(x)|$ implies $x = (-f(x)) \vee f(x)$. Further, we have $f(x) + x \geq 0$, $-f(x) + x \geq 0$. Then $f(f(x) + x) = f^2(x) + f(x) = x + f(x) = f(x) + x$, $f(-f(x) + x) = -f^2(x) + f(x) = -x + f(x) = -(-f(x) + x)$. Therefore $f(x) + x \in A_1$, $-f(x) + x \in B_1$. Let $x_1 = f(x) + x$, $x_2 = -f(x) + x$. Thus $x' = 2x = x_1 + x_2$ where $x_1 \in A_1$, $x_2 \in B_1$.

Let $x' = a + b$ for some $a \in A_1$, $b \in B_1$. Thus $a + b = x_1 + x_2$. In view of Theorem 8 (v) we obtain $a - x_1 = x_2 - b$, $a - x_1 \in A$, $x_2 - b \in B$. According to Theorem 8 (vii), $A \cap B = \{0\}$. Hence $a - x_1 = x_2 - b = 0$. Therefore $x_1 = a$, $x_2 = b$.

Thus any element $x' \in G^+$ can be uniquely represented in the form $x' = x_1 + x_2$ where $x_1 \in A_1$, $x_2 \in B_1$.

Let $y, z \in G^+, y \leq z$. Then $y = y_1 + y_2$, $z = z_1 + z_2$, $z - y = (z - y)_1 + (z - y)_2$ for some $y_1, z_1, (z - y)_1 \in A_1$, x_2, y_2, $(z - y)_2 \in B_1$. Thus $(z - y)_1 + (z - y)_2 = z - y = z_1 + z_2 - (y_1 + y_2) = (z_1 - y_1) + (z_2 - y_2)$ and hence $-(z_1 - y_1) + (z - y)_1 = (z_2 - y_2) - (z - y)_2$. Since $-(z_1 - y_1) + (z - y)_1 \in A,\ (z_2 - y_2) - (z - y)_2 \in B,\ A \cap B = \{0\}$, we have $z_1 - y_1 = (z - y)_1 \geq 0, z_2 - y_2 = (z - y)_2 \geq 0$. This yields $z_1 \geq y_1$, $z_2 \geq y_2$. In view of Theorem 8 we conclude $G^+ = A_1 \times B_1$.

Let $g \in G$. Thus $g = h - k$, where $h, k \in G^+$. Hence $h = h_1 + h_2, k = k_1 + k_2$, where $h_1, k_1 \in A_1$, h_2, $k_2 \in B_1$. Then $g = h_1 + h_2 - k_2 - k_1 = h_1 - k_1 + h_2 - k_2$. Let $g_1 = h_1 - k_1,\ g_2 = h_2 - k_2$. Thus $g = g_1 + g_2$ where $g_1 \in A, g_2 \in B$.

Let $g = c + d$ for some $c \in A$, $d \in B$. Thus $g_1 + g_2 = c + d$ and hence $-c + g_1 = d - g_2$. Since $-c + g_1 \in A$, $d - g_2 \in B$, $A \cap B = \{0\}$, we obtain $-c + g_1 = d - g_2 = 0$. Hence $c = g_1 = h_1 - k_1 = g_A, d = g_2 = h_2 - k_2 = g_B$.

Let $p, q \in G, p \leq q$. Then $p = p_A + p_B,\ q = q_A + q_B$. Further, we have $0 \leq q - p = (q - p)_{A_1} + (q - p)_{B_1}$. Thus $q - p = (q_A + q_B) - (p_A + p_B) = (q_A - p_A) + (q_B - p_B) = (q - p)_{A_1} + (q - p)_{B_1}$. This implies $q_A - p_A = (q - p)_{A_1} \geq 0,\ q_B - p_B = (q - p)_{B_1} \geq 0$. Hence $q_A \geq p_A,\ q_B \geq p_B$.

In view of Theorem 8 we conclude $G = A \times B$.

Further, according to Theorem 8 we have $f(x') = f(x'_A + x'_B) = f(x'_A) + f(x'_B) = x'_A - x'_B$.

Theorem 10 *Let G be a directed group and $G = P \times Q$ a direct decomposition of G. Then $|z| = |z_P| + |z_Q|$ for each $z \in G$.*

Proof. Let $x \in |z|$. Then from $x \geq z, x \geq -z$ we obtain $x_P \geq z_P, x_Q \geq z_Q$, $x_P \geq -z_P, x_Q \geq -z_Q$. Thus $x_P \in U(z_P, -z_P) = |z_P|$, $x_Q \in U(z_Q, -z_Q) = |z_Q|$. Hence $x = x_P + z_Q \in |z_P| + |z_Q|$. Therefore $|z| \subseteq |z_P| + |z_Q|$.

Let $y \in |z_P| + |z_Q|$. Thus $y = y_1 + y_2$ where $y_1 \in U(z_P,\ -z_P), y_2 \in U(z_Q, -z_Q)$. This yields $y = y_1 + y_2 \in U(z, -z) = |z|$. Hence $|z_P| + |z_Q| \subseteq |z|$.

Theorem 11 *Let G be a directed group and $G = P \times Q$ a direct decomposition of G with Q abelian. Let $f(x) = x_P - x_Q$ for each $x \in G$. Then f is a stable weak isometry in G.*

Proof. Let $x, y \in G$. In view of Theorem 12 we have $|f(x) - f(y)| = |x_P - x_Q - (y_P - y_Q)| = |x_P - x_Q + y_Q - y_P| = |x_P - y_P - (-y_Q + x_Q)| = |x_P - y_P - (x_Q - y_Q)| = |(x - y)_P - (x - y)_Q)| = |(x - y)_P| + |-(x - y)_Q)| = |(x - y)_P| + |(x - y)_Q)| = |(x - y)_P + (x - y)_Q)| = |x - y|$. Clearly $f(0) = 0$. Therefore f is a stable weak isometry in G.

From Theorems 9 and 11 we get the following corollary.

Corollary 3 *Let G be a directed group such that if $g' \in G^+$, then $g' = 2g$ for some $g \in G^+$. Then there exists a one-to-one correspondence between stable weak isometries in G and direct decompositions of G with abelian second factor.*

Theorem 12 *Let G be a 2-isolated directed group. Let f and g be stable weak isometries in G. Let $A_1^f = \{x \in G^+; f(x) = x\}$, $A_1^g = \{x \in G^+; g(x) = x\}$, $B_1^f = \{x \in G^+; f(x) = -x\}$, $B_1^g = \{x \in G^+; g(x) = -x\}$. If $A_1^f = A_1^g$ and $B_1^f = B_1^g$, then $f(x) = g(x)$ for each $x \in G$.*

Proof. Let $z \in G^+$. Then from $|z| = |z - 0| = |f(z) - f(0)| = |f(z)|$ we obtain $z \geq f(z)$, $z \geq -f(z)$. Thus $-f(z) + z \geq 0$, $f(z) + z \geq 0$. Let $z_1 = f(z) + z$, $z_2 = -f(z) + z$. Then $f(z_1) = f(f(z)+z) = f^2(z)+f(z) = z+f(z) = f(z)+z = z_1$, $f(z_2) = f(-f(z)+z) = -f^2(z)+f(z) = -z + f(z) = -(-f(z) + z) = -z_2$. Hence $2z = z_1 + z_2$ where $z_1 \in A_1^f$, $z_2 \in B_1^f$.

Let $2z = a + b$ where $a \in A_1^f$, $b \in B_1^f$. Thus $z_1 + z_2 = a + b$. Hence $z_1 - z_2 = f(z_1) + f(z_2) = f(z_1 + z_2) = f(a + b) = f(a) + f(b) = a - b$. Then from $-z_2 - z_1 = -b - a$ and $z_1 - z_2 = a - b$ we get $-2z_2 = -2b$. In view of Theorem 8 (vi) we get $2(z_2 - b) = 0$, $2(b - z_2) = 0$. Since G is a 2-isolated group, we have $z_2 \geq b$, $b \geq z_2$. Therefore $z_2 = b$. Then $z_1 = a$.

Hence the element $2z$ can be uniquely represented in the form $2z = z_1 + z_2$, where $z_1 \in A_1^f$, $z_2 \in B_1^f$.

Analogously we can show that $2z$ can be uniquely represented in the form $2z = z_1' + z_2'$ where $z_1' \in A_1^g$, $z_2' \in B_1^g$, $z_1' = g(z) + z$, $z_2' = -g(z) + z$.

If $A_1^f = A_1^g$ and $B_1^f = B_1^g$, then $z_1 = z_1'$, $z_2 = z_2'$. Thus $f(z) + z = g(z) + z$, $-f(z) + z = -g(z) + z$. This implies $f(z) = g(z)$.

Let $x \in G$. Since G is a directed group, we have $x = x_1 - x_2$ where $x_1, x_2 \in G^+$. Then $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = g(x_1) - g(x_2) = g(x_1 - x_2) = g(x)$.

CONCLUSION

In the paper some properties of weak isometries in directed groups are described. It is shown that directed groups G and H are og-isomorphic if and only if there exists an og-isomorphism θ of $I(G)$ and $I(H)$ carrying translations onto translations. Further it is shown that stable weak isometries in some directed groups are directly related to the structure of these directed groups. Namely, there exists a one-to-one correspondence between stable weak isometries in a directed group G with the property that for each $g' \in G^+$ there exists $g \in G^+$ such that $g' = 2g$ and direct decompositions of G with abelian second factor.

References

[1] Holland, W. C.: *Intrinsic metrics for lattice oredered groups.* Algebra Universalis **19**, 1984, p. 142–150. ISSN 0002-5240

[2] Jakubík, J.: *Isometries of lattice ordered groups.* Czechoslovak Math. J. **30**, 1980, p. 142–152. ISSN 0011-4642

[3] Jakubík, J.: *On isometries of non-abelian lattice ordered groups.* Math. Slovaca **31**, 1981,p. 171–175. ISSN 0139-9918

[4] Jakubík, J.: *Weak isometries of lattice ordered groups.* Math. Slovaca **38**, 1988, p. 133–138. ISSN 0139-9918

[5] Jasem, M.: *Isometries in Riesz groups.* Czechoslovak Math. J. **36**, 1986, p. 35–43. ISSN 0011-4642

[6] Jasem, M.: *Isometries in non-abelian multilattice groups.* Math. Slovaca **46**, 1996, p. 491–496. ISSN 0139-9918

[7] Jasem, M.: *Weak isometries in directed groups.* Math. Slovaca **44**, 1994, p. 39–43. ISSN 0139-9918
[8] Jasem, M.: *Weak isometries in partially ordered groups.* Acta Math. Univ. Comen. **63**, 1994, p. 259–265. ISSN 0862-9544
[9] Jasem, M.: *Weak isometries and direct decompositions of partially ordered groups.* Tatra Mt. Math. Publ. **5**, 1995, p. 131–142. ISSN 1210-3195
[10] Jasem, M.: *Intrinsic metric preserving maps on partially ordered groups.* Algebra Universalis **36**, 1996, p. 135–140. ISSN 0002-5240
[11] Novák, M.: *EL-semihypergroups in which the quasi-ordering is not antisymmetric.* Mathematics, Information Technologies and Applied Sciences 2017, Post-conference Proceedings of Extended Versions of Selected papers. Brno: University of Defence in Brno, 2017, p. 183-192. ISBN 978-80-7582-026-6.
[12] Rachůnek, J.: *Isometries in ordered groups.* Czechoslovak Math. J. **34**, 1984, p. 334–341. ISSN 0011-4642
[13] Swamy, K. L. N.: *Isometries in autometrized lattice ordered groups.* Algebra Universalis **8**, 1978, p. 59–64. ISSN 0002-5240
[14] Swamy, K. L. N.: *Isometries in autometrized l-groups II.* Math. Sem. Notes Kobe Univ. **8**, 1977, p. 211–214. ISSN 0385-633X

ABOUT ONE MATEMATICAL MODEL OF MARKET DYNAMICS WITH TIME-DELAY

Denis Khusainov[1], Kseniia Fedorova[1], Josef Diblik[2], Jaromir Bastinec[2]
[1]Taras Shevchenko National University of Kyiv, Faculty of Computer Science and Cybernetics, Department of Modelling of Complex Systems, Glushkova st. 4d, 03187 Kyiv, Ukraine
dkh@unicyb,kiev.ua, fedorova.ksm@gmail.com
[2] Brno University of Technolgy (BUT), Faculty of Electrical Engineering and Communication (FEEC), Department of Mathematics, Technicka 2848/8, 616 00 Brno, Czech Republic
bastinec@feec.vutbr.cz, diblik@feec.vutbr.cz

Abstract: *One of the most important things in real time is modelling of dynamics of economic growth. And one of the determining components of growth is stability of market relations. As a rule, building of mathematical models is based on the balance of products and the possible development of manufacture over a certain time interval.*
As a rule, the mathematical model has equilibrium positions and it is interesting to figure out conditions when this position is asymptotically stable. The state of system at the moment is essentially depend on prehistory. Because of this facts model of free competition market was developed and investigated, found out which components and how affect trade.

Keywords: differential equations, market model, equilibrium point, characteristic equation, delay.

INTRODUCTION

In work [1] the mathematical model of dynamics of pricing in market of free competition was offered. The model had the form of a system of ordinary differential equations with a fractional-rational right-hand side. In this paper, system was written in a universal vector-matrix form. The stability of the equilibrium position is studied, i.e. compromise price.
It should be noted that in modelling of processes in economics, ecology, in social phenomena, in contrast to Newtonian mechanics, the influence of aftereffect plays an important role. The state of system at the moment is essentially depend on prehistory. Laws adopted by the state at present time, begin to act after a certain period of time, called delay time. And more adequate apparatus for modelling dynamic processes are systems with aftereffect, in particular, a system of differential equations with retarded argument [2-4]. Therefore, work considers the influence of delay on stability of pricing dynamics.

1. MODEL OF FREE COMPETITION MARKET

Consider the mathematical model of market dynamics described by the following system of ordinary differential equations [1]

$$\frac{dp_j(t)}{dt} = V_j\left(p_j^*, p_j^0, p_j(t)\right) + C_j\left(p_0, p(t)\right) + D_j\left(p_j^{**}, p_j^0, p(t)\right) + G_j\left(q_j^0, q(t)\right). \tag{1}$$

Here $V_j\left(p_j^*, p_j^0, p_j(t)\right)$, $p_j(t) > p_j^*$, $j = \overline{1,n}$ functions that characterize the influence of seller, $D_j\left(p_j^{**}, p_j^0, p(t)\right)$, $p_j(t) < p$, functions that characterize the influence of buyer, $C_j\left(p_0, p(t)\right)$,

$p(t)=(p_1(t),p_2(t),...,p_n(t))$ functions that characterize the impact of competition, $G_j(q_j^0,q(t))$, $q(t)=(q_1(t),q_2(t),...,q_n(t))$, $j=\overline{1,n}$ functions that characterize the influence of external factors (state, laws, etc.).

The entered variables have the following content: $p_j(t)$ - price of j - th good, which is sold at the moment $t>0$, p_j^0 - equilibrium price for j - th good, $q_j(t)$ - number of units of j - th good, which is sold at the moment $t>0$, q_j^0 - equilibrium quantity of j - th good, p_j^* - lower price threshold of j - th good (prime cost of good), p_j^{**} - upper price threshold j - th good (buyer's capabilities), $p_j' = p_j^0 - p_j^*$ - allowable price difference for j - th good for seller, $p_j'' = p_j^{**} - p_j^0$ - allowable price difference for j - th goods for the buyer.

In normal trading, need be maintained $p_j(t)>p_j^*$, $j=\overline{1,n}$, i.e the seller should not suffer losses. As the functions $V_j(p_j^*,p_j^0,p_j(t))$, $C_j(p^0,p(t))$, $D_j(p_j^{**},p_j^0,p(t))$, $G_j(q_j^0,q(t),p_j^0,p_j(t))$ we can propose the following functions of fractional-rational form [1]

$$V_j(p_j^*,p_j^0,p_j(t))=-v_j p_j' \frac{p_j(t)-p_j^0}{p_j(t)-p_j^*},$$

$$C_j(p^0,p(t))=-\sum_{i=1,i\neq j}^{n} c_{ji} p_j^0\left(\frac{p_j(t)-p_j^0}{p_j^0}-\frac{p_i(t)-p_i^0}{p_i^0}\right), \tag{2}$$

$$D_j(p_j^{**},p_j^0,p(t))=-d_j p_j'' \frac{p_j(t)-p_j^0}{p_j^{**}-p_j(t)}.$$

$$G_j(q_j^0,q(t),p_j^0,p_j(t))=r_j\left[p_j(t)\left(1-e_j\frac{p_j(t)-p_j^0}{p_j^0}+\sum_{i=1,i\neq j}^{n} e_{ji}\frac{p_i(t)-p_i^0}{p_i^0}\right)-p_j^0\right],$$

$p_j^* < p_j(t) < p_j^{**}$, $q_j^* < q_j(t) < q_j^{**}$, v_j, d_j, c_{ji}, r_j, e_j, e_{ji} - weight coefficients of influence functions. $i,j=\overline{1,n}$.

The proposed system (1) with functions (2) is a system where right-hand side is the sum of a linear, quadratic, and fractional-rational summands. The system has a equilibrium point (a price that suits both sides) $p_i(t)=p_i^0$, $i=\overline{1,n}$. It is interesting to see under what conditions this point is asymptotically stable, i.e. the small perturbations of the system parameters do not change the qualitative characteristics.

2. SYSTEM WITH DELAY ON THE PLANE.

We consider the system (1), (2) on the plane. It has the form

$$\frac{dp_1(t)}{dt}=-v_1p_1'\frac{p_1(t)-p_1^0}{p_1(t)-p_1^*}-d_1p_1''\frac{p_1(t)-p_1^0}{p_1^{**}-p_1(t)}-c_{12}p_1^0\left(\frac{p_1(t)-p_1^0}{p_1^0}-\frac{p_2(t)-p_2^0}{p_2^0}\right)+$$
$$+r_1\left[p_1(t)\left(1-e_1\frac{p_1(t)-p_1^0}{p_1^0}+e_{12}\frac{p_2(t)-p_2^0}{p_2^0}\right)-p_1^0\right],$$

$$\frac{dp_2(t)}{dt}=-v_2p_2'\frac{p_2(t)-p_2^0}{p_2(t)-p_2^*}-d_2p_2''\frac{p_2(t)-p_2^0}{p_2^{**}-p_2(t)}-c_{21}p_2^0\left(\frac{p_2(t)-p_2^0}{p_2^0}-\frac{p_1(t)-p_1^0}{p_1^0}\right)+$$

$$+r_2\left[p_2(t)\left(1-e_2\frac{p_2(t)-p_2^0}{p_2^0}+e_{21}\frac{p_1(t)-p_1^0}{p_1^0}\right)-p_2^0\right].$$

As a rule, between the time of passing law and current moment of time, there is a certain gap, which can be called "information lag". Besides, the influence of competition is not shown immediately. Therefore, the functions $G_j(\bullet)$ and $C_j(\bullet)$ in dependence (2) can be considered, depending on prehistory and having the form

$$C_j(\bullet)=-\sum_{\substack{i=1,\\ i\neq j}}^{n}c_{ji}p_j^0\left(\frac{p_j(t-\tau)-p_j^0}{p_j^0}-\frac{p_i(t-\tau)-p_i^0}{p_i^0}\right),$$

$$G_j(\bullet)=r_j\left[p_j(t-\tau)\left(1-e_j\frac{p_j(t-\tau)-p_j^0}{p_j^0}+\sum_{i=1,i\neq j}^{n}e_{ji}\frac{p_i(t-\tau)-p_i^0}{p_i^0}\right)-p_j^0\right],\ i,j=\overline{1,n}. \qquad (3)$$

And on the plane, we will have a system of two differential-difference equations with delay

$$\frac{dp_1(t)}{dt}=-v_1p_1'\frac{p_1(t)-p_1^0}{p_1(t)-p_1^*}-d_1p_1''\frac{p_1(t)-p_1^0}{p_1^{**}-p_1(t)}-c_{12}p_1^0\left(\frac{p_1(t-\tau)-p_1^0}{p_1^0}-\frac{p_2(t-\tau)-p_2^0}{p_2^0}\right)+$$

$$+r_1\left[p_1(t-\tau)\left(1-e_1\frac{p_1(t-\tau)-p_1^0}{p_1^0}+e_{12}\frac{p_2(t-\tau)-p_2^0}{p_2^0}\right)-p_1^0\right],$$

$$\frac{dp_2(t)}{dt}=-v_2p_2'\frac{p_2(t)-p_2^0}{p_2(t)-p_2^*}-d_2p_2''\frac{p_2(t)-p_2^0}{p_2^{**}-p_2(t)}-c_{21}p_2^0\left(\frac{p_2(t-\tau)-p_2^0}{p_2^0}-\frac{p_1(t-\tau)-p_1^0}{p_1^0}\right)+$$

$$+r_2\left[p_2(t-\tau)\left(1-e_2\frac{p_2(t-\tau)-p_2^0}{p_2^0}+e_{21}\frac{p_1(t-\tau)-p_1^0}{p_1^0}\right)-p_2^0\right]. \qquad (4)$$

The system has an equilibrium point (a price that suits both sides) $p_1(t)=p_1^0$, $p_2(t)=p_2^0$. It is interesting to see under which conditions this rest point is asymptotically stable. Let's make a replacement

$$p_1(t)=p_1^0(1+x_1(t)),\ p_2(t)=p_2^0(1+x_2(t)).$$

In this case, the study of steady state of equilibrium $p_1(t)=p_1^0$, $p_2(t)=p_2^0$ of the system (4) reduces to study of zero position of equilibrium $x_1(t)=0$, $x_2(t)=0$ of the system of perturbation equations

$$\frac{dx_1(t)}{dt}=x_1(t)\left[-\frac{v_1p_1'}{x_1(t)p_1^0+p_1'}+\frac{d_1p_1''}{x_1(t)p_1^0-p_1''}\right]-c_{12}[x_1(t-\tau)-x_2(t-\tau)]+$$

$$+r_1[(1+x_1(t-\tau))(1-e_1x_1(t-\tau)-e_{12}x_2(t-\tau))-1],$$

$$\frac{dx_2(t)}{dt}=x_2(t)\left[-\frac{v_2p_2'}{x_2(t)p_2^0+p_2'}+\frac{d_2p_2''}{x_2(t)p_2^0-p_2''}\right]-c_{21}[x_2(t-\tau)-x_1(t-\tau)]+$$

$$+r_2[(1+x_2(t-\tau))(1-e_{21}x_1(t-\tau)-e_2x_2(t-\tau))].$$

Or

$$\frac{dx_1(t)}{dt}=x_1(t)\left[-\frac{v_1p_1'}{x_1(t)p_1^0+p_1'}+\frac{d_1p_1''}{x_1(t)p_1^0-p_1''}\right]+x_1(t-\tau)[-c_{12}+r_1(1-e_1)]+$$

$$+x_2(t-\tau)(c_{12}+r_1e_{12})+r_1[e_{12}x_2(t-\tau)-e_1x_1(t-\tau)]x_1(t-\tau),$$

$$\frac{dx_2(t)}{dt} = x_2(t)\left[-\frac{\nu_2 p_2'}{x_2(t)p_2^0 + p_2'} + \frac{d_2 p_2''}{x_2(t)p_2^0 - p_2''}\right] + x_2(t-\tau)[-c_{21} + r_2(1-e_2)] +$$
$$+ x_1(t-\tau)(c_{21} + r_2 e_{21}) + r_2[e_{21}x_1(t-\tau) - e_2(t-\tau)]x_2(t-\tau). \qquad (5)$$

We transform the resulting system to universal form. Using vector-matrix notations, we write

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} -c_{12} + r_1(1-e_1) & c_{12} + r_1 e_{12} \\ c_{21} + r_2 e_{21} & -c_{21} + r_2(1-e_2) \end{bmatrix}\begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} +$$
$$+ \frac{1}{2}\begin{bmatrix} x_1(t-\tau) & x_2(t-\tau) & 0 & 0 \\ 0 & 0 & x_1(t-\tau) & x_2(t-\tau) \end{bmatrix}\begin{bmatrix} -2e_1 r_1 & r_1 e_{12} \\ r_1 e_{12} & 0 \\ 0 & r_2 e_{21} \\ r_2 e_{21} & -2e_2 r_2 \end{bmatrix}\begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} -$$
$$-\left(\begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix}\begin{bmatrix} p_1^0 & 0 \\ 0 & p_2^0 \end{bmatrix} + \begin{bmatrix} p_1' & 0 \\ 0 & p_2' \end{bmatrix}\right)^{-1}\begin{bmatrix} \nu_1 p_1' & 0 \\ 0 & \nu_2 p_2' \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} +$$
$$+\left(\begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix}\begin{bmatrix} p_1^0 & 0 \\ 0 & p_2^0 \end{bmatrix} - \begin{bmatrix} p_1'' & 0 \\ 0 & p_2'' \end{bmatrix}\right)^{-1}\begin{bmatrix} d_1 p_1'' & 0 \\ 0 & d_2 p_2'' \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We introduce the following notation

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\ x(t-\tau) = \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix},\ X_0(t) = \begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix},$$
$$X(t-\tau) = \begin{pmatrix} x_1(t-\tau) & x_2(t-\tau) & 0 & 0 \\ 0 & 0 & x_1(t-\tau) & x_2(t-\tau) \end{pmatrix},$$
$$A = \begin{bmatrix} -c_{12} + r_1(1-e_1) & c_{12} + r_1 e_{12} \\ c_{21} + r_2 e_{21} & -c_{21} + r_2(1-e_2) \end{bmatrix},\ B = \frac{1}{2}\begin{bmatrix} -2e_1 r_1 & r_1 e_{12} \\ r_1 e_{12} & 0 \\ 0 & r_2 e_{21} \\ r_2 e_{21} & -2e_2 r_2 \end{bmatrix}, \qquad (6)$$
$$P_0 = \begin{bmatrix} p_1^0 & 0 \\ 0 & p_2^0 \end{bmatrix}, P' = \begin{bmatrix} p_1' & 0 \\ 0 & p_2' \end{bmatrix},\ P'' = \begin{bmatrix} p_1'' & 0 \\ 0 & p_2'' \end{bmatrix}, V = \begin{bmatrix} \nu_1 p_1' & 0 \\ 0 & \nu_2 p_2' \end{bmatrix},\ D = \begin{bmatrix} d_1 p_1'' & 0 \\ 0 & d_2 p_2'' \end{bmatrix}.$$

Then the system takes universal vector-matrix form

$$\frac{d}{dt}x(t) = Ax(t-\tau) + X(t-\tau)Bx(t-\tau) - [X_0(t)P_0 + P']^{-1}Vx(t) + [X_0(t)P_0 - P'']^{-1}Dx(t). \qquad (7)$$

3. INVESTIGATION OF STABILITY OF STATIONARY POSITION OF SYSTEM WITH A DELAY ON THE PLANE.

Equilibrium positions of system with delay are determined in the same way as for system without delay. They have the form $x_1(t) \equiv 0$, $x_2(t) \equiv 0$. Let us carry out the linearization of system (7) in the neighborhood of zero equilibrium position.

We rewrite system (7) in form

$$\frac{d}{dt}x(t) = Ax(t-\tau) + X(t-\tau)Bx(t-\tau) - F(x(t))x(t), \qquad (8)$$

Where

$$F(x(t)) = -[X_0(t)P_0 + P']^{-1}V + [X_0(t)P_0 - P'']^{-1}D.$$

Since the matrix function $F(x(t))$ has a form

$$F(x(t))=\begin{bmatrix} F_{11}(x_1(t),x_2(t)) & F_{12}(x_1(t),x_2(t)) \\ F_{21}(x_1(t),x_2(t)) & F_{22}(x_1(t),x_2(t)) \end{bmatrix},$$

then the system (8), linearized at zero point, will be present in the form

$$\frac{d}{dt}x(t)=\tilde{A}x(t)+Ax(t-\tau),\ \tilde{A}=-\left.\frac{D(F(x(t))x(t))}{D(x(t))}\right|_{x(t)=0}.$$

We compute the Jacobian

$$\frac{D(F(x)x)}{D(x)}=\begin{bmatrix} x_1\frac{\partial}{\partial x_1}F_{11}+F_{11}+x_2\frac{\partial}{\partial x_1}F_{12} & x_1\frac{\partial}{\partial x_2}F_{11}+F_{12}+x_2\frac{\partial}{\partial x_2}F_{12} \\ x_1\frac{\partial}{\partial x_1}F_{21}+F_{21}+x_2\frac{\partial}{\partial x_1}F_{22} & x_1\frac{\partial}{\partial x_2}F_{21}+F_{22}+x_2\frac{\partial}{\partial x_2}F_{22} \end{bmatrix}=$$

$$\begin{bmatrix} x_1\frac{\partial}{\partial x_1}F_{11}+x_2\frac{\partial}{\partial x_1}F_{12} & x_1\frac{\partial}{\partial x_2}F_{11}+x_2\frac{\partial}{\partial x_2}F_{12} \\ x_1\frac{\partial}{\partial x_1}F_{21}+x_2\frac{\partial}{\partial x_1}F_{22} & x_1\frac{\partial}{\partial x_2}F_{21}+x_2\frac{\partial}{\partial x_2}F_{22} \end{bmatrix}+\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

And at the point $O(0,0)$ we obtain

$$\left.\frac{D(F(x(t)))}{D(x(t))}\right|_{x_1(t)=0,x_2(t)=0}=\begin{bmatrix} F_{11}(0,0) & F_{12}(0,0) \\ F_{21}(0,0) & F_{22}(0,0) \end{bmatrix}=F(0).$$

Thus the matrix $\tilde{A}$ has a form

$$\tilde{A}=-(P')^{-1}V-(P'')^{-1}D=-\begin{bmatrix} p_1' & 0 \\ 0 & p_2' \end{bmatrix}^{-1}\begin{bmatrix} v_1p_1' & 0 \\ 0 & v_2p_2' \end{bmatrix}-\begin{bmatrix} p_1'' & 0 \\ 0 & p_2'' \end{bmatrix}^{-1}\begin{bmatrix} d_1p_1'' & 0 \\ 0 & d_2p_2'' \end{bmatrix}=$$

$$=\begin{bmatrix} -v_1-d_1 & 0 \\ 0 & -v_2-d_2 \end{bmatrix}.$$

We denote

$$p_{11}=-v_1-d_1,\ q_{11}=-c_{12}+r_1(1-e_1),\ p_{12}=c_{12}+r_1e_{12},\ p_{21}=c_{21}+r_2e_{21},$$
$$p_{22}=-v_2-d_2,\ q_{22}=-c_{21}+r_2(1-e_2).$$

Characteristic equation of linearized system has the form

$$\det\left(\tilde{A}+e^{-\lambda\tau}A-\lambda E\right)=\det\begin{bmatrix} p_{11}+e^{-\lambda\tau}q_{11}-\lambda & p_{12} \\ p_{21} & p_{22}+e^{-\lambda\tau}q_{22}-\lambda \end{bmatrix}=$$

$$=\lambda^2-\left[(p_{11}+p_{22})+e^{-\lambda\tau}(q_{11}+q_{22})\right]\lambda+\left[p_{11}p_{22}+(p_{11}q_{22}+p_{22}q_{11})e^{-\lambda\tau}+q_{11}q_{22}e^{-2\lambda\tau}\right]=0.$$

Necessary condition for stability of system with delay is asymptotic stability of system without delay. By putting the value of time-delay $\tau=0$, we obtain that characteristic equation of system without delay on the plane has the form

$$\lambda^2-\left[(p_{11}+p_{22})+(q_{11}+q_{22})\right]\lambda+\left[p_{11}p_{22}+(p_{11}q_{22}+p_{22}q_{11})+q_{11}q_{22}\right]=0.$$

Moreover, necessary and sufficient condition for stability on the plane is the positivity of the coefficients of the characteristic equation, i.e.

$$p_{11}+p_{22}+q_{11}+q_{22}<0,\ p_{11}p_{22}+p_{11}q_{22}+p_{22}q_{11}+q_{11}q_{22}>0.$$

Example. It should be noted that the initial position for systems with delay is determined not by the initial point, but by the initial function. Systems with delay are systems of differential equations in an infinite-dimensional banach space. Consider the following example.

$$\dot{x}_1(t) = (-\frac{30}{13x_1+2} - \frac{6}{1.5-13x_1} + 2.25)x_1(t)$$
$$-0.75x_2(t-\tau) + 0.5x_1^2(t-\tau) + 0.75x_1(t-\tau)x_2(t-\tau),$$
$$\dot{x}_2(t) = 2.1x_1(t) - (\frac{25}{7.5x_2+2.5} + \frac{6}{2-7.5x_2} + 0.9)x_2(t-\tau)$$
$$+0.3x_2^2(t-\tau) + 0.6x_1(t-\tau)x_2(t-\tau).$$

The corresponding matrices have the following form

$$A = \begin{bmatrix} 2.25 & -0.75 \\ 2.1 & -0.9 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0.375 \\ 0.375 & 0 \\ 0 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}, \; P_0 = \begin{bmatrix} 13 & 0 \\ 0 & 7.5 \end{bmatrix}, \; P' = \begin{bmatrix} 2 & 0 \\ 0 & 2.5 \end{bmatrix}, \; P'' = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix},$$
$$V = \begin{bmatrix} 30 & 0 \\ 0 & 25 \end{bmatrix}, \; D = -\begin{bmatrix} 8 & 0 \\ 0 & 6 \end{bmatrix}.$$

The system of perturbation equations has the form

$$\frac{dx_1(t)}{dt} = x_1(t)\left[-\frac{\nu_1 p_1'}{x_1(t)p_1^0 + p_1'} + \frac{d_1 p_1''}{x_1(t)p_1^0 - p_1''}\right] + x_1(t-\tau)\left[-c_{12} + r_1(1-e_{11})\right] +$$
$$+ x_2(t-\tau)(c_{12} + r_1 e_{12}) + r_1\left(e_{12}x_2(t-\tau) - e_{12}x_1(t-\tau)\right)x_1(t-\tau),$$

$$\frac{dx_2(t)}{dt} = x_2(t)\left[-\frac{\nu_2 p_2'}{x_2(t)p_2^0 + p_2'} + \frac{d_2 p_2''}{x_2(t)p_2^0 - p_2''}\right] + x_2(t-\tau)\left[-c_{21} + r_2(1-e_2)\right] +$$
$$+ x_1(t-\tau)(c_{21} + r_2 e_{21}) + r_2\left(e_{21}x_1(t-\tau) - e_2 x_2(t-\tau)\right)x_2(t-\tau).$$

System, linearized in the neighborhood of zero point, has the form:

$$\begin{cases} \dot{x}_1(t) = -11x_1(t) + 2.25x_1(t-\tau) - 0.75x_2(t-\tau) \\ \dot{x}_2(t) = -7x_2(t) + 2.1x_1(t-\tau) - 0.9x_2(t-\tau) \end{cases}.$$

Its characteristic equation has the form

$$\lambda^2 + \lambda(18.5 - 0.15e^{-\lambda\tau}) - 5.85e^{-\lambda\tau} - 2.025e^{-2\lambda\tau} + 78.575 = 0.$$

In the absence of delay, characteristic equation will be

$$\lambda^2 + 18.35\lambda + 70.805 = 0.$$

It has roots with negative real part. Hence, there exists $\tau_0 > 0$, that when $\tau < \tau_0$ system of equations with delay will also have an asymptotically stable equilibrium position.

CONCLUSION

We always have competition on market, and sellers of similar goods or suppliers of such services always depend on each other. For this reason, the system of dynamics of free competition market was investigated. The goal was to determine when this system has a stable equilibrium position. In case of markets, equilibrium means a situation when sellers and buyers are collectively satisfied with current combination of prices and sales or purchases, and thus have no incentive to change existing situation. If, for some reason, equilibrium price has not been established, then forces are emerging in market aimed at establishing such price. The ideal economic equilibrium implies the conditions of perfect competition and absence of external effects.

Nevertheless, in a real economy such conditions are not observed: there is no perfect market, there are side effects of entrepreneurial activity, cyclical and structural fluctuations, unemployment, inflation. All of them deduce economy from equilibrium state. Moreover, in economy term "delay" is very popular, that is why we investigate the system of delay difference equations. In such way, economic system ca be brought into a state of equilibrium that will correspond to market realities.

The real equilibrium is an equilibrium that is established in economy in conditions of imperfect competition in the presence of external and internal factors of influence on the market. Each of the market participants can influence the trade with a certain coefficient. In the course of the study of the obtained system, it was established that for the equilibrium of the system, the total influence of the seller and the buyer should be more than the influence of the state. This model can be used in real business to minimize risks, investigate market and increase profits.

References

[1] Kalitin B. *First order mathematical models of competitive market*. – Minsk, BSU, 2012. – 131 c. (in Russian)

[2] Hale J. *Theory of functional differential equations*. - Springer, New York, 1977.

[3] Elsgol'ts L.E., Norkin S.B. *Introduction to the theory of differential equations with deviating argument*.– Academic Press, New York, 1973.

[4] Khusainov D., Diblík J., Růžičková M. *Linear dynamical systems with aftereffect. Representation of solutions, sustainability, management, stabilization.* - Kyiv, Taras Shevchenko National University of Kyiv: GP Information Analytical Agency, 2015. – 252 c. (in Russian)

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E-LEARNING AND MASTERING OF ELECTROCARDIOGRAPHY AND BLOOD-PRESSURE MEASUREMENT

Martin Kopecek, Petr Voda, David Kordek, Pravoslav Stransky
Department of Medical Biophysics, Medical Faculty in Hradec Kralove, Charles University
Simkova 870, 500 03 Hradec Kralove, Czech Republic
kopecema@lfhk.cuni.cz, vodap@lfhk.cuni.cz, kordekd@lfhk.cuni.cz, str@lfhk.cuni.cz,

Abstract: *Electrocardiography and blood-pressure measurement is one of the most important medical examination methods used in clinics. An innovative laboratory was created in order to provide students with the practical Electrocardiography (ECG) and the blood pressure (BP) experience using the latest e-learning multiple-step learning concepts. This educational course focuses on students of General medicine and Dentistry and expanding their knowledge in fields ECG, BP. The course content is presented in three different levels of details and students can choose from the highest to the lowest level of difficulty according to their needs. Based on students' feedback, multiple-step-learning, e-learning course ECG, BP for the study year 2016/2017 was modified and enhanced with new step-by-step lab instructions. The impact of enhancement of the course ECG, BP to the final student results has been analysed.*

Keywords: biophysics, electrocardiography, blood pressure measurement, education, E-learning, analysis

INTRODUCTION

Students, during their studies at the Faculty of Medicine, obtain, through the education specialization of General Medicine and Dentistry, basic knowledge of physical and biophysical principles of physiological processes in human body. Biophysics leads students in their first year of studies to logical reasoning in finding solutions to the tasks built on basis of physics. This subject is not one of the most popular and, to students, it is therefore necessary to give a hand. Complex processes should be explained to students by attractive and creative forms of education. Possibilities of physics in medicine are far-reaching and students should learn the correct orientation in the maze of physical concepts, methods, equipment and processes. The latest techniques and the most modern devices are necessary to include into specific courses for the preparation of future physicians. The preparation should directly correspond with possibilities of their future workplace, and current trends in modern medicine.

The cornerstone for attractiveness in field of physics is the use of modern information and communication technologies and practical demonstrations that have specific practical outcomes. Department of Medical Biophysics has been involved in research and development of e-learning systems. These were found more effective in education of medical systems [1], [2] and - especially for laboratory experiments [3]. For example, in laboratory practicum in areas such as nitinol stents [4], materials for dentistry [5], [6] and other several topics, e-learning courses were created to deepen knowledge and combine an interesting education environment with practical examples. Also, knowledge of basic branches of the statistics according to Bezrouk [7] and design of the experiments [8, 9] is an important part of the practical training of undergraduate students for their future research.

This paper describes the outcomes from an innovated interactive e-learning course ECG, BP in specialized laboratory using multiple-step learning (MSL) [10] concept of education. Mandatory practicum is based on basic knowledge about human circulatory system where students are virtual patient and physician in one person and they compare and see the task from both sides. The laboratory is designed to show how to measure ECG and blood pressure in the real environment and tries to increase the popularity [11] of biophysics as such. Attractiveness of real solutions can be increased by special software means, like in [12] and this can be also modified for the measurement ECG, BP. Real data to post-processing are used. Real output from ECG test is also printed sample with some basic curves.
The goal of the research was to answer hypothesis and assumption that proved to be crucial when testing success of the test results from part of the subject Medical Biophysics. We expected change of quality of the e-learning step-by-step lab instructions and data-sheets, the results were statistically significantly different. As a side effect, we also attempted to monitor the time if there is no positive effect and shortening the laboratory task.
There are many scientific articles, where techniques, methods and use of ECG test and BP measurement are described (a simple search for topic "CG, BP" in the Web of Science database produces 914 results in the period 2000-2018). This work is new and it is unique in accessing these methods, tests for educational purposes for students of medicine in ECG, BP measurement.

1 MATERIALS AND METHODS

1.1 ECG in Modern Conception of Learning and Mastering

As it was written in the previous text, the course in the laboratory is mandatory and it is the main part at the biophysical practical laboratories. A group for ECG, BP course usually contents 3-5 students.
The structure of the course can be split into several parts, all in the e-learning form (Fig.1): Theoretical preparation as *Introduction* (study materials – basic theory and presentation are prepared in the system MSL Moodle); *Manuals and data-sheets* (how to be prepared on the lab and basic knowledge about devices); *Basic questions* and *Sample test* (own verification of knowledge from theory); *External links*; *Online chat* (individual questions and answers); *Instruction* for all tasks (ECG, blood pressure measurement); *Micro-test* (written before the start of the lab); *Protocol ECG and BP* (working with the data, control mechanism); *Questionnaire* (students feedback).

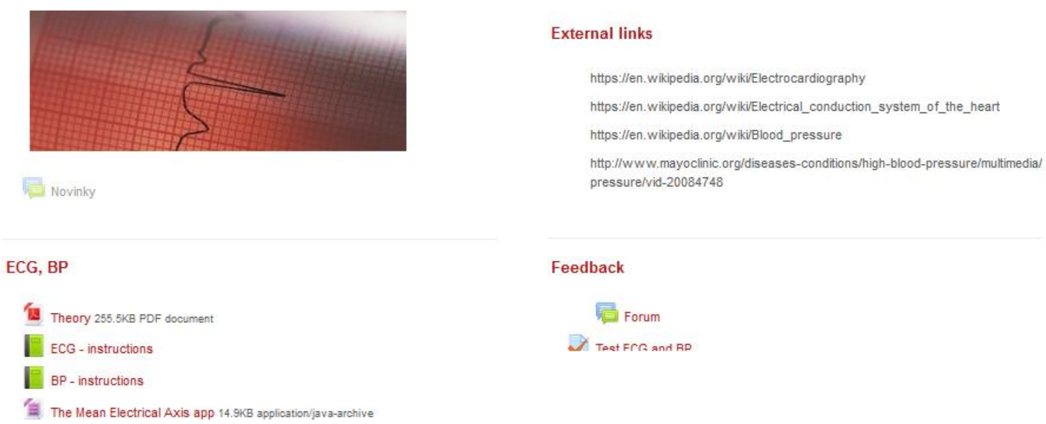
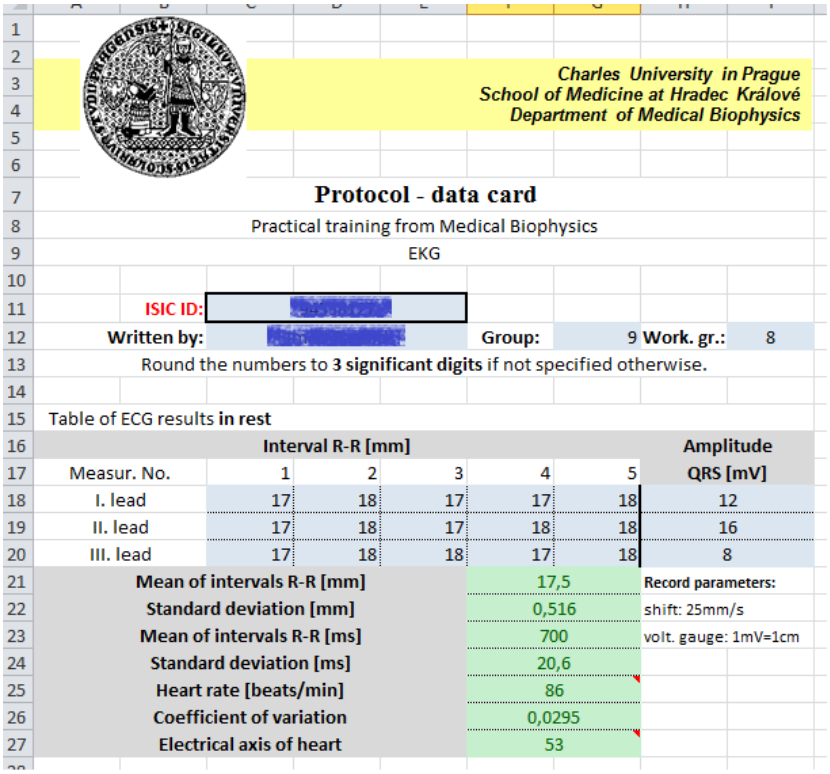


Fig. 1. Categorization of the topics in the MSL system Moodle
Source: own

1.2 E–protocols

For a fair environment, MS excel protocol sheet with an auto-evaluation of measured and calculated results was prepared. Students must use the basic physical and mathematical equations which correspond to the individually obtained data to receive immediate positive or eventually negative feedback. Each student receives individual results of their own work and can be confronted in the group with or without the teacher's assistance. Protocol consists of predefined free frames where data have to be written in three significant digits forms. An automatic MS Excel macro system must be sometimes upgraded for better accuracy by Excel's VBA environment. Tolerance of the result is ±1%. Positive answer colors the cell by green, negative answer by red as it is shown in Fig. 2. Gray cells are only informative and blocked for students (not allowed for any text). For a middle calculation, they can use other MS Excel lists which are free to use.


Charles University in Prague
School of Medicine at Hradec Králové
Department of Medical Biophysics

Protocol - data card
Practical training from Medical Biophysics
EKG

ISIC ID: [redacted]
Written by: [redacted] Group: 9 Work. gr.: 8
Round the numbers to 3 significant digits if not specified otherwise.

Table of ECG results in rest

	Interval R-R [mm]					Amplitude QRS [mV]
Measur. No.	1	2	3	4	5	
I. lead	17	18	17	17	18	12
II. lead	17	18	17	18	18	16
III. lead	17	18	18	17	18	8

Mean of intervals R-R [mm]	17,5	Record parameters:
Standard deviation [mm]	0,516	shift: 25mm/s
Mean of intervals R-R [ms]	700	volt. gauge: 1mV=1cm
Standard deviation [ms]	20,6	
Heart rate [beats/min]	86	
Coefficient of variation	0,0295	
Electrical axis of heart	53	

Fig. 2. Interactive protocol of ECG (at example all data in columns are highlighted green so this part of the protocol is correct, and it is possible to "Send" the protocol into the database)
Source: own

When all required cells are green, then the protocol can be sent by the button to the system database. The protocol in MS Excel is connected with real online ambulant and database system PC Doctor. If all credentials – usually ISIC ID, student's name, group number and results fit the limits, student can check the success points in PC Doctor SW in his own patient/physician card (each student has an individual account as a patient and a physician as well).

The biggest benefit of this "live" and interactive protocol evaluation is the speed of control mechanism of correct answers. Calculations can be, in case of need, solved with the helping hand of teacher and the solution can be found together with significant educational impact.

1.3 ECG, BP – Innovation of Measurement Instructions

New step-by-step manuals, data-sheets were prepared for the study year 2016/2017. The instructions are **more detailed and illustrated** by specific pictures step-by-step according to the measurement procedure. The lab data-sheet is online in e-learning form as well as directly on paper form on the table during the measurement. The specific devices and their functionality, SW and program control are explained more into the deep but still the need to think and seek solutions of the task is maintained. The most important parts are highlighted as it is shown in Fig. 3.

Noninvasive blood pressure measurement

Used devices and tools: analogue tonometer with mercury manometer, digital tonometer Omron, digital tonometer Arm-in

Workflow:

1. Measurements at rest
   - Measurement results can be written on paper or directly into your record in the PC doctor.
     - **I.** Measure the blood pressure using the **classic analogue mercury tonometer** (measured on the schoolmate) according to the procedure described in the theory.

       - Repeat the measurement **3x at rest**.
     - **II.** Measure the blood pressure using the **Omron M4 (oscillometric method) tonometer** (measured on the schoolmate) according to the procedure described in the theory.

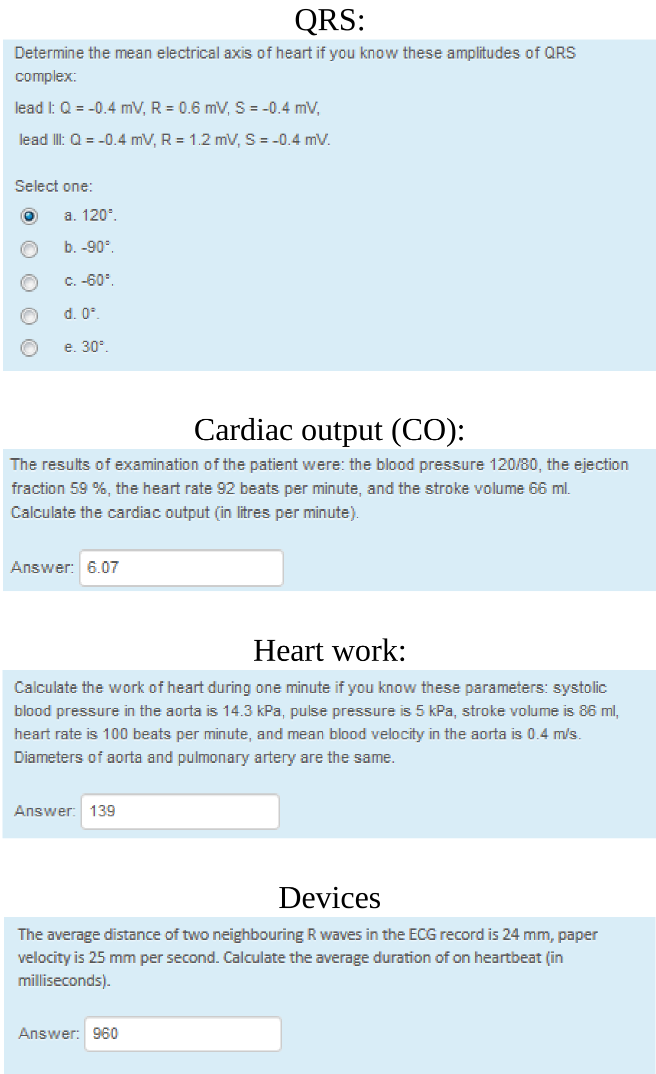
       - Repeat the measurement **3x at rest**.

Fig. 3. Step by step instruction in manual for BP
Source: own

1.4 Statistical Evaluation

The result from the students' final tests for our research were compared, processed, and statistically analyzed using MS Excel 2007 (Microsoft Corp, Redmond WA, USA) and NCSS 2007 (NCSS LLC, Kaysville, UT, USA. www.ncss.com). **The chi-squared test** was used and the value of **P<0.05** was considered as statistically significant.

The impact of change step-by-step instruction of data-sheets was evaluated using the **cumulative levels of success**, related to the topic ECG, BP, in final exam tests 2016/17 **vs** 2015/16 (before modifications). Significant increase of the cumulative level of success in the final exam test 2016/2017 vs 2015/2016 was considered as positive education impact. These two study years were compared. In these tests, the topic ECG, BP was represented by 4 types of questions: QRS complexes - ECG and bipolar leads description; Cardiac Output – volumes, heart rate; Heart work - power and work of the heart; Devices - blood pressure, calculations. The level of success in each test was based on the ratio of number of correct answers vs the total number of answers. Below, for better understanding some examples are shown from the final exam test at Fig. 4.



QRS:

Determine the mean electrical axis of heart if you know these amplitudes of QRS complex:

lead I: Q = -0.4 mV, R = 0.6 mV, S = -0.4 mV,

lead III: Q = -0.4 mV, R = 1.2 mV, S = -0.4 mV.

Select one:

- a. 120°.
- b. -90°.
- c. -60°.
- d. 0°.
- e. 30°.

Cardiac output (CO):

The results of examination of the patient were: the blood pressure 120/80, the ejection fraction 59 %, the heart rate 92 beats per minute, and the stroke volume 66 ml. Calculate the cardiac output (in litres per minute).

Answer: 6.07

Heart work:

Calculate the work of heart during one minute if you know these parameters: systolic blood pressure in the aorta is 14.3 kPa, pulse pressure is 5 kPa, stroke volume is 86 ml, heart rate is 100 beats per minute, and mean blood velocity in the aorta is 0.4 m/s. Diameters of aorta and pulmonary artery are the same.

Answer: 139

Devices

The average distance of two neighbouring R waves in the ECG record is 24 mm, paper velocity is 25 mm per second. Calculate the average duration of on heartbeat (in milliseconds).

Answer: 960

Fig. 4. Example of the formulation of the questions at final test from 4 selected topics
Source: own

2 RESULTS

The greatest benefit brought by the innovative e-learning course ECG, BP was evidently the time saving. This measuring and testing laboratory ECG, BP is relatively difficult and time consuming – approximately three teaching hours in total. The new course structure reduced the time by almost 30 minutes. Students appreciated this very much. The rest of the time is usually used for a preparation of protocols.

The individual level of success (Fig. 5) for specific types of questions as **QRS, Heart work** and **Devices** show **insignificant but distinct positive change** in the final test 2016/17 when compared with the final test 2015/16. Figure 5 shows that there is also negative but insignificant change in the level of success in question **CO.**

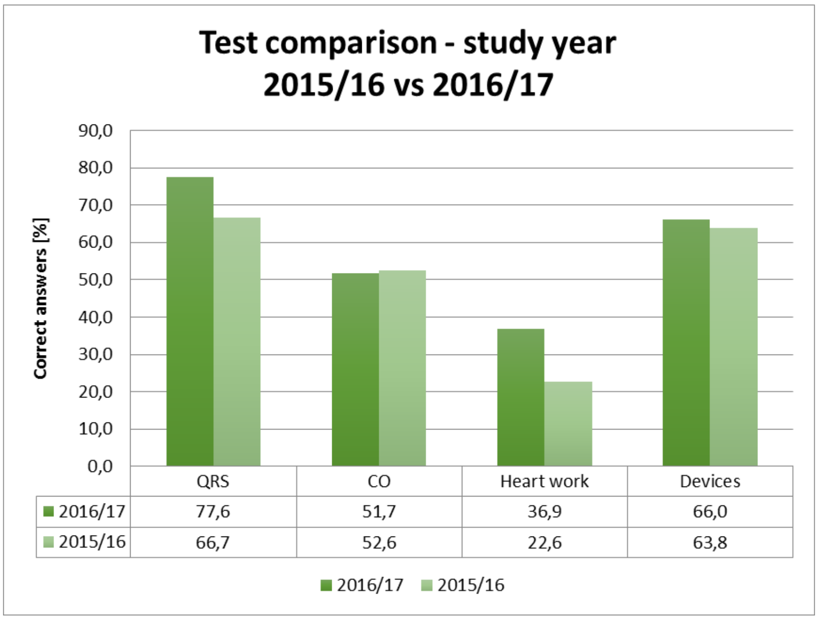


Fig. 5. Comparison of individual levels of success for specific final test and type of a question
Source: own

Totally, the **cumulative level** of success of the test 2016/17 is **better** (higher), although particularly insignificant, when compared with the test 2015/16 as can be seen in Table 1.

School Year		**Number of answers**
2015/16	correct	160
	total	282
	ratio [%]	56,7
2016/17	correct	187
	total	307
	ratio [%]	60,9
p - value		**0,03156**

Table 1. Final exam test cumulative levels of success - comparison of 2015/16 vs 2016/17, related to topic ECG, BP, represented by the ratio [%] of the number of correct answers vs the total number of answers.
Source: own

3 CONCLUSION

The innovative ECG, BP course combines e-learning and distance method of theoretical preparation with practical use of specific devices. The course modification from the school year 2016/2017 shows better results (success in the final test) than in the previous version (till 2015/2016). Also, time of preparation of the measurement and its own implementation has been significantly shortened. This is considered to be one of the major benefits of adjusting the course. As a result, the teacher has a larger time-frame to explain more complex concepts and also students have more time to memorize basic procedures. The e-learning course teaches students to use their theoretical knowledge to prepare and build real solutions that can be determined and evaluated. Teamwork and discussion of final results simulate the environment for their future careers. Students significantly improve their knowledge of biophysics through an entertaining and modern form of e-learning MSL teaching.

Our innovative solution of ECG preparation, BP laboratory uses general knowledge of other work based at the MSL concept and extends it into specific practical use. This work uniquely and originally intervenes in the practical learning of ECG, BP. The positive impact of the innovated laboratory is evident and we will certainly work on further improvements to both sites of MSL concept and the manuals for laboratory tasks. Research in the field of the correct interpretation of backgrounds and manuals for students has proved to be crucial in this work and has substantial practical impact on teaching. Further studies including larger samples from multiple medical schools and in objective assessment of skills performance might facilitate an accurate evaluation impact of course innovations on the quality of teaching and also define opportunities for improvement.

References

[1] Hanus, J., Nosek, T., Zahora, J., Bezrouk, A., Masin, V. On-line Integration Of Computer Controlled Diagnostic Devices And Medical Information Systems In Undergraduate Medical Physics Education For Physicians. *Physica Medica-European Journal of Medical Physics*, vol. 29, No. 1, 2013, p. 83–90. ISSN 1120-1797.

[2] Hanus, J., Zahora, J., Masin, V., Bezrouk, A. On-Line Incorporation of Study and Medical Information System in Undergraduate Medical Education. In: *6th International Conference of Education, Research and Innovation (iceri 2013)*, Seville, Valenica: Iated-Int Assoc Technology Education & Development. 2013, p. 1500–1507. [Cit. 2018-04-15]. ISBN 978-84-616-3847-5.

[3] Zahora, J., Hanus, J., Jezbera, D., Bezrouk, A. Remotely Controlled Laboratory and Virtual Experiments in Teaching Medical Biophysics. In: *6th International Conference of Education, Research and Innovation (iceri 2013)*, Seville, Valenica: Iated-Int Assoc Technology Education & Development. 2013, p. 900–906. [Cit. 2018-04-15]. ISBN 978-84-616-3847-5.

[4] Zahora, J., Bezrouk, A., Hanus, J., Models of stents - Comparison And Applications, *Physiological Research*, vol. 56, 2007, p. 115–121. ISSN 0862-8408.

[5] Bezrouk, A., Balsky, L., Smutny, M., Selke Krulochova, I., Zahora, J., Hanus, J., Meling, T.R. Thermomechanical Properties Of Nickel-Titanium Closed-Coil Springs And Their Implications For Clinical Practice. *American Journal of Orthodontics and Dentofacial Orthopedics*, vol. 146, No. 3, 2014, p. 319–327. ISSN 0889-5406.

[6] Bezrouk, A., Balsky, L., Selke Krulichova, I., Smutny, M., Zahora, J., Selke, G.W., Meling, T.R. Nickel-titanium Closed-coil Springs: Evaluation Of The Clinical Plateau. *Revista de Chimie (Bucharest)*, vol. 68, No. 5, 2017, p. 1137-1142. ISSN 0034-7752.

[7] Bezrouk, A., Fiala, Z., Kotingova, L., Selke-Krulichova, I., Kopecna, M., Vavrova, K. SAMPA: A Free Software Tool For Skin And Membrane Permeation Data Analysis. *Toxicology In Vitro*, vol. 44, 2017, p. 361-371. ISSN 0887-2333.

[8] Vagaska, A., Gombar, M. The Application of Design of Experiments to Analyze Operating Conditions of Technological Process. In: *Mathematics, Information Technologies and Applied Sciences 2016, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2016, p. 105-111. [Online]. [Cit. 2018-04-15]. Available at: <http://mitav.unob.cz/data/MITAV 2016 Proceedings.pdf>. ISBN 978-80-7231-400-3.

[9] Fabo, P., Pavlikova, S. Interactive School Experiments In The PSE Graphical Environments. In: *Mathematics, Information Technologies and Applied Sciences 2015, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2015, p. 67-77. [Online]. [Cit. 2018-04-15]. Available at: <http://mitav.unob.cz/data/MITAV 2015 Proceedings.pdf>. ISBN 978-80-7231-436-2.
[10] Shengye Y., Shiguang S., Xilin Ch., Wen G., Jie Ch. Matrix-Structural Learning (MSL) of Cascaded Classifier from Enormous Training Set. In: *Computer Vision and Pattern Recognition CVPR '07, 2007 IEEE Conference on Computer Vision and Pattern Recognition*, Minneapolis, USA, 2007, p. 1-8. [Online]. [Cit. 2018-04-16]. Available at:<https://www.researchgate.net/publication/221362865_Matrix-Structural_Learning_MSL_of_Cascaded_Classifier_from_Enormous_Training_Set>. ISBN 1-4244-1179-3.
[11] Stuchlikova, L., Benko, P., Janicek, F., Pohorelec, O., Hrbacek, J. Engineering Education And Science & Technology Popularization Among Youngsters Supported By IT. In: *Mathematics, Information Technologies and Applied Sciences 2017, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2017, p. 226-234. [Online]. [Cit. 2018-04-15]. Available at: <http://mitav.unob.cz/data/MITAV 2017 Proceedings.pdf>. ISBN 978-80-7582-026-6.
[12] Fechova, E. Raising Attractiveness In Teaching Technical Subjects By Using Software Means. In: *Mathematics, Information Technologies and Applied Sciences 2015, post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2015, p. 78-87. [Online]. [Cit. 2018-04-15]. Available at: <http://mitav.unob.cz/data/MITAV 2015 Proceedings.pdf>. ISBN 978-80-7231-436-2.

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ON SEQUENCES PRESERVING THE CONVERGENCE OF INFINITE NUMERIC SERIES

Renata Masarova
Faculty of Materials Science and Technology in Trnava, Slovak University of Technology in Bratislava
Ulica Jána Bottu č. 2781/25, 917 24 Trnava, Slovak Republic, renata.masarova@stuba.sk

Abstract: *This paper is motivated by the results of rearrangements of conditionally convergent numeric series. From them we get to an inherent question: What are the properties of the set of all sequences* (ε_n) *where* $\varepsilon_n = \pm 1$*, that preserve the convergence of the series?*
Let $(a) = \sum a_n$ *be a conditionally convergent numeric series and X the set of all sequences* (ε_n)*, where* $\varepsilon_n = 1$ *or* $\varepsilon_n = -1$*. By* $CPS(a)$ *we denote the set of all sequences from space X, for which the series* $\sum \varepsilon_n a_n$ *is once again convergent and CPS we denote a set of all sequences from X, that preserve the convergence of all conditionally convergent numeric series. In this paper we study the porosity of* $CPS(a)$ *and CPS in X.*

Keywords: the numeric series, the sequences, convergence preserving sequences, porous set.

INTRODUCTION

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. No possible rearrangement will change its convergence or sum. They can be changed only by changing its members, as is shown in paper [5].
The situation is different if the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Using a rearrangement of its members we can either get a convergent series with a predetermined sum or a divergent series. Examples of different sums with changes of the harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ can be found in paper[4].
If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then every sequences of numbers ± 1 preserves its convergence. Then we can concern ourselves with the set of all sums of the series $\sum_{n=1}^{\infty} \varepsilon_n a_n$. This problem is the focus of papers [6], [7], [8].
Let $(a) = \sum a_n$ be a convergent numeric series and $x = (\varepsilon_n)$ a sequence of numbers $\varepsilon_n = \pm 1$. We say that the sequence $x = (\varepsilon_n)$ preserves the converge of series $(a) = \sum a_n$, if the series $\sum \varepsilon_n a_n$ is once again convergent.
This work follows up on the author's contribution to the MITAV 2017 conference. It was focused on the following problem:
By CPS we denote the set of all sequences that preserve the convergence of every conditionally convergent series. What is the porosity of set CPS in set X?
It is evident that there are multiple sequences that preserve the convergence of a single conditionally convergent series. Is this set porous in set X as well?

1 BASIC TERMS AND DEFINITIONS

1.1 Set porosity

The term porous set can be found in papers [9] and [10].
Let (X, ρ) be a metric space, $M \subseteq X$.
Let $x \in X, r \in R^{+}$. By $K(x,r)$ we denote a sphere with the center x and radius r, that is

$$K(x,r) = \{y \in X: \rho(x,y) < r\}.$$

Let us set

$$\gamma(M,x,r) = \sup\{t > 0: \exists z \in X, [K(z,t) \subseteq K(x,r)] \wedge [K(z,t) \cap M] = \emptyset\}.$$

Then

$$\Gamma(M,x) = \limsup_{r \to 0^{+}} \frac{\gamma(M,x,r)}{r}.$$

From the definition we can see that $\Gamma(M,x) \in \langle 0,1 \rangle$.
The set M is porous in point x, if $\Gamma(M,x) > 0$.
If there exists a number $c \in \langle 0,1 \rangle$, such that $\Gamma(M,x) \geq c$, the set M is referred to as c-porous in point x.
We refer to set M as σ-porous ($c - \sigma$-porous) in point x, if $M = \bigcup_{n=1}^{\infty} M_n$, where M_n is porous (c-porous) in point x for every $n = 1, 2, \ldots$.
From these definitions it is evident that:
If set $M \subset X$ is porous in every point of space X, then M is nowhere dense in X.
If the set $M \subset X$ is σ-porous in every point of space X, then M is in the first category in X.

1.2 Space X

Let X be the set of all sequences $x = (\varepsilon_n)$, where $\varepsilon_n = 1$ or $\varepsilon_n = -1$. On this set we can establish the Baire metric $\rho(x,y)$ as follows:
Let $x = (\varepsilon_n), y = (\theta_n)$ be two sequences from set X. Then
i) if $x = y \Rightarrow \rho(x,y) = 0$
ii) if $x \neq y \Rightarrow \rho(x,y) = \frac{1}{m,}, m = min\{n: \varepsilon_n \neq \theta_n\}$.
It is proven (e.g. [8]) that space (X, ρ) is a complete metric space.

1.3 Space CPS

Let $\sum_{n=1}^{\infty} a_n$ be any continually convergent series.
By CPS we denote the set of all sequences of space X that preserve the convergence of every continually convergent series.

The following theorems have been proven for this set:

Theorem 1.1. Sequence $x = (\varepsilon_n)_{n=1}^{\infty}$ is in set CPS if and only if there exist a natural number m, such that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ is composed of at most m constant blocks. [1]

Note: A finite (respectively infinite) sequence $(\varepsilon_i)_{i=1}^{s}$ such that $\varepsilon_i = \varepsilon_j$ for every $i, j = 1,2, \ldots, s$ where $s \in R \cup \{\infty\}$ is called a constant block.

Let us denote $H_m = \{x \in X: \varepsilon_j = \varepsilon_{m+1}\ pre\ j \geq m+1\}$.
Then $CPS = \bigcup_{m=1}^{\infty} H_m$.

For sets defined as such it is proven in paper [3] that:
Theorem 1.2. For every $m \in N$ the set H_m is nowhere dense in every point of space X.
Theorem 1.3. The set CPS is dense and of the first Baire category in X.

1.4 Space $CPS(a)$

Let $(a) = \sum a_n$ be a conditionally convergent series. By $CPS(a)$ we denote the set of those sequences $x = (\varepsilon_n)$ from space X that preserve the convergence of these series, i.e. those for which the series $\sum \varepsilon_n a_n$ is once again convergent.
It is evident that $CPS \subseteq CPS(a)$.

2 MAIN RESULTS

The set H_m is nowhere dense, from which arises the question of its porosity. In work [3] we can find the following theorem:
Theorem 2.1.. For every $m \in N$ the set H_m is $1-$porous in every point of space X.
Proof: Let $x = (\varepsilon_n)_{n=1}^{\infty}$ be any point of space X, $K(x,\delta)$ is the vicinity of this point, let

$$H_m = \{x \in X : \varepsilon_j = \varepsilon_{m+1}\ pre\ j \geq m+1\}.$$

Let us choose $s \in N$ such that $\frac{1}{s} \leq \delta < \frac{1}{s-1}$ and $s \geq m+1$.

Let us construct point $y = (\xi_n)_{n=1}^{\infty}$ from X as follows:

$$\xi_n = \begin{cases} \varepsilon_n, & n = 1, 2, \dots, s \\ (-1)^j \varepsilon_s, & n = s+j, j = 1, 2, \dots \end{cases}$$

A sequence defined as such $y = (\xi_n)_{n=1}^{\infty} \in K(x,\delta)$ and $y \notin H_m$.
Let us create a sphere $K\left(y, \frac{1}{s+1}\right)$.
If any sequence $z = (\eta_n)_{n=1}^{\infty} \in K\left(y, \frac{1}{s+1}\right)$, then $\eta_n = \xi_n$ pre $n = 1,2,\dots,s+1$ and $z \in K(x,\delta)$ (from the definition of s).

$$K\left(y, \frac{1}{s+1}\right) \subseteq K(x,\delta) \text{ and } K\left(y, \frac{1}{s+1}\right) \cap H_m = \emptyset$$

$$\gamma(H_m, x, \delta) \geq \frac{1}{s+1}$$

From the definition of s we get

$$\frac{\gamma(H_m, x, \delta)}{\delta} \geq \frac{\frac{1}{s+1}}{\frac{1}{s-1}} = \frac{s+1}{s-1}$$

If $\delta \to 0^+$ then $s \to \infty$ a

$$\Gamma(H_m, x) = \limsup_{\delta \to 0^+} \frac{\gamma(H_m,x,\delta)}{\delta} = 1.$$

Point x is any point of space X, i.e. the set H_m is 1-porous in every point of space X.

From this theorem we immediately get (in paper[3]):
Theorem 2.2. The set CPS is $\sigma - 1$-porous in every point of space X.
Proof: It is sufficient to remember that $CPS = \bigcup_{m=1}^{\infty} H_m$.

Theorem 2.3. Let $(a) = \sum a_n$ be a conditionally convergent series. Then the set $CPS(a)$ is a set of type $F_{\sigma\delta}$ (i.e. $CPS(a) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} F_{nm}$, where F_{nm} is a closed set).
Proof: Let $(a) = \sum a_n$ be a conditionally convergent series.

The sequence $x = (\varepsilon_n)_{n=1}^{\infty}$ is from the set $CPS(a)$ (i.e. the series $\sum \varepsilon_n a_n$ converges) if and only if

$$\forall k \geq 1\, \exists m\, \forall n \geq m\, \forall p \geq 1\colon \left|\varepsilon_{n+1}\, a_{n+1} + \cdots + \varepsilon_{n+p}\, a_{n+p}\right| \leq \frac{1}{k}.$$

Let us denote

$$F(k,n,p) = \left\{x = (\varepsilon_n) \in X\colon \left|\varepsilon_{n+1}\, a_{n+1} + \cdots + \varepsilon_{n+p}\, a_{n+p}\right| \leq \tfrac{1}{k}\right\}.$$

From the characterization of the set $CPS(a)$ we get

$$CPS(a) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \bigcap_{p=1}^{\infty} F(k,n,p).$$

The set $F(k,n,p)$ is closed for every $k, n, p \in N$, therefore the set $CPS(a)$ is of type $F_{\sigma\delta}$.

When proving porosity we need to limit ourselves to a special class of conditionally convergent series.

Theorem 2.4. Let the conditionally convergent series $(a) = \sum a_n$ meet the condition $\limsup_{n\to\infty}\left(|a_{n+1}| + \cdots + \left|a_{n+q(n)}\right|\right) > 0$, where $\left(q(n)\right)_{n=1}^{\infty}$ is a sequence of natural numbers, for which $\left(q(n)\right) \leq B \cdot n$, $B \in R^{+}$. Then the set $CPS(a)$ is $\sigma - \frac{1}{1+B}$-porous in every point of space X.

Proof: Let $\sum a_n$ be a conditionally convergent series and (ε_n) is such a sequence of numbers ± 1, that the series $\sum \varepsilon_n a_n$ converges.

We denote (as in the previous proof)

$$F(k,n,p) = \left\{x = (\varepsilon_n) \in X\colon |\varepsilon_{n+1}\, a_{n+1} + \cdots + \varepsilon_{n+1}\, a_{n+1}| \leq \frac{1}{k}\right\}$$

and the set

$$H(m.k) = \bigcap_{n=m}^{\infty} \bigcap_{p=1}^{\infty} F(k,n,p).$$

We show that the set $H(m,k)$ is $\frac{1}{1+B}$-porous for every m.

Let $x^0 = ({\varepsilon_n}^0) \in X$ and $\delta > 0$.

From the preconditions of the theorem there exists $\vartheta > 0$ and such a sequence of indices (n_k), that the following holds true:

$\left|a_{n_j+1}\right| + \cdots + \left|a_{n_j+q(n_j)}\right| > \vartheta$ for $j = 1,2, \ldots.$

For $k > k_0$ ($\frac{1}{k_0} = \vartheta$) then

$\left|a_{n_j+1}\right| + \cdots + \left|a_{n_j+q(n_j)}\right| > \frac{1}{k}$ for $j = 1,2, \ldots.$

We find an index j_0, for which $\frac{1}{n_{j_0}} \leq \delta < \frac{1}{n_{j_0}-1}$.

Then $K\left(x_0, \frac{1}{n_{j_0}}\right) \subseteq K(x_0, \delta)$.

We create a sequence $y = (\eta_l) \in X$ as follows:

$$\eta_l = \begin{cases} {\varepsilon_l}^0, & l = 1,2, \ldots, n_{j_0} \\ {\varepsilon_l}^0 \cdot \operatorname{sign} a_l, & l > n_{j_0} \end{cases}.$$

From the definition of sequences $x^0 = ({\varepsilon_n}^0)$ and $y = (\eta_l)$ we get $\rho(x^0, y) \leq \frac{1}{n_{j_0}+1}$, from which

$$K\left(y, \frac{1}{n_{j_0}}\right) \subseteq K\left(x_0, \frac{1}{n_{j_0}}\right).$$

We create a sphere $K\left(y, \frac{1}{n_{j_0}+q(n_{j_0})}\right)$ such that

$$K\left(y, \frac{1}{n_{j_0}+q(n_{j_0})}\right) \subseteq K\left(y, \frac{1}{n_{j_0}}\right).$$

Let $z = (\xi_d)$ be any point of this sphere, i.e. $z \in K\left(y, \frac{1}{n_{j_0}+q(n_{j_0})}\right)$.

Then $\rho(z, y) < \frac{1}{n_{j_0}+q(n_{j_0})}$ results in $\xi_i = \eta_i$ for $i = 1, 2, \dots, n_{j_0} + q(n_{j_0})$.

In regards to the definition of the sequence y for $j > n_{j_0}$ we have

$\left|\xi_{n_{j_0}+1} a_{n_{j_0}+1} + \cdots + \xi_{n_{j_0}+q(n_{j_0})} a_{n_{j_0}+q(n_{j_0})}\right| = \left|a_{n_{j_0}+1}\right| + \cdots + \left|a_{n_{j_0}+q(n_{j_0})}\right| > \frac{1}{k}$.

That means that $z \notin F\left(k, n_{j_0}, q(n_{j_0})\right)$.

Therefore

$$K\left(y, \frac{1}{n_{j_0}+q(n_{j_0})}\right) \cap F\left(k, n_{j_0}, q(n_{j_0})\right) = \emptyset.$$

In view of the definition of $H(m, k)$

$K\left(y, \frac{1}{n_{j_0}+q(n_{j_0})}\right) \cap H(m, k) = \emptyset$ for every m and any $k > k_0$.

Consequently

$$\gamma(x^0, H(m, k), \delta) \geq \frac{1}{n_{j_0}+q(n_{j_0})}.$$

From the choice of the number n_{j_0} we get

$$\frac{\gamma(x^0, H(m, k), \delta)}{\delta} \geq \frac{\frac{1}{n_{j_0}+q(n_{j_0})}}{\delta} \geq \frac{\frac{1}{n_{j_0}+q(n_{j_0})}}{\frac{1}{n_{j_0}-1}} = \frac{n_{j_0}-1}{n_{j_0}+q(n_{j_0})} =$$

$$= \frac{1-\frac{1}{n_{j_0}}}{1+\frac{q(n_{j_0})}{n_{j_0}}} \geq \frac{1-\frac{1}{n_{j_0}}}{1+B}.$$

When we take $\delta \to 0^+$, then $n_{j_0} \to \infty$ and therefore

$$\Gamma\left(x^0, H(m, k)\right) = \limsup_{\delta \to 0^+} \frac{\gamma(x^0, H(m, k), \delta)}{\delta} \geq \frac{1}{1+B} > 0.$$

We have proven that the set $H(m, k)$ is $\frac{1}{1+B}$-porous in every point of space X.

Then the set

$$CPS(a) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} H(m, k)$$

is $\sigma - \frac{1}{1+B}$-porous in every point of space X.

CONCLUSION

From the theorems in chapter 2 it is evident that even if the set $CPS(a)$ is larger than the set CPS. In this paper, we have proven that the set $CPS(a)$ is porous for a special class of series (more precisely $\sigma - \frac{1}{1+B}$-porous) in every point of the space of sequences of numbers ± 1. Another key result in this paper is the topological characterization of the set $CPS(a)$. For every conditionally convergent series $(a) = \sum a_n$ the set $CPS(a)$ is of type $F_{\sigma\delta}$ (i.e. $CPS(a) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} F_{nm}$, where F_{nm} is a closed set).

References

[1] COWEN, C.C., DAVIDSON, K.R. Rearanging the Alternating Series. *Amer. Math. Monthly* 87, 1976, p. 817-819.

[2] MASÁROVÁ, R. Topologická dimenzia číselných postupností. *7. konference o matematice a fyzice na vysokých školách technických Brno*, 2011, p. 287-290. ISBN 978-80-7231-815-5.

[3] MASÁROVÁ, R. Porosity of a set of sequences preserving the convergence of numerical series . *MITAV 2017, Brno*, 2017, ISBN 978-80-7231-417-1.

[4] POTŮČEK, R. Sums of generalized alternating harmonic series with periodically repeated numerators (1,1) and (1,1,a) . *MITAV 2014, Brno*, 2014, p. 83-88. ISBN 978-80-7231-987-7.

[5] POTŮČEK, R. A Formula for the Sum of the Series of Reciprocals of the Polynomial of Degree Two with Different Positive Integer Roots . *MITAV 2016, Brno*, 2016, p. 73-83. ISBN 978-80-7231-400-3.

[6] ŠALÁT, T. O istých priestoroch radov s Bairovskou metrikou. *Mat.-fyz. čas.* VII, 1957, p. 193-205.

[7] ŠALÁT, T. O istých vlastnostiach radov s kladnými členmi. *Acta Fac. Rerum Natur. Univ. Com., Mathematica* II, 1957, p. 71-75.

[8] ŠALÁT, T. O niektorých problémoch z teorie nekonečných radov. *Acta Fac. Rerum Natur. Univ. Com., Mathematica* III, 1958, p. 29-39.

[9] TKADLEC, J. Constructions of non-σ-porous Sets of Real Line. *Real. Anal. Exchange* 9, 1984, p. 473-482.

[10] ZAJÍČEK, L. Sets of σ-porosity and sets of σ-porosity (q). *Čas.pro pěst. mat.* 101, 1976, p. 350-359.

GENERALIZED INVERSES OF CYCLES

Soňa Pavlíková, Naďa Krivoňáková
Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology Bratislava, Slovakia
sona.pavlikova@stuba.sk, nada.krivonakova@stuba.sk

Abstract: *Research into graph spectra includes the study of upper and lower bounds on certain important eigenvalues, such as the largest and the smallest positive ones. A way to obtain good lower bounds on the smallest positive eigenvalue of a graph is to consider its 'inverse' graph. The inverse of a graph with a non-singular adjacency matrix may be defined to be the edge-labeled graph determined by the inverse matrix. This way the spectrum of the inverse graph will be reciprocal to the spectrum of the original graph. This property will be retained even if the matrix is singular but one has to take a generalized inverse (a special case of both Drazin and Moore-Penrose inverses) instead.*
In this contribution we determine inverses of cycles of any length. As a cycle of length $n \geq 3$ has a non-singular adjacency matrix if and only if n is not divisible by 4, the task includes determination of both classical and generalized inverses (the latter for n a multiple of 4).

Keywords: Spectrum of a graph, eigenvalue, adjacency matrix, inverse matrix, singular matrix, group inverse, cycle.

INTRODUCTION

Graphs considered in this paper are all simple, finite and undirected, except that we allow at most one loop attached at a vertex. We will further assume that each edge e of a graph G carries a non-zero real *label* $\alpha(e)$, and the pair (G, α) will be called a *edge-labeled graph*. Let A be an adjacency matrix of (G, α), which means that for any two vertices u, v of G the uv-th entry of A is $\alpha(e)$ if $e = uv$ is an edge of G, and 0 otherwise.

If A is non-singular, the *inverse* of (G, α) is the edge-labeled graph (H, β) whose adjacency matrix is A^{-1}, the inverse of A. We thus assume that G and H share the same vertex set, and $e = uv$ is an edge of H if the uv-th entry of A^{-1} is non-zero, in which case this uv-th entry is the label $\beta(e)$ of e. Obviously, the inverse defined this way is unique up to graph isomorphism preserving edge-labels.

Inverses of edge-labeled graphs as introduced above were studied, for example, in [4, 7, 8, 12]. Edge-labels in [4, 8] were even allowed to be elements of a (not necessarily commutative) ring, and a formula for an inverse graph to (G, α) was given in both papers in the special case of bipartite graphs G with a unique perfect matching and with multiplicatively invertible α-labels on matched edges.

It turns out that the above approach may be extended to edge-labeled graphs with a *singular* adjacency matrix by taking the so-called group inverse of a matrix (coinciding with the Moore-Penrose or Drazin inverse) for symmetric matrices. To introduce it, recall that a real symmetric $n \times n$ matrix A is orthogonally diagonalizable. This means that there is an orthogonal matrix P

such that $PAP^T = D$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ is the diagonal matrix of eigenvalues of A, with $k = \mathrm{rank}(A)$ non-zero eigenvalues $\lambda_1, \ldots, \lambda_k$. If we let D^* denote the diagonal matrix $\mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_k^{-1}, 0, \ldots, 0)$, the *group inverse* or the *generalized inverse* A^* of A is simply given by $A^* = PD^*P^T$. Observe that both A and A^* are expressed as conjugates of the respective diagonal matrices by the same orthogonal matrix P. Note also that A^* is symmetric and that $A^* = A^{-1}$ if A is non-singular.

We define accordingly the *generalized inverse* of an edge-labeled graph (G, α) with adjacency matrix A to be the labeled graph (G^*, α^*) whose adjacency matrix A^* is the generalized inverse of A. As earlier, G and G^* have the same vertex set, and $e = uv$ is an edge of G^* if and only if the uv-th entry of A^* is non-zero, in which case this entry is the label $\alpha^*(e)$ of the edge e. Clearly, G^* is well defined up to isomorphism preserving edge labels.

An important stream of research in graph spectra is the study of upper and lower bounds on certain important eigenvalues, such as the largest and the smallest positive ones; interest in this comes from chemistry and we will give particulars in the next section. In the light of what has been introduced above, a way to obtain lower bounds on the smallest positive eigenvalue of a graph is to consider its inverse graph corresponding to the (generalized) inverse of the adjacency matrix of the original graph. Then, any upper bound of the spectral radius of the inverse graph yields a lower bound of the smallest positive eigenvalue of the original graph.

In this paper we will determine the (generalized) inverse graphs of cycles. By elementary algebraic combinatorics is well known that a cycle has non-singular adjacency matrix if and only if its length is not a multiple of 4, and in such a case one inverts the cycle by simply inverting its adjacency matrix. For lengths divisible by 4, however, one needs to consider the actual generalized inverses instead; we will present detailed calculations in both cases.

1 MOTIVATION

Molecules in chemistry are represented by graphs in a natural way. We will focus on organic molecules, where only carbon atoms and their bonds are represented by vertices and (possibly multiple) edges; hydrogen atoms are ignored in diagrammatic representations of this type. Suppose that such a graph has n vertices, so that its n eigenvalues (all real) can be ordered in the form $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. By known facts in theoretical chemistry (see e.g. [5]), the energy E_H of the highest occupied molecular orbital (HOMO) corresponds to the eigenvalue $\lambda_H = \lambda_k$ where $k = n/2$ for n even and $k = (n+1)/2$ for n odd, and the energy E_L of the lowest unoccupied molecular orbital (LUMO) corresponds to the subsequent eigenvalue $\lambda_L = \lambda_{k+1}$. The correspondence in both cases means that both E_H and E_L are multiples of λ_k and λ_{k+1} by the *same multiplicative constant*, that is, $E_H = \beta\lambda_H$ and $E_L = \beta\lambda_L$ for a constant factor $\beta < 0$; see [5] for background in physical chemistry.

The so-called *HOMO-LUMO separation gap* is the difference between the E_L and E_H energies; by the above we have $E_L - E_H = -\beta(\lambda_H - \lambda_L)$; this difference is non-negative because $\beta < 0$. As regards meaning of this parameter in physical chemistry, by Aihara [1, 2] a large HOMO-LUMO separation gap implies high kinetic stability and low chemical reactivity of a molecule, as it is energetically unfavorable to add electrons to a high-lying LUMO orbital. By experimental observations, the HOMO-LUMO separation gap appears to be decreasing with the size of the graph (cf. Bacalis and Zdetsis [3]).

Molecules called *properly closed shells* in chemistry are distinguished by the property that $\lambda_H > 0 > \lambda_L$, cf. [5]. This property is believed to guarantee stability of carbon molecules known as *fullerenes*, represented by planar trivalent graphs with faces of length 5 and 6, cf. [6]. If G is a graph representing a properly closed shell and if $\lambda^+(G)$ and $\lambda^-(G)$ denote the smallest positive and the largest negative eigenvalue of G, then the HOMO-LUMO separation gap for the molecule is a multiple of $\lambda^+(G)-\lambda^-(G)$. With a slight abuse of terminology we may then call the difference $\lambda^+(G) - \lambda^-(G)$ the *HOMO-LUMO separation gap* of G and denote it $\lambda_{H-L}(G)$.

Spectra, and hence also the HOMO-LUMO separation gap, of concrete graphs of a manageable order can be computationally estimated by means of a range of relaxation techniques available for determination of the spectral radius. These, however, do not give any information regarding asymptotic behaviour of spectral characteristics for infinite families of graphs. Moreover, there appear to be just a few methods for estimating the smallest positive eigenvalue of a matrix, compared with a larger number of techniques for bounding the largest positive eigenvalue. But by taking the (generalized) inverse of a particular 'chemical graph' one may hope to increase the number of techniques for estimating its smallest positive eigenvalue by applying bounds on the largest positive eigenvalue of the inverse graph instead. We reiterate that this is due to the fact that the spectrum of the inverse graph (classical or generalized) is reciprocal to the spectrum of the original graph, and so upper bounds of the spectral radius of the inverse graph give lower bounds of the smallest positive eigenvalue of the original graph.In recent papers [11, 10] Pavlíková and Ševčovič investigated qualitative and quantitative properties of the HOMO-LUMO separation gap in the context of its maximization with respect to the structure of the graph. They proposed a fast and efficient method for constructing optimal graphs structure that maximizes the HOMO-LUMO separation gap for a class of graphs.

2 GENERALIZED INVERSES OF CYCLES

In our analysis we will use the following result proved in [9].

Proposition 2.1 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be symmetric $n \times n$ matrices. Then, B is the generalized inverse of A if and only if A and B have the same null-spaces, and every eigenvector $f : [n] \to \mathbb{R}$ of A corresponding to a non-zero eigenvalue of A satisfies*

$$\sum_{j\in[n]} b_{ij} \sum_{k\in[n]} a_{jk} f(k) = f(i) \text{ for every } i \in [n] \,. \tag{1}$$

Let C_n be a cycle of length $n \geq 3$ with vertex set $V = \{0, 1, ..., n-1\}$, where each $u \in V$ is joined by an edge to the vertices $u - 1$ and $u + 1$ (mod n); because of this modularity we will identify V with elements of the cyclic group Z_n whenever convenient. For an arbitrary $\tau \in \sqrt[n]{1}$, the set of the n complex n-th roots of unity, let f_τ be the function mapping V into the field of complex numbers by the rule $f_\tau(u) = \tau^u$ for every $u \in V$. Letting $N(u)$ denote the (two) neighbours of u on the cycle, for all $u \in V$ we have

$$\sum_{v\in N(u)} f_\tau(v) = (\tau^{-1} + \tau) f_\tau(u) \tag{2}$$

so that the real numbers $\tau^{-1} + \tau$ for $\tau \in \sqrt[n]{1}$ represent all eigenvalues of C_n (including multiplicity) and the functions f_τ considered as vectors of length n are generators of the corresponding eigenspaces. Note that this principle extends to any circulant graph, that is, to a Cayley graph of a cyclic group.

We will assume that rows and columns of the adjacency matrix A_n of the cycle C_n is indexed by elements of V in the natural order. As A_n is circulant and symmetric, so is its generalized inverse, which we denote B_n. Any circulant matrix is completely determined by its first row, and hence to determine B_n it is sufficient to describe its first row. We will begin with the case when A_n is invertible, that is, when n is not divisible by 4.

Theorem 2.1 *Let $n \geq 3$ be not divisible by 4. Then, the entries $b_{0,i}$ $(0 \leq i \leq n-1)$ of the first row of the inverse B_n of the adjacency matrix A_n of a cycle of length n are determined as follows:*

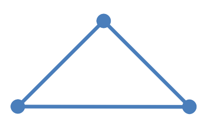
1. if $n \equiv 1 \pmod 4$, then $b_{0,i} = 1/2$ for $i \equiv 0, 1 \pmod 4$ and $b_{0,i} = -1/2$ for $i \equiv 2, 3 \pmod 4$;

2. if $n \equiv 3 \pmod 4$, then $b_{0,i} = -1/2$ for $i \equiv 0, 3 \pmod 4$ and $b_{0,i} = 1/2$ for $i \equiv 1, 2 \pmod 4$;

3. if $n \equiv 2 \pmod 4$, then $b_{0,i} = 0$ if i is even, and $b_{0,i} = (-1)^{(i-1)/2}/2$ if i is odd.

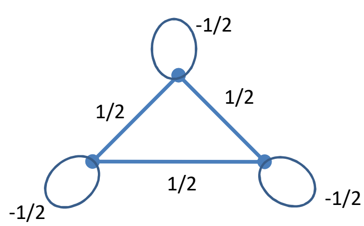
Proof. By a straightforward calculation one can verify that $A_n B_n = B_n A_n = I$. $\square$

Example 2.1 *Let us consider a cycle with the lenght $n = 3, 5, 6$. In this cases the lenght is not dividable by 4. On the figures bellow you can see a cycle, an adjecency matrix, an inverse matrix and an inverse graph of the cycle.*



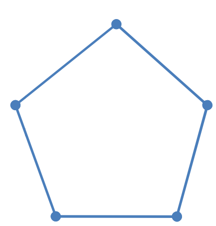
$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The cycle C_3 and its adjacency matrix.



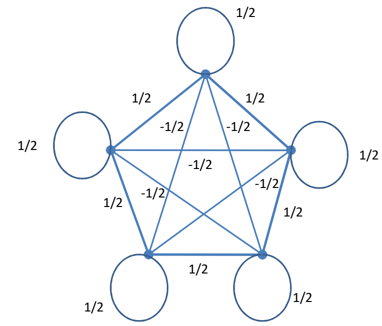
$$B_3 = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

The graph C_3^{-1} and its adjacency matrix.



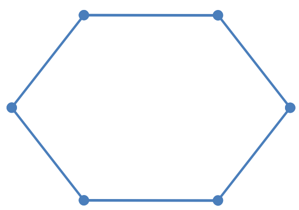
$$A_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The cycle C_5 and its adjacency matrix.



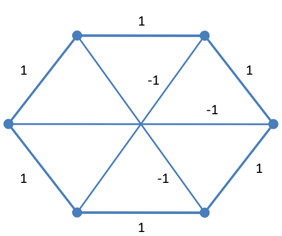
$$B_5 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

The graph C_5^{-1} and its adjacency matrix.



$$A_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The cycle C_6 and its adjacency matrix.



$$B_6 = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

The graph C_6^{-1} and its adjacency matrix.

We continue with the case when n is divisible by 4. In this case the adjacency matrix A_n of the cycle C_n is singular, so we determine the first row of its generalized inverse matrix B_n.

Theorem 2.2 *Let $n \geq 4$ be divisible by 4. Then, the entries $b_{0,i}$ $(0 \leq i \leq n-1)$ of the first row of the generalize inverse B_n of the adjacency matrix A_n of a cycle of length n satisfy $b_{0,i} = 0$ for i even, and for odd i one has*

$$b_{0,i} = (-1)^{\frac{i-1}{2}}\left(\frac{1}{2} - \frac{i}{n}\right) \quad \text{if} \quad 1 \leq i \leq \frac{n}{2} - 1\,, \quad \text{and} \quad b_{0,i} = b_{0,n-i} \ \text{ if } \ i > \frac{n}{2}. \tag{3}$$

Proof. In the notation introduced above, we saw at the beginning of this section that for every $\tau \in \sqrt[n]{1}$ and for the eigenvector f_τ of the cycle C_n corresponding to the eigenvalue $\tau + \tau^{-1}$ we have $f_\tau(i) = \tau^i$; in particular, $f_\tau(0) = 0$ for evert $\tau \in \sqrt[n]{1}$. Now, $\tau + \tau^{-1} = 0$ if and only if $\tau^2 + 1 = 0$, that is, if and only if τ is one of the two primitive 4-th roots of unity; note that here we use the assumption that n is a multiple of 4. Let $\sqrt[n]{1^*}$ be the set $\sqrt[n]{1}$ with the two primitive 4-th roots of unity removed. Then, for every $\tau \in \sqrt[n]{1^*}$ we have, by the equation (2) with $N(i) = \{i-1, i+1\}$ mod n, the following equation

$$\sum_i b_{0,i}(\tau^{i-1} + \tau^{i+1}) = 1, \quad \tau \in \sqrt[n]{1^*} \tag{4}$$

By Proposition 2.1, by uniqueness of a generalized inverse and by its circularity, our assertion will be proved if we show that the coefficients $b_{0,i}$ given by (3) satisfy the equation (4) and that the matrices A_n and B_n have the same null-spaces.

We begin by showing that the coefficients $b_{0,i}$ from (3) satisfy (4). This is equivalent to proving that

$$b_{0,1}(\tau^0 + \tau^2) + b_{0,3}(\tau^2 + \tau^4) + \cdots + b_{0,n-1}(\tau^{n-2} + \tau^n) = 1 \tag{5}$$

for every $\tau \in \sqrt[n]{1^*}$. Rearranging terms in (5) yields

$$(b_{0,1} + b_{0,n-1}) + (b_{0,1} + b_{0,3})\tau^2 + (b_{0,3} + b_{0,5})\tau^4 + \cdots + (b_{0,n-3} + b_{0,n-1})\tau^{n-2} = 1$$

and using the definition of $b_{0,i}$ from (3) for odd i we obtain after further rearrangements

$$\tau^2 - \tau^4 + \tau^6 - \tau^8 \pm \ldots - \tau^{n-4} + \tau^{n-2} = 1 \tag{6}$$

But on the left-hand side of (6) we have a sum of the first $k = (n-2)/2$ terms of a geometric series $a_j = a_1 q^{j-1}$ with first term $a_1 = \tau^2$ and quotient $q = -\tau^2$. Since $1 - q = 1 + \tau^2$ is non-zero for $\tau \in \sqrt[n]{1^*}$ we may use the formula $\sum_{j=1}^k a_j = a_1(1-q^k)/(1-q)$ in this case, which is easily seen to evaluate to 1, the value on the right-hand side of (6). This establishes (4) for $\tau \in \sqrt[n]{1^*}$.

It remains to show that the matrices A_n and B_n have the same null-spaces; by dimension of the eigenspaces of A_n and B_n corresponding to non-zero eigenvalues it is sufficient to show that the null-space of A_n is contained in the null-space of B_n. But observe that the null-space of A is generated by the two independent vectors $f_{tau}(j) = \tau^j$ for τ a primitive 4-th root of unity. This means that the two vectors evaluate (for $j = 0, 1, \ldots, n-1$) for the two values of τ to

$$(1, \tau, -1, \tau^{-1}, 1, \tau, -1, \tau^{-1}, \ldots, 1, \tau, -1, \tau^{-1}) \tag{7}$$

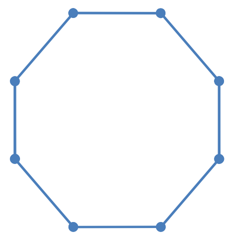
To prove that the vector given by (7) is in the null-space of the *circulant* matrix B_n it suffices to show that every cyclic shift of this vector is orthogonal to the first row of B_n, that is, to the vector $(b_{0,i})_{i=0}^{n-1}$ given by (3). But observe that the latter vector has, for $n = 4m$, the following structure:

$$(0, b_{0,1}, 0, b_{0,3}, \ldots, 0, b_{0,2m-3}, 0, b_{0,2m-1}; 0, b_{0,2m-1}, 0, b_{0,2m-3}, \ldots, 0, b_{0,3}, 0, b_{0,1}) \tag{8}$$

Orthogonality of the vectors (7) and (8) now follows from the fact that the primitive 4-th roots τ of unity considered here satisfy $\tau + \tau^{-1} = 0$. Orthogonality of the vector in (8) to the vector obtained from (7) by the cyclic shift by one coordinate to the right follows easily by the structure of (8). The remaining two cyclic shifts of the vector (7) to be checked reduce to the previous two cases. This completes the proof. □

We illustrate theorem 2.2 on example.

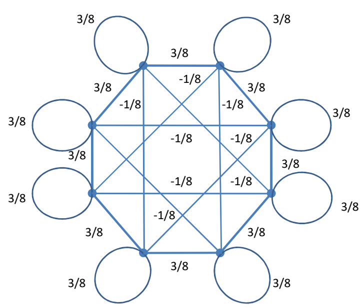
Example 2.2 *Let $n = 8$; in this case the length of the cycle is divisible by 4, so adjacency matrix of the cycle is singular.*



$$A_8 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The cycle C_8 and its adjacency matrix.

Generalized inverse matrix $\mathbf{B_8}$ of the adjacency matrix $\mathbf{A_8}$ is



$$B_8 = \frac{1}{8} \begin{pmatrix} 3 & 0 & -1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 3 & 0 & -1 & 0 & -1 & 0 & 3 \\ 3 & 0 & 3 & 0 & -1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 3 & 0 & -1 & 0 & -1 \\ -1 & 0 & 3 & 0 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 & 0 & 3 & 0 & -1 \\ -1 & 0 & -1 & 0 & 3 & 0 & 3 & 0 \\ 0 & -1 & 0 & -1 & 0 & 3 & 0 & 3 \end{pmatrix}$$

The graph C_8^{-1} and its adjacency matrix.

CONCLUSION

The result presented in this this paper is an example how to calculate generalized inverse matrices of adjacency matrices of graphs. We demonstrated the approach on calculating generalized (and also classical) inverse matrices to those obtained from cycles. These appear to be a first and interesting non-trivial example of this situation which may be of interest from the point of view of applications in chemistry. Constructions of the corresponding (edge-labeled) generalized inverse graphs is an obvious consequence of our results. We note that the method used in this paper can be extended to other families of graphs with singular adjacency matrices.

References

[1] Aihara, J. I.: Reduced HOMO-LUMO Gap as an Index of Kinetic Stability for Polycyclic Aromatic Hydrocarbons, J. Phys. Chem. A 183 (1999), 74877495.

[2] Aihara, J. I.: Weighted HOMO-LUMO energy separation as an index of kinetic stability for fullerenes, Theor. Chem. Acta 102 (1999), 134138.

[3] Bacalis, N. C., Zdetsis, A. D.: Properties of hydrogen terminated silicon nanocrystals via a transferable tight-binding Hamiltonian, based on ab-initio results, J. Math. Chem. 46 (2009), 962970.
[4] R. B. Bapat, and E. Ghorbani, Inverses of triangular matrices and bipartite graphs, Linear Algebra and its Applications 447 (2014), 68–73.
[5] Fowler, P. W.: HOMO-LUMO Maps for Chemical Graphs, MATCH Commun. Math. Comput. Chem. 64 (2010), 373390.
[6] Manolopoulos, D. E., Woodall, D. R., Fowler, P. W., Electronic stability of fullerenes: eigenvalue theorem for leapfrog carbon clusters, J. Chem. Soc. Faraday Trans. 88 (1992), 24272435.
[7] S. K. Panda and S. Pati, On the inverse of a class of bipartite graphs with unique perfect matching, Electr. J. Lin. Algebra 29 (2015), Art. 8.
[8] S. Pavlíková, A note on inverses of labeled graphs, Australas. J. Combin. 67 (2017), 222–234.
[9] S. Pavlíková, and J. Širáň, Inverting non-invertible trees, Linear Algebra and its Applications, submited (2016).
[10] Pavlíková, S., Ševčovič, D., On Construction of Upper and Lower Bounds for the HOMO-LUMO Spectral Gap by Means of Semidefinite Relaxation Techniques, Numerical Algebra, Control and Optimization, 9(1) (2019), 53–69.
[11] Pavlíková, S., Ševčovič, D., Maximization of the Spectral Gap for Chemical Graphs by means of a Solution to a Mixed Integer Semidefinite Program, Computer Methods in Materials Science, 4 (2016), 169–176.
[12] D. Ye, Y. Yang, B. Manda, and D. J. Klein, Graph invertibility and median eigenvalues, ArXiv:1506.04054v1 (2015).

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COMPARATIVE ANALYSIS AND NEW FIELD OF APPLICATION OF LANCHESTER'S COMBAT MODELS

Andriy Shatyrko, Bedřich Půža, Veronika Novotná
Brno University of Technology, Faculty of Business Management,
616 00 Brno, Czech Republic,
shatyrko.a@gmail.com, puza@fbm.vutbr.cz, novotna@fbm.vutbr.cz

Abstract: Lanchester's equations and their solutions, as continuous differential equations, have been studied in first decades of XX century. New approach with the use of the discrete form of Lanchester's equations, which represent by dynamical systems of difference equations was proposed at the beginning of XXI. This was caused both by the desire to build more adequate models and the improvement of computer technology. We illustrate these models, their solutions and their comparability using historic combat examples, and some student case studies. We show how these well-studied models can be applied in a completely different field - the modeling of economic dynamics problems. We formulate new statements of problems. Namely, models with time-delay argument.

Keywords: Lanchester's models, continuous and discrete forms, control, economic models, behavior of solutions.

INTRODUCTION

The scope of mathematical modeling application is constantly expanding and deepening. Constantly complicated as the well-known classical statements of problems for the purpose of their greater adequacy of reality, and absolutely new previously unexplored models appear [1-4]. This process does not depend on what tasks it is necessary to investigate: technical, engineering, social, economic or other. But no matter how the process evolved, still most of the mathematical models are representable in the form of systems of ordinary differential equations (ODE), partial differential equations (PDE), difference equations (DDE), and functionally differential equations (FDE) [5-7]. Because, in the majority, they allow an analytical solution. And if it can't be found, then with success to solve similar problems you can approach using numerical methods and computer technology [8,9].
We will focus our attention on one of the most acute social phenomena - military actions. And let us consider one of the simplest formulations of similar problems. In the most general terms Lanchester model [10-14] can be described by the equation:

$$\begin{cases} \dfrac{dx}{dt} = ax + bxy + cy + d \\ \dfrac{dy}{dt} = ey + fyx + gx + h \end{cases} \tag{1}$$

Here a and e - constants that determine the rate of non-combat losses; b and f - rate losses due to the action area; c and g - losses from exposure to the enemy at the forefront; d and h - reserves approaching or receding [15]. There are next more wellknown models.
1. The model is actually Lancaster (there are only coefficients b and f). In this case, the number of victims is proportional to the number of meetings between the individuals of the opposing parties (the product of the number of parties). The most relevant such interaction is

when two parties are located in the common territory (partisan war, repression, hostility of two ethnic groups, etc.)
2. Model of Osipov (there are only coefficients c and g). The number of victims is proportional to the number of opposite parties. This may be a classic military engagement when the two sides come into contact only at the forefront.
3. Peterson model (coefficients a and e). The number of victims is determined by the number of each party. This may be a model of the Cold War, for example, when more of its submarines are fighting alert, the more they die.
4. Brekney Model (coefficients a and f , or b and e). Victims of one side are proportional to the number of meetings, and the other side - the number of its opponent. The model was created under the impression of fighting in Vietnam and quite satisfactorily describes the conflict in which one side is a classic battle, and the second - a guerrilla.
The greatest applicability Lanchester equation found in the form [10-12]:

$$\begin{cases} \dfrac{dx}{dt} = -ax - cy \pm d \\ \dfrac{dy}{dt} = -ey - gx \pm h \end{cases} \tag{2}$$

1 MAIN PART

1.1 Combat models in continuous case

Let the fighting take two sides x and y. Their size at a time t, which is measured in days, starting from the first day of combat operations, denote through $x(t)$ and $y(t)$ respectively.
We also assume, according to [10-12,16] that $x(t)$ and $y(t)$ change continuously and, moreover, they are differentiated as functions of time. Of course, these assumptions are a simplification of the real situation, because $x(t)$ and $y(t)$ are integers. But at the same time it is clear that, with a sufficiently large numerical composition of each side, an increase in the number of one or two persons gives from a practical point of view an infinitesimal value compared to the already existing composition. Therefore, it can be assumed that at small time intervals the numerical composition also varies by small numbers (not integers). These conditions, of course, are not enough to make specific formulas for $x(t)$ and $y(t)$ as functions of time t. However, we can specify a number of factors that allow us to describe the rate of change in the number of opposing sides.
Further, we will use the following notation: a, b, c, d, g, h - nonnegative constants, which characterizing the rate of influence of various factors on the losses in manpower and both parties $x(t)$ and $y(t)$; $P(t)$ and $Q(t)$ - terms taking into account the possibility of an approach to strengthening forces during the day; x_0, y_0 - the number of forces before the start of combat operations. We will demonstrate three models built by Lanchester.
The first of these is the description of the fighting between regular troops, so-called "direct fire" model, and it has the form:

$$\begin{cases} \dfrac{dx}{dt} = -ax - by + P(t) \\ \dfrac{dy}{dt} = -cx - dy + Q(t) \end{cases} \tag{3}$$

In the future, this system will be called the differential system of type (**A**).
The second model describes the fighting between partisan connections ("guerilla warfare"). We will call it a differential system of type (**B**).

$$\begin{cases} \dfrac{dx}{dt} = -ax - gxy + P(t) \\ \dfrac{dy}{dt} = -dy - hxy + Q(t) \end{cases} \tag{4}$$

Finally, the third model ("mix warfare" model), which will be called differential system type (**C**) describes a mixed type of fighting, involving both regular units and partisan

$$\begin{cases} \dfrac{dx}{dt} = -ax - gxy + P(t) \\ \dfrac{dy}{dt} = -cx - dy + Q(t) \end{cases} \tag{5}$$

We will illustrate the above systems.
Quadratic law (equation type **A**). Suppose that the regular forces of two opposing forces are fighting in the simplified situation, where the losses are not associated with such actions, is absent. And then, if both sides do not receive reinforcements, the mathematical model is reduced to the following form:

$$\frac{dx}{dt} = -by,\ \frac{dy}{dt} = -cx \tag{6}$$

Dividing in (6) the second equation by first, we obtain that

$$\frac{dy}{dx} = \frac{cx}{by} \tag{7}$$

Integrating the differential equation (7), we obtain at equality

$$b[y^2(t) - y_0^2] = c[x^2(t) - x_0^2] \tag{8}$$

The relation (8) explains why the system (6) corresponds to a quadratic law model. If denoted by K a constant $by_0^2 - cx_0^2$,

$$by^2 - cx^2 = K \tag{9}$$

then the equation obtained from equation (8), defines a hyperbole (a pair of straight lines if $K = 0$), and we can more exactly classify the system (6). It is such a system that can be called a differential system with hyperbolic law.
In fig.1 we present the geometric interpretation of functional dependence for different value K.

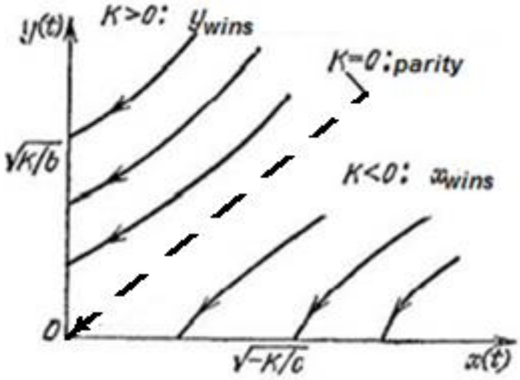

Fig.1. The balance of the forces of the problem type **A**.

For answer the question of who wins in the constructed model (6), we will firstly agree to say that the party $x(t)$ wins (hence $y(t)$) if it is the first to destroy the fighting forces of the party $y(t)$ (respectively $x(t)$). From the subsequent analysis, it is not difficult to see the effects of quadratic dependence. For example, the change in the attitude of forces from $y/x_0 = 1$ to $y/x_0 = 2$ gives a fourfold advantage to the forces $y(t)$

Linear law (equation type **B**). The equation of dynamics which simulating the combat actions of the two opposing sides can be easily solved, if the losses are excluded, not related to military actions, and neither side receives reinforcements. Under such restrictions, the differential system of type (**B**) takes the form

$$\frac{dx}{dt} = -gxy \quad \frac{dy}{dt} = -hxy\,. \tag{10}$$

Having made mathematical calculations similar to the previous one, we can make the plot of linear dependence of forces at different value $gy - hx = L$ (See Fig.2.)

Parabolic law (equation type **C**). In model (**C**) guerrilla forces resist regular parts. We will again make simplifying assumptions that the two opposing sides are not provided with reinforcements and do not bear losses not related to military action. In this case, we have a differential system of the type

$$\frac{dx}{dt} = -gxy \quad \frac{dy}{dt} = -cx \tag{11}$$

In accordance with the previous calculations, denoting by $M = gy_0^2 - 2cx_0$, we obtain plot of parabolic law (Fig.3).

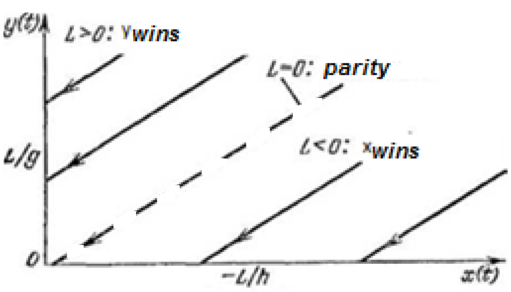

Fig.2. The balance of the forces of the problem type **B**.

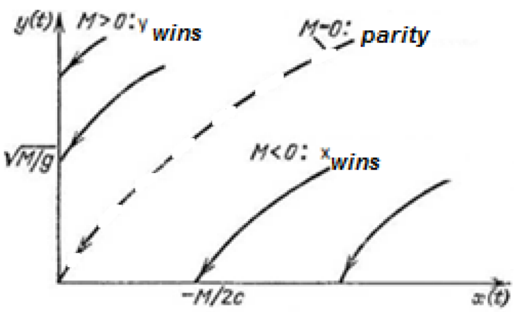

Fig.3. The balance of the forces of the problem type **C**

1.2 Combat models in discrete form

Combat is fought over continuous time, there are typically discrete starting, pause, and stopping points. Often models of combat employ discrete time simulation. Dynamical systems can always be solved by iteration, which make them quite attractive for use in both computer modeling and simulations of combat. However, we can gain some powerful insights with those discrete equations that have analytical solutions. This particular dynamical system of equations for Lanchester's direct fire model does have an analytical solution. In this directions good results (analytical and computer simulation) were obtained at [17,18].

Quadratic law (type **A**). In order to obtain a discrete model for the system (3) we use the simplest method of Euler's approximation by writing the usual derivatives as discrete values:

$$\frac{dy}{dt} = f(t,y) = \lim_{\Delta t \to 0} \frac{y_{n+1} - y_n}{\Delta t} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$f(t_n, y_n) \approx \frac{y_{n+1} - y_n}{\Delta t}, \quad y_{n+1} \approx y_n + \Delta t\left[f(t_n, y_n)\right].$$

Thus, according to [17,18], the continuous model (3) can be approximated to the form:

$$\begin{cases} X(n+1) = X(n) - bY(n) - aX(n) + P(t), \ X(0) = X_0 \\ Y(n+1) = Y(n) - cX(n) - dY(n) + Q(t), \ \ Y(0) = Y_0 \end{cases}$$

Assume that the regular forces of the two opposing forces conduct hostilities in a simplified situation where losses that are not related to such actions are absent. And then, if both sides do not receive reinforcements, the mathematical model is reduced to the following form

$$\begin{cases} x(n+1) = x(n) - b \cdot y(n), \\ y(n+1) = y(n) - c \cdot x(n). \end{cases} \tag{12}$$

Further, the coefficients b and c we will record as that k_1 and k_2, that is, as indicators of the rate of combat losses.

Let us rewrite the system (12) in the matrix form:

$$X_{n+1} = \begin{bmatrix} 1 & -k_1 \\ -k_2 & 1 \end{bmatrix} X_n, \ X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{13}$$

We will use the matrix solution method using eigenvalues and eigenvectors to find the analytical solution of equation (13). We construct the characteristics polynomial and calculate an eigenvalues

$$\lambda_{1.2} = 1 \pm \sqrt{k_1 k_2}$$

Under some assumption [17,18] we will obtain correspondent eigenvectors

$$V_1 = \begin{bmatrix} k_1 \\ \sqrt{k_1 k_2} \end{bmatrix}, V_2 = \begin{bmatrix} -k_1 \\ \sqrt{k_1 k_2} \end{bmatrix}$$

or

$$V_1 = \begin{bmatrix} \dfrac{1}{\sqrt{k_1 k_2}} \\ k_1 \end{bmatrix}, V_2 = \begin{bmatrix} \dfrac{-1}{\sqrt{k_1 k_2}} \\ k_1 \end{bmatrix}.$$

Now, knowing all the parameters, we can write a solution of the system (12) ((13)) in the form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ \sqrt{\dfrac{k_2}{k_1}} \end{pmatrix} (1+\sqrt{k_1 \cdot k_2})^t + c_2 \begin{pmatrix} 1 \\ \sqrt{\dfrac{k_2}{k_1}} \end{pmatrix} (1-\sqrt{k_1 \cdot k_2})^t, \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \quad (14)$$

It's easy to make sure that the conditions for winning the party X are the next: $\sqrt{k_1 k_2} \cdot x_0 > k_1 y_0$, where x_0, y_0 - the number of forces before the start of combat operations (initial values). Similarly, however with the opposite sign (<) - a condition of winning the side Y. If, however the sign (=), we will have a draw.

We find the correlation of the forces of the parties. One of own vectors is $\begin{pmatrix} k_1 \\ \sqrt{k_1 k_2} \end{pmatrix}$.

Hence

$$\frac{k_1}{\sqrt{k_1 k_2}} = \frac{x_0}{y_0}, \qquad \text{or} \qquad \sqrt{k_1 k_2}\, x_0 = k_1 y_0, \qquad \text{or} \qquad \frac{k_2}{k_1} = \left(\frac{y_0}{x_0}\right)^2.$$

Having simplified the previous relations and solved relative x, we get a line:

$$y = \frac{\sqrt{k_1 k_2}}{k_1} x.$$

Formally, all the lines thus obtained will be invested in the scheme of Fig. 1

1.3 Illustrative examples

We will use descreete Lanchester model for calculate some examples.

Example 1. (Historical) The Battle of Trafalgar [17,19,20].
In classical naval warfare, two fleets (Blue against Red) would sail parallel to each other (see Fig. 4) and fire broadside at one another until one fleet was annihilated or gave up. The Blue fleet represents the British and the Red fleet represents the French-Spanish fleet. In such an engagement, the fleet with superior firepower will inevitably win. To model this battle, we begin with the system of difference equations that models the interaction of two fleets in combat.

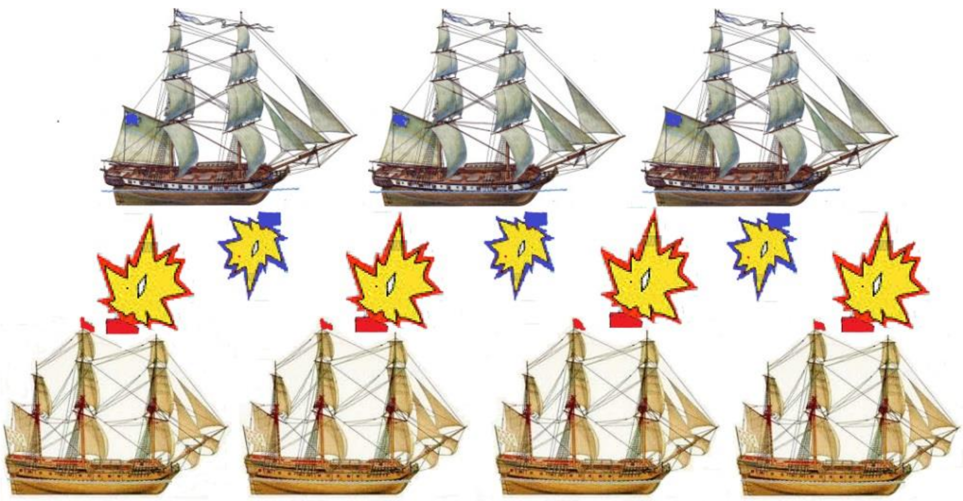

Fig. 4.

We iterate these dynamical systems equations to obtain the numbers in the table to determine who wins the engagement. We graph this information as illustrated in Fig. 5.
But, history gave us the opposite result. Admiral Nelson defied conventional warfare, ordered his captains to split the British fleet and spear the enemy's line, called "crossing the T," to create a "pell-mell battle," which has been called the "Nelson Touch."

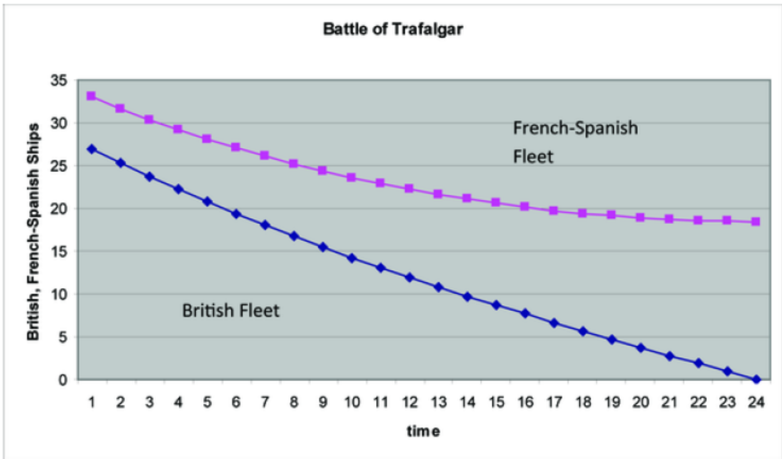

Fig. 5. [17] Battle of Trafalgar under normal battle strategies showing the victory of the French-Spanish fleet. In this example, Admiral Nelson has 27 ships while the allied French and Spanish fleet had 33 ships. As we can see, Admiral Nelson is expected to lose all 27 of his ships while the allied fleet will lose only about 14 ships.

We can test this new strategy that was used by Admiral Nelson at the Battle of Trafalgar using our discrete combat model. Admiral Nelson decided to move away from the course of linear battle of the day and use a "divide and conquer" strategy. Nelson decided to break his fleet into two groups of size 13 and size 14. He also divided the enemy fleet into three groups: a force of 17 ships (called B), a force of 3 ships (called A) and a force of 13 ships (called C). We can assume these as the head, middle, and tail of the enemy fleet. His plan was to take the 13 ships and attack the middle 3 ships. Then have his reserve 14 ships rejoin the attack and attack the larger force B, and then turn to attack the smaller force C. How did Nelson's strategy prevail? Assuming all other variables remain constant other than the order of the attacks against the differing size forces, we find the Admiral Nelson and the British fleet now win the battle sinking all French-Spanish ships with the British fleet having 13 or 14 ships remaining. The easiest method to obtain these results ware by iteration. We used three battle formulas. We stop each battle when one of the values gets close to zero (before going negative). This is displayed in Fig. 6.

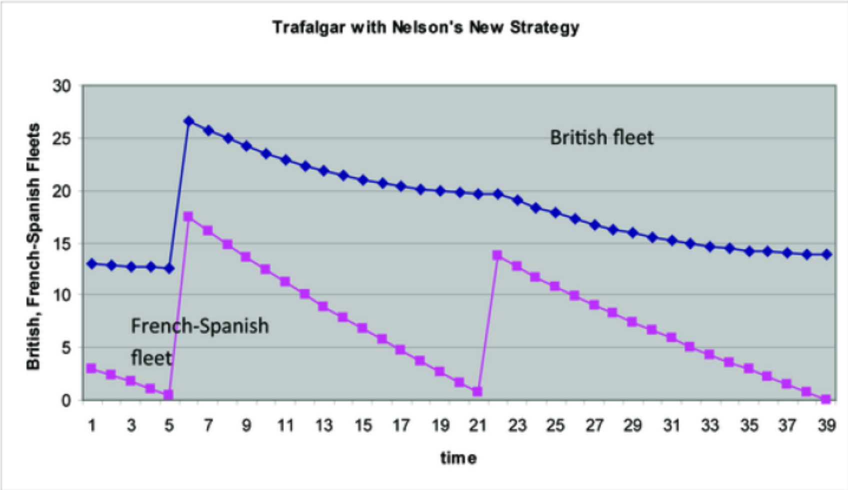

Fig. 6. [17] British prevail with Nelson's new strategy

Example 2. The Tank battle (all parameters takes from [21]).

Side 1 (Red): 800 tanks, that drive the defense ($k_1 = 1$). Coefficient of effectiveness $c = 1 \cdot 1 \cdot 0.5 \cdot 0.04 = 0.02$

Side 2 (Blue): 2000 tanks, that drive the attack ($k_1 = 0.8$).Coefficient of effectiveness $d = 1 \cdot 0.8 \cdot 0.5 \cdot 0.04 = 0.0016$

The battle takes place fog: ($k_3 = 0.5$). Second World War ($C = 0.04$).

Task: find duration of the battle?

Using the above results and Maple Tools we obtain next plots

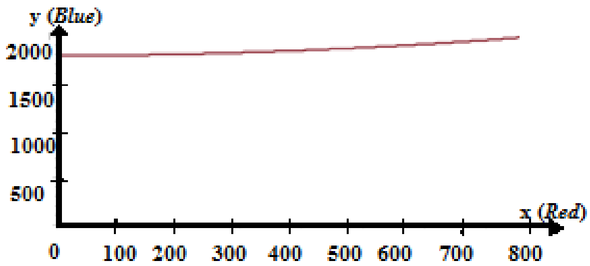

Fig. 7. Illustration balance of forces to Ex.2.

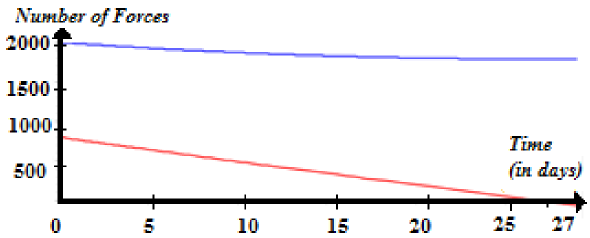

Fig. 8. Illustration number of vehicles in time for Ex.2

From Fig.8 we can see that the duration of battle will be **27** days.

Example 3. ("mix warfare" model) [18,21]

Side 1 (Red): 800 defense soldiers. $c = 1.5 \cdot 1 \cdot 0.3 \cdot 0.12 = 0.054$

Side 2 (Green): 2000 guerilla, which drive the attack.

$d = 1 \cdot 0.8 \cdot 0.3 \cdot 0.12 \cdot 2/9 = 0.00064$

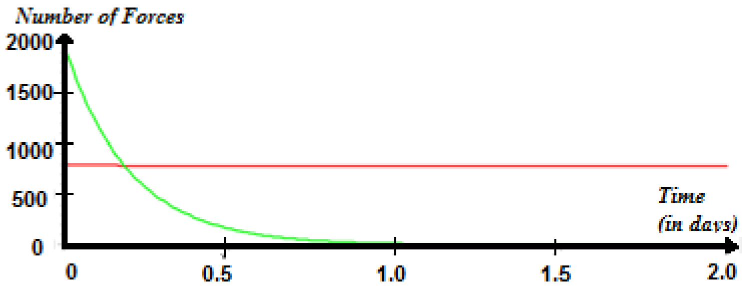

Fig. 9. Illustration of the number of combat units to Ex. 3
Guerilla (Green) have no chances, the battle ended in 0.9 days.

1.4 Comparison between differential and difference form of Lanchester's model

We propose an Example to illustrate such compere [17,18].
Example 4. Let's consider the battle between red (R), and the blue (B) side
A) Discrete model

$$\begin{cases} B(n+1) = B(n) - 0.1 \cdot R(n), \ B(0) = 100 \\ R(n+1) = R(n) - 0.05 \cdot B(n), \ R(0) = 50 \end{cases}$$

$$k_1 = -0.1 \quad k_2 = -0.05 \quad \sqrt{k_1 k_2} = 0.0707 \quad \frac{\sqrt{k_1 k_2}}{k_1} = 0.7070$$

$$V_1 = 1.0707 \qquad V_2 = 0.9293$$

$$x(t) = c_1 \begin{pmatrix} -1 \\ 0.707 \end{pmatrix} (1.0707)^t + c_2 \begin{pmatrix} 1 \\ 0.707 \end{pmatrix} (0.9293)^t$$

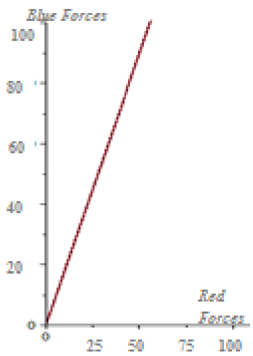

Fig. 10. Illustration balance of forces to Ex.4

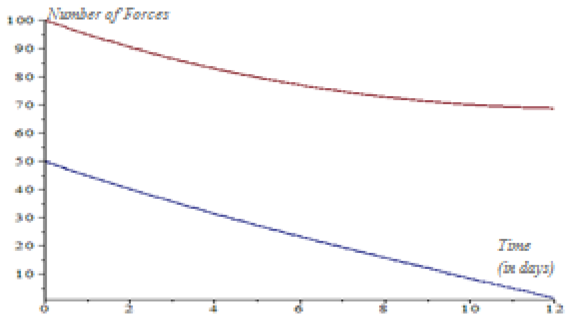

Fig. 11. [17,18] Illustration of the number of combat units to Ex.4

B) Continuous model. Denote $x(t)$ - red (R), and $y(t)$ the blue (B)

$$\frac{dx}{dt} = -0.1y,$$

$$\frac{dy}{dt} = -0.05x$$

$$x(0) = 100, \quad y(0) = 50$$

We obtained the system solution

$$x(t) = 14.644\exp(0.0707t) + 85.355\exp(-0.707)$$
$$y(t) = -10.355\exp(0.0707t) + 60.355\exp(-0.707)$$

We can compeare these two results by plots (Fig. 12).

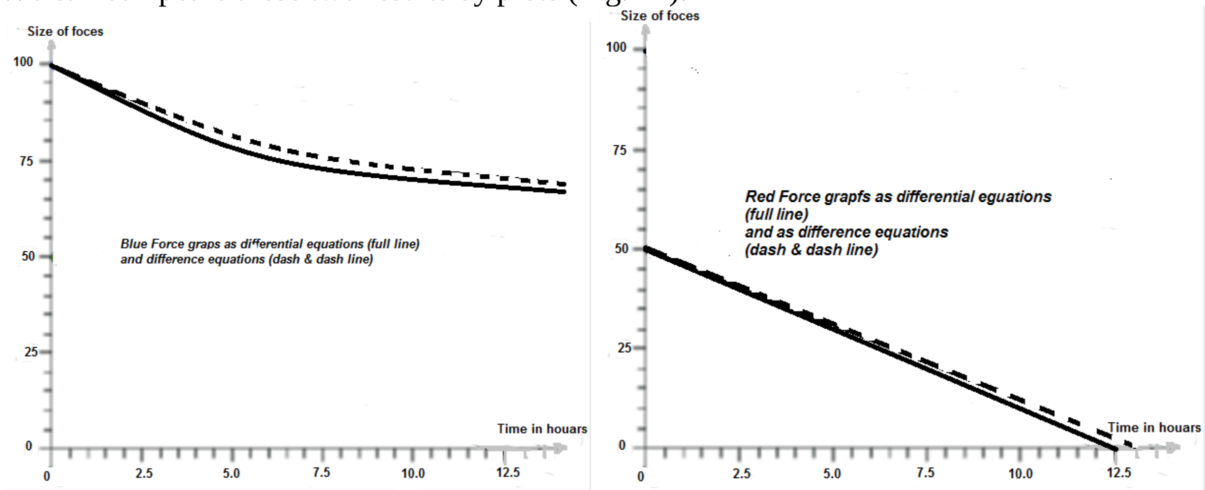

Fig. 12. Solution plots of Blue vs Red forces
via differential equations and difference equations
showing graphically that they are practically the same

2 NEW FILD OF APPLICATION LANCHESTER'S COMBAT MODELS

2.1 Combat models as a task of control problem

Let's give one of the tasks of optimal distribution of resources in dynamic systems on the example of a model of battle of two parties [22]. The dynamics of competitive production of similar products from the same raw materials in the common market can be described by the systems

$$\frac{dx_1}{dt} = -bx_2 + u(t)$$
$$\frac{dx_2}{dt} = -ax_1 + v(t) \quad ,$$

де $x_1(t)$ – quantity of goods produced by the party A in time $t \in [t_0, t_1]$, $x_2(t)$ – quantity of goods produced by the party B; $u(t)$, $v(t)$ – the rate of depreciation (or the possibility of obtaining raw materials from other sources) for A and B accordingly; a, b – average production efficiency of the parties A and B accordingly; $T = t_1 - t_0$ – the given time of the production process.

Let's be known:

$$x_1(t_0) = x_1^0$$
$$x_2(t_0) = x_2^0$$

and value $v(t)$.

The tasks of optimal distribution of resources:

need to be find the control function $u^0(t)$

with restriction: $0 \le u(t) \le \bar{u}, \int_{t_0}^{t_1} u(t)dt \le \bar{\bar{u}}$, ($\bar{u}, \bar{\bar{u}}$ – given values),

such that the selected quality functional $Q(u(t))$ reached its extremum.

As the criterion for the best control can be selected definite purpose of competitive production, for example:

$Q = x_2(t_1) \to \min\limits_u$ – at the end of the manufacturing process, the party B has manufactured and launched on the market fewer products;

$Q = x_1(t_1) \to \max\limits_u$ – a purpose of A - to produce and realize the largest quantity of products on the market by the end of the production process.

We can offer other some criteria of optimality.

2.2 Time delay in combat models

In real processes, there are almost always elements that cause late effects. Physical and technical reasons for delays may include transport delays, delays in information transmission, delays in decision making, and so on. Other factors are also possible. Therefore, the mathematical models represented by functional-differential equations describe the most part of the dynamic objects more adequately [7,23-28].

In real life at the confrontation between two hostile parties, the side that starts the second always does it not simultaneously with the first one. That is, the answer comes with some delay in time. As far as the authors are concerned, to date, such situation is not modeled in terms of models of combat operations of Lanchester type, or others.

For simplicity of presentation, we return to "direct fire" model (3) and rewrite it in the next form

$$\begin{cases} \dfrac{dx(t)}{dt} = -ax(t) - by(t-\tau) + P(t) \\ \dfrac{dy(t)}{dt} = -cx(t-\tau) - dy(t) + Q(t) \end{cases}$$

Here $\tau = const > 0$ - time lag corresponding to a delayed reaction.

In this case, under the same restrictions, system (6) can be represented as follows

$$\begin{cases} \dfrac{dx(t)}{dt} = -by(t-\tau), \\ \dfrac{dy(t)}{dt} = -cx(t-\tau) \end{cases}$$

It is well known that the presence of lag in the system of equations can radically change the behavior of the solutions and significantly affect the quality of the phase portrait [9,10].

2.2.1 General solution of Cauchy problem for linear non-homogeneous systems with time delay

Following the results of [28], we consider a linear inhomogeneous system with a time-delay argument in the form

$$\dot{x}(t) = Bx(t-\tau) + f(t), \tag{15}$$

The solution of the Cauchy problem of system (15) has the form of a sum of two solutions

$$x(t) = x_0(t) + x_c(t).$$

Where

$x_0(t)$ a solution of the Cauchy problem for a homogeneous system with delay satisfying given next conditions

$$\dot{x}(t) = Bx(t-\tau),\ x_0(t) = \varphi(t),\ -\tau \le t \le 0$$

$x_c(t)$ - a solution of the Cauchy problem for a nonhomogeneous system with the next initial conditions

$$x_c(t) \equiv 0,\ -\tau \le t \le 0$$

And it can be finally represented in the form

$$x(t) = \exp_\tau\{B,t\}\varphi(-\tau) + \int_{-\tau}^{0} \exp_\tau\{B, t-\tau-s\}\varphi'(s)ds, \tag{16}$$

where the matrix function $\exp_\tau\{B,t\}$, , which called “delay exponential”, has the form

$$\exp_\tau\{B,t\} = \begin{cases} 0 & , \quad -\infty < t < -\tau, \\ I & , \quad -\tau \le t < 0, \\ I + B\dfrac{t}{1!} & , \quad 0 \le t < \tau, \\ I + B\dfrac{t}{1!} + B^2\dfrac{(t-\tau)^2}{2!} & , \quad \tau \le t < 2\tau \\ \cdots & , \quad \cdots \\ I + B\dfrac{t}{1!} + \cdots + B^k\dfrac{[t-(k-1)\tau]^k}{k!} & , \quad (k-1)\tau \le t < k\tau \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases} \tag{17}$$

When $\tau \to 0$, convergence takes place on the interval $t \ge 0$

$$\lim_{\tau \to 0}\{\exp_\tau\{B,t\}\} = e^{Bt}$$

where

$$e^{Bt} = \sum_{k=0}^{\infty} B^k \frac{t^k}{k!}$$

Remark. In the initial conditions of the integral representation (3), it is necessary to have the continuous differentiability of the initial vector function $\varphi(t)$. If we take the integral by parts, we obtain integral representation of the solution

$$x(t) = \exp_\tau\{B, t-\tau\}\varphi(0) + B\int_{-\tau}^{0} \exp_\tau\{t-2\tau-s\}\varphi(s)ds \tag{18}$$

For a partial solution of a non-homogeneous system, the Cauchy formula holds. The solution of the non-homogeneous system (15), which satisfies the zero initial conditions,

$$x(t) \equiv 0, \; -\tau \le t \le 0,$$

has the form

$$x_0(t) = \int_0^t \exp_\tau \{B, t-\tau-s\} f(s) ds. \tag{19}$$

Using relations (18), (19) we can return to our Lanchasters models.

2.2.2 Direct fire Lanchesters type model with time-delay argument

Let us consider the next modification of the model type **A**.

$$\dot{x}(t) = -by(t-\tau) + d,$$
$$\dot{y}(t) = -cx(t-\tau) + e.$$

Here b, c, d, e - are positive constants.

In this case matrix B and vector f for the system (15) have the next form

$$B = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}, \qquad f(t) = f = \begin{pmatrix} d \\ e \end{pmatrix}.$$

For the matrix B we can simply calculates its power:

$$B^2 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}\begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix},$$

$$B^3 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} = \begin{bmatrix} 0 & -b^2c \\ -bc^2 & 0 \end{bmatrix},$$

$$B^4 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}\begin{bmatrix} 0 & -b^2c \\ -bc^2 & 0 \end{bmatrix} = \begin{bmatrix} b^2c^2 & 0 \\ 0 & b^2c^2 \end{bmatrix},$$

$$B^5 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}\begin{bmatrix} b^2c^2 & 0 \\ 0 & b^2c^2 \end{bmatrix} = \begin{bmatrix} 0 & -b^3c^2 \\ -b^2c^3 & 0 \end{bmatrix},$$

$$B^6 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}\begin{bmatrix} 0 & -b^3c^2 \\ -b^2c^3 & 0 \end{bmatrix} = \begin{bmatrix} b^3c^3 & 0 \\ 0 & b^3c^3 \end{bmatrix},$$

$$B^7 = \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}\begin{bmatrix} b^3c^3 & 0 \\ 0 & b^3c^3 \end{bmatrix} = \begin{bmatrix} 0 & -b^4c^3 \\ -b^3c^4 & 0 \end{bmatrix}, \ldots,$$

$$B^{2n}=\begin{bmatrix}0 & -b\\ -c & 0\end{bmatrix}\begin{bmatrix}0 & -b^n c^{n-1}\\ -b^{n-1}c^n & 0\end{bmatrix}=\begin{bmatrix}b^n c^n & 0\\ 0 & b^n c^n\end{bmatrix},$$

$$B^{2n+1}=\begin{bmatrix}0 & -b\\ -c & 0\end{bmatrix}\begin{bmatrix}0 & b^n c^n\\ b^n c^n & 0\end{bmatrix}=\begin{bmatrix}0 & -b^{n+1}c^n\\ -b^n c^{n+1} & b^n c^n\end{bmatrix}.$$

Let is consider relation (3), and will make change variable $t-\tau-s=\xi$. Than the boundaries of the integral will be the next $s=-\tau \Rightarrow \xi=t$, $s=0 \Rightarrow \xi=t-\tau$. Let is $n\tau \le t<(n+1)\tau$, than $n\tau \le t-\tau<\xi<(n+1)\tau$ and the relation (3) we can rewrite as

$$x(t)=\exp_\tau\{B,t\}\varphi(-\tau)+\int_{t-\tau}^{t}\exp_\tau\{B,\xi\}\varphi'(t-\tau-\xi)d\xi \tag{20}$$

Since the vector function $f(t)$ is a constant, than

$$x_0(t)=\left[\int_0^t \exp_\tau\{B,t-\tau-s\}ds\right]f$$

Let will make the same change of variable $t-\tau-s=\xi$, than $s=0\to\xi=t-\tau$, $s=t\to\xi=-\tau$ and last integral will have next form

$$x_0(t)=\left[\int_{-\tau}^{t-\tau}\exp_\tau\{B,\xi\}ds\right]f$$

At the time interval $(n-1)\tau \le t-\tau<n\tau$ it can be rewrite as

$$x_0(t)=\left[\int_{-\tau}^{0}\exp_\tau\{B,\xi\}ds\right]f+\left[\int_{0}^{\tau}\exp_\tau\{B,\xi\}ds\right]f+\left[\int_{\tau}^{2\tau}\exp_\tau\{B,\xi\}ds\right]f+\ldots+$$
$$+\left[\int_{(n-1)-\tau}^{t-\tau}\exp_\tau\{B,\xi\}ds\right]f.$$

And since the “delay exponential” is the continuous function, finally, we will have

$$x_0(t)=\left[\int_{-\tau}^{0}Ids\right]f+\left[\int_{0}^{\tau}\left\{I+B\frac{s}{1}\right\}ds\right]f+\left[\int_{\tau}^{2\tau}\left\{I+B\frac{s}{1!}+B^2\frac{(s-\tau)^2}{2!}\right\}ds\right]f+\ldots+$$
$$+\left[\int_{(n-1)-\tau}^{t-\tau}\exp_\tau\{B,\xi\}ds\right]f.$$

Even more significant is the fact of belated reaction of one of the parties, that is, the presence of time-delay argument in “guerilla-warfare” and so-called “mix warfare” models. Consideration of the issues posed in this section goes beyond of the scope this article, and it is interesting for future scientific research.

CONCLUSION

The article examines various aspects of the combat operations which described by Lanchester type models. The first part (Ch.1.1) is devoted to classical problems in a continuous case. The next part (Ch.1.2) deals with more advanced options for presenting similar tasks. On the example of “direct fire” model, the problem is formulated in a discrete case. It is shown that in this case there exists both analytical and numerical coincidence of results. The following (Ch.1.3) are historical and student examples of solving such problems. In fact, the entire first chapter is of interest primarily as a possible part of text-book for students, with the aim of demonstrating various mathematical approaches to solve specific problems, and to teach their of this methodology. The second part of the work is of interest in sense of future practical (transfer of known results from one branch to another) and scientific research (qualitative research of warfare models or models of dynamic competition in the market) in the case of describing these processes in terms of functional differential equations with a deviation argument.

References

[1] David G. Luenberger. *Introduction to Dynamic Systems: Theory, Models, and Applications*. John Wilay & Sons Inc., 1979.

[2] Chiang, A.C. *Fundamental Methods of Mathematical Economics*, 3rd edition. London: McGrawHill, 1984.

[3] Weidlich, W. *Sociodynamics: A Systematic Approach to Mathematical Modeling in the Social Sciences*. The Harwood Academic Publishers. 2000.

[4] Michael Olinick. *Mathematical Modeling in the Social and Life Sciences*. – Wiley, 2014. – 588p. ISBN 978-1-118-64269-6

[5] M. Braun, *Differential Equations and Their Applications*, 2nd Ed. New York: Springer-Verlag, 1978.

[6] J. David Logan. *Applied Partial Differential Equations.* 3rd Edition. Springer, 2015.

[7] J. Hale, *Theory of Functional Differential Equations*. New York: Springer, 1977.

[8] Harold T. Davis. *Introduction to Nonlinear Differential and Integral Equations* (Dover Books on Mathematics). Paperback. 2010

[9] Kondratiuk, S., Kruchinin, K., Krak, I., Kruchinin, S. *Information technology for security system based on cross platform software*. NATO Science for Peace and Security Series A: Chemistry and Biology, Part F1, **(**2018) pp. 331-339.

[10] Lanchester F.W. *Aircraft in warfare: The dawn of the fourth arm*. - London: Lanchester Press. 1916.

[11] Lanchester, F.W. *Mathematics in warfare* // In J. Newman (Ed.), The world of mathematics. New York: Simon & Shuster. 1956, pp.214-246.

[12] Coleman, C.S. *Combat models*. In W. Lucas (Series Ed.) & M. Braun, C. S. Coleman, & D. Drew (Vol. Eds.), Differential equation models: Vol. 1. Models in applied mathematics. New York: Springer Verlag, 1983.

[13] Deitchman, S.J. *A Lanchester model of guerilla warfare* // Operations Research, 10, 1962. pp.818–827. doi:10.1287/opre.10.6.818

[14] Schaffer, M.B. *Lanchester models of guerilla engagement.* // ORSA Journal , 16(3), 1968. pp.457–488.

[15] Dupuy T.N. *Understanding War. History And Theory Of Combat*. 2nd Ed. - Nova Publishers, 1998. - 312p.

[16] Amel’kin V.V. *Differential equations in applications.* - Nauka. Moscow, 1987 (in Russian)

[17] William P.Fox Investigating the Solutions to Discrete Forms of Lanchester's Combat Models. In *Innovations in Information Systems for Business Functionality and Operations Management (Ed. Wang, J.)*. -Hershey, PA: IGI Global. -2012. doi:10.4018/978-1-4666-0933-4
[18] Dem'yanenko O., Shatyrko A. The information system for qualitative investigation of the combat models // II International scientific-practic Conference Information technology and interaction. Kyiv.-2015. pp.204-206. (in Ukrainian)
[19] Fox, W.P. *Battle of Trafalgar: An example of modeling with technology*. / In Proceedings of the Twentieth Annual International Conference on Technology in Collegiate Mathematics, San Antonio, TX (pp. 9397). Addison Wesley. 2008.
[20] Trautteur, G., & Virgilio, R. *An agent based computational model for the battle of Trafalgar: A comparison between analytical and simulative methods of research.* / In Proceedings of the Twelfth IEEE International Workshop on Enabling technologies: Infrastructure for Collaborative Enterprises (pp. 16). Washington, DC: IEEE Computer Society. 2003.
[21] Everson, M. *The Clash Of Civilizations. Proposed Clash Combat System*. 2007. URL: http://Home.Akademie.De/Dtoussaint/Clash/Clash_Combat.Htm
[22] Krak Yurii, Shatyrko Andriy *Control Theory for Informatics Students. Textbook.* TSNUSK. Kyiv. 2015. 175p. (in Ukraine).
[23] El'sgol'ts, L.E., Norkin S.B. *Introduction to the theory of the differential equations with deviating argument*. – Academic Press, New York, 1973.
[24] Khusainov D.Ya., Shatyrko A.V. *Lyapunov functions method in stability investigation of functional - differential systems*. – Kiev, Publ. by Kiev National University, 1997. – 236p. (in Russian).
[25] Diblík, J., Mencáková, K. *Formula for Explicit Solutions of a Class of Linear Discrete Equations with Delay* / Post-conference proceedings of extended versions of selected papers Conference MITAV-2016. University of Defence, Brno, CZ – 2016. pp.42-55. ISBN:978-80-7231-400-3
http://mitav.unob.cz/data/MITAV%202016%20Proceedings.pdf
[26] Šafařík, J., Diblík, J. *Weakly Delayed Difference Systems in R^3 and their Solution* / Post-conference proceedings of extended versions of selected papers Conference MITAV-2016. University of Defence, Brno, CZ - 2016 pp.84-104. ISBN: 978-80-7231-400-3 http://mitav.unob.cz/data/MITAV%202016%20Proceedings.pdf
[27] Jaromír Baštinec and Josef Diblík. *Two Classes of Positive Solutions of a Discrete Equation* / Post-conference proceedings of extended versions of selected papers Conference MITAV-2017. University of Defence, Brno, CZ - 2017 pp.21-32. ISBN 978-80-7582-026-6 http://mitav.unob.cz/data/MITAV%202017%20Proceedings.pdf
[28] Khusainov D.Ya., Diblik J., Ruzickova M. Linear dynamical systems with aftereffect. Solutions representation, stability, control, stabilization. – Kyiv, Inform.–Analit. Agency Publ., 2015. 252p. (in Russian).

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EXAMPLES OF HOMOTHETY CURVATURE HOMOGENEOUS SPACES

Alena Vanžurová
Institute of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, 602 00 Brno, Czech Republic. E-mail: vanzurova.a@fce.vutbr.cz; vanzurova.a@upol.cz

Abstract: *First we distinguish between curvature homogeneity and homothety curvature homogeneity. Curvature homogeneous manifolds are Riemannian spaces whose curvature tensor is, in some sense, "the same" in all points, while for homothety curvature homogeneous spaces, cuvatures (and their covariant derivatives) in two points are related in a more general way. Trivial examples of curvature homogeneous spaces are homogeneous spaces and connected locally homogeneous manifolds. First non-trivial examples were discovered by K. Sekigawa and for a long time, only a few classes of such examples which are not locally homogeneous have been known. We study here an interesting class of metrics, given in arbitrary dimension, which are not locally homogeneous, and which generalize a 3-dimensional example originally given by K. Sekigawa. We also examine examples of "Sekigawa type" from the view-point of homothety curvature homogeneity.*

Keywords: Riemannian manifold, curvature tensor, curvature homogeneous manifold, locally homogeneous space.

INTRODUCTION

The notion of curvature homogeneous space was introduced by by I. M. Singer [24] in 1960 in his study of (locally) homogeneous Riemannian manifolds. Singer proved that a Riemannian space is (locally) curvature homogeneous if and only if the Riemann curvature tensor and its covariant derivatives up to some order $k_M + 1$ are the same at each point. The number k_M, the so-called Singer number, depends on the manifold and is always smaller (or equal) than $n(n-1)/2$ (where $n = \dim M$). A better estimate was later given by Gromov, namely $k_M \leq 3n/2 - 1$.

Curvature homogeneous manifolds are Riemannian or pseudo-Riemannian manifolds ([18]) whose curvature tensor of type (0,4) is "the same" at all points in some sense (in the coordinate expression, have the same components). Homogeneous spaces and connected locally homogeneous manifolds belong to trivial examples of such spaces. First non-trivial examples were discovered by K. Sekigawa. For a long time, only sporadic classes of examples have been known of curvature homogeneous spaces which are not locally homogeneous. The first classification results for dimension 3 were published in [12], [2], [17], and classes of explicit examples were found in [26] and [13]. We study here an interesting class of cuvature homogeneous metrics, given in arbitrary dimension, which are not locally homogeneous, and which generalize an example posed originally by K. Sekigawa in dimension 3, [21], [22], [23]. We examine examples of Sekigawa type from the view-point of homothety curvature homogeneity.

1 CURVATURE HOMOGENEOUS AND LOCALLY HOMOGENEOUS SPACES

Let M denote a (smooth) n-dimensional manifold, TM its tangent bundle, T_pM the tangent space in $p \in M$, and $\mathcal{X}(M)$ the module of smooth vector fields of M. By (M, g) we mean a smooth Riemannian manifold together with a positive metric g; the corresponding Riemannian (Levi-Civita) connection of (M, g) is denoted by ∇; $\nabla g = 0$ holds, and the torsion tensor vanishes. We denote here by R the curvature tensor in type (1,3) of the manifold (M, ∇), that is, for any triple of vector fields $X, Y, Z \in \mathcal{X}(M)$, $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ holds. If $p \in M$ then R_p denotes the value of R at the point p.

The curvature tensor $\mathcal{R}$ in type (0,4) of the Riemannian space (M, g) is related to the tensor R by means of the metric tensor: for vector fields X, Y, Z, W from $\mathcal{X}(M)$, $\mathcal{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$; $\mathcal{R}_p$ denotes its value in $p \in M$.

A smooth Riemannian manifold (M, g) is called *curvature homogeneous* if it satisfies the condition

$\mathbf{P(1)}$: for any pair of points p and q in M, there exists a linear isometry $F\colon T_pM \to T_qM$ such that its pullback F^* satisfies $F^*(\mathcal{R}_q) = \mathcal{R}_p$.

That is, $F^*(\mathcal{R}_q)(X, Y, Z, W) = \mathcal{R}_q(FX, FY, FZ, FW)$ for vector fields X, Y, Z, W on M; [24], [3], [15], [16], [29], [30] and the references therein.

The study of curvature homogeneous spaces was initiated by I.M. Singer in 1960, and at the beginning, only trivial examples were known. It took some time till classes of examples were constructed of curvature homogeneous spaces which are *not* locally homogeneous. Recall that a Riemannian manifold (M, g) is a *locally homogeneous space* if the pseudogroup of local isometries of the manifold acts transitively on it. (M, g) is locally homogeneous if and only if there exists a symmetric linear connection $\hat{\nabla}$ satisfying $\hat{\nabla} g = 0$, $\hat{\nabla} T = 0$ and $\hat{\nabla} R = 0$. In other words, there is a metric connection with parallel torsion and curvature. Such a connection is called *Ambrose-Singer* connection (AS-connection) in [28]. Recall that the torsion tensor T of $(M, \hat{\nabla})$ is the tensor field of type (1,2) given by $T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y]$ for X, $Y \in \mathcal{X}(M)$. To each AS-connection it is attached an algebraic object, an *infinitesimal model*. Vice versa, the so-called Nomizu construction associates to each infinitesimal model, and therefore to each AS-connection, a particular Lie algebra g, its subalgebra h and a reductive decomposition $g = V \oplus h$; h is a subalgebra of g satisfying $[h, V] \subset V$. Let G be the simply connected Lie group with the Lie algebra g. Let H be the connected Lie subgroup of G with the Lie algebra h. If H is closed in G then the infinitesimal model is called *regular*. Due to the assumptions for the corresponding Lie groups G, H just mentioned, we can construct the homogeneous space G/H and show that the locally homogeneous space (M, g) is locally isometric to G/H endowed with a suitable G-invariant Riemannian metric; the converse also holds, [28].

If the Riemannian manifold is a *complete* locally homogeneous space then the universal Riemannian covering of the manifold is globally homogeneous. If this is the case then (M, g) is locally isometric to a Riemannian homogeneous space G/H endowed with a G-invariant metric (H is a closed Lie subgroup of the Lie group G), [28]. But this is no longer true if we *drop the completeness*: there exist noncomplete Riemannian manifolds that are not locally isometric to any Riemannian homogeneous space, [28].

I.M. Singer also introduced in [24] the following condition. A space (M, g) is said to be *curvature homogeneous up to order* r if it satisfies the property $\mathbf{P(r)}$ where

$\mathbf{P(r)}$: For every $p, q \in (M, g)$ there exists a linear isometry $F\colon T_pM \to T_qM$ such that $F^*((\nabla^k\mathcal{R})_q) = (\nabla^k\mathcal{R})_p$ for $k = 0, 1, \ldots, r$.

It is obvious that connected locally homogeneous spaces are curvature homogeneous of all orders. On the other hand, as we already mentioned, the main result of [24] says that there is always a finite number $k_M \leq n(n-1)/2$ such that, if (M, g) is curvature homogeneous up to order k_M, then it is automatically locally homogeneous. See also [19] and [3]. In contrast, we are interested here in spaces that are not locally homogeneous.

By a *model space* we mean a connected homogeneous space $(\bar{M}, \bar{g})$, that is, a Riemannian manifold with transitive isometry group. We say that a Riemannian manifold (M, g) has the same curvature tensor as the model space $(\bar{M}, \bar{g})$ if for each point $p \in M$, there is a linear isometry $F\colon T_pM \to T_o\bar{M}$ such that $F^*(\bar{R}_o) = R_p$ where o is a fixed point of $\bar{M}$.

Note that the above concepts can be formulated even in the case of affine manifold (M, ∇). A symmetric (=torsion-free) affine connection ∇ on a (connected) manifold M is *locally homogeneous* if and only if for every pair $x, y \in M$ there are neighborhoods U of x and V of y and there exists an affine transformation $f : (U, \nabla_{|U}) \to (V, \nabla_{|U})$ sending x to y, $f(x) = y$. We say that ∇ is *locally symmetric* if and only if $\nabla R = 0$. A locally symmetric connection is locally homogeneous, a locally homogeneous connection is curvature homogeneous of any order, [20].

2 CURVATURE HOMOGENEITY IN DIMENSION 3

In what follows, metrics and functions are supposed to be real analytic.

Let $R_{ijk\ell}$ be the components of $\mathcal{R}$ with respect to any local moving frame, let ϱ_{jk} denote components of the Ricci tensor, and let τ be the scalar curvature (which arises by contraction of the Ricci tensor).

Let $C_{ijk\ell}$ denote components of the Weyl tensor of conformal curvature which is defined by (cf. [7], [25], [4], [27] and the heading "Weyl tensor" in WIKIPEDIA; note that the formula in different sources might be written with different signs depending on the sign convention used for curvature):

$$\begin{aligned} C_{ijk\ell} = R_{ijk\ell} &+ \frac{1}{n-2}(g_{ik}R_{j\ell} - g_{i\ell}R_{jk} + g_{j\ell}R_{ik} - g_{jk}R_{i\ell}) \\ &+ \frac{1}{(n-1)(n-2)}(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}). \end{aligned} \tag{1}$$

As well known, in dimension 3, the Weyl tensor vanishes, $C \equiv 0$, [25]. Therefore components of $\mathcal{R}$ can be expressed from the above formula as

$$\begin{aligned} R_{ijk\ell} = &\frac{1}{n-2}(g_{ik}\varrho_{j\ell} - g_{i\ell}\varrho_{jk} + g_{j\ell}\varrho_{ik} - g_{jk}\varrho_{i\ell}) \\ &+ \frac{\tau}{(n-1)(n-2)}(g_{i\ell}g_{jk} - g_{ik}g_{j\ell}). \end{aligned} \tag{2}$$

The curvature tensor R of a 3-dimensional Riemannian manifold is uniquely determined by the corresponding Ricci tensor ϱ and the metric g, as we can see from the above formula (2). As a consequence, we get

Proposition 1 *A Riemannian manifold* (M, g) *is curvature homogeneous if and only if the Ricci eigenvalues* ϱ_1*,* ϱ_2*,* ϱ_3 *are* constant *at all points.*

The following results can be proved using Cauchy-Kowalewski theorem:

Theorem 1 *([12]) All real analytic Riemannian manifolds with the prescribed constant Ricci eigenvalues* $\varrho_1 = \varrho_2 \neq \varrho_3$ *depend, up to a local isometry, on two arbitrary (real analytic) functions of one variable.*

Theorem 2 *([17]) All real analytic Riemannian manifolds with the prescribed distinct constant Ricci eigenvalues* $\varrho_1 > \varrho_2 > \varrho_3$ *depend, up to a local isometry, on three arbitrary (real analytic) functions of two variables.*

On an open subset of $\mathbb{R}^3$, the prescribed triplets of constant Ricci eigenvalues can be realized only on spaces which are not locally homogeneous. Explicit examples of this kind can be found in [12] and [13]. The classification of all triplets of distinct real numbers which can be realized as Ricci eigenvalues on a 3-dimensional locally homogeneous space was made in [14]. We conclude that the Riemannian spaces (M, g) with prescribed constant Ricci eigenvalues are "usually" not locally homogeneous, with some rare exceptions. In [13], the authors constructed so-called generalized Yamato examples which are explicit for each choice of the triplet $\varrho_1 > \varrho_2 > \varrho_3$ of prescribed Ricci eigenvalues. See [26] for the original construction by K. Yamato where some restrictions are put on the triplets $\varrho_1 > \varrho_2 > \varrho_3$, and all constructed metrics are complete.

The following is known [23] (see [9] for a shorter and more direct as well as clear proof):

Proposition 2 *Each 3-dimensional Riemannian manifold* (M, g) *satisfying both the conditions* $P(0)$ *and* $P(1)$ *is locally homogeneous.*

Recall that an analogous result was found even in dimension four (see [127], [128] from [3]):

Proposition 3 *Each 4-dimensional Riemannian manifold* (M, g) *which is curvature homogeneous up to order one is locally homogeneous.*

A Riemannian manifold satisfying the conditions $P(0)$, $P(1)$ and not locally homogeneous is not known yet. Notice that in the pseudo-Riemannian case, the situation is quite different [5].

3 HOMOTHETY CURVATURE HOMOGENEITY

Here we work in arbitrary dimension again. We say that a Riemannian manifold (M, g) is *homothety curvature homogeneous* if for any pair of points p, q there is a curvature-preserving linear homothety $f\colon T_pM \to T_qM$, i.e. such that $f^*(R_q) = R_p$ where R_p and R_q denote the (1,3)-curvature tensors in p and q, respectively, [15], [16], [29].

Recall that under a *linear homothety* $f\colon V \to W$ of vector spaces with coefficient $\lambda > 0$ we mean a composition of linear isometry $F\colon V \to W$ and a homothety $H\colon W \to W$ with the coefficient λ. Note that $(f^*(R_q))(X, Y, Z) = f^{-1}(R_q(fX, fY, fZ))$, and $f^*(R_q) = R_p$ is equivalent with $f(R_p(X, Y)Z) = R_q(fX, fY)fZ$ for all $X, Y, Z \in T_p(M)$. The following implication can be proved [15]:

Proposition 4 *If (M, g) is curvature homogeneous then it is also homothety curvature homogeneous.*

Let us sketch the proof in components. Let (M, g) be curvature homogeneous. Let $p, q \in M$ be a pair of points, and let $F\colon T_pM \to T_qM$ be a $(0,4)$-curvature preserving linear isometry of tangent spaces, that is, $F^*(g_q) = g_p$ and $F^*(\mathcal{R}_q) = \mathcal{R}_p$ hold. Now take a fixed orthonormal basis $\langle e_1, \dots, e_n\rangle$ in T_pM and choose in T_qM the corresponding orthonormal frame $\langle Fe_1, \dots, Fe_n\rangle$. Introduce the components of metric and curvature in p by $g_p(e_i, e_j) = g_{ij} = \delta_i^j$, and curvature components in p, q by $R_p(e_k, e_\ell)e_j = \sum_m R^m_{jk\ell}(p)e_m$, $R_{ijk\ell}(p) = \mathcal{R}_p(e_i, e_j, e_k, e_\ell) = g_p(R_p(e_k, e_\ell)e_j, e_i)$, $R_q(Fe_k, Fe_\ell)Fe_j = \sum_m R^m_{jk\ell}(q)Fe_m$, and similarly for $R_{ijk\ell}(q)$.

We get

$$R^i_{jk\ell}(p) = \delta^i_m R^m_{jk\ell}(p) = \textstyle\sum_m g_{im}R^m_{jk\ell}(p) = g_p(R^m_{jk\ell}(p)e_m, e_i) = g_p(R_p(e_k, e_\ell)e_j, e_i) = R_{ijk\ell}(p)$$

$$= \mathcal{R}_p(e_i, e_j, e_k, e_\ell) = \mathcal{R}_q(Fe_i, Fe_j, Fe_k, Fe_\ell) = R_{ijk\ell}(q) = g_q(R_q(Fe_k, Fe_\ell)Fe_j, Fe_i)$$

$$= g_q(R^m_{jk\ell}(q)Fe_m, Fe_i) = R^m_{jk\ell}(q)g_q(Fe_m, Fe_i) = R^m_{jk\ell}(q)g_p(e_m, e_i) = R^m_{jk\ell}(q)\delta^i_m = R^i_{jk\ell}(q).$$

Therefore the curvature tensor of type $(1,3)$ is preserved under F. Hence (M, g) is homothety curvature homogeneous.

Let us call "generic" those Riemannian manifolds for which the Ricci eigenvalues are distinct at all points. The following was proved [29], [15]:

Theorem 3 *In dimension 3, all generic real analytic homothety curvature homogeneous manifolds depend, up to a local isometry, by 1 arbitrary real analytic function of 3 variables and 3 arbitrary real analytic functions of 2 variables.*

Proposition 5 *The converse of Proposition 4 does not hold.*

Indeed, the concept of homothety curvature homogeneity is more general, which can be seen "theoretically" already in dimension three if we compare Theorem 3 with Theorem 1 and Theorem 2. Moreover, the class of spaces of Sekigawa type described in the next section gives counterexamples in any dimension.

For practical purposes, homothety curvature homogeneity can be equivalently characterized as follows, [15]:

Proposition 6 *Let (M, g) be a smooth Riemannian manifold and let $\mathcal{R}$ or R denote its curvature tensor field of type (0,4), or of type (1,3), respectively. Then the following two conditions are equivalent:*

(a) For each $q \in M$, there is a linear homothety $f_q\colon T_pM \to T_qM$ ($p \in M$) such that $R_p = f_q^*(R_q)$, that is, (M, g) is homothety curvature homogeneous.

(b) There is a smooth function φ on M such that $\varphi(p) = 0$ and for each $q \in M$, $\mathcal{R}_p = e^{2\varphi(q)}F_q^*(\mathcal{R}_q)$ where $F_q\colon T_pM \to T_qM$ is a linear isometry.

We can extend the definition to higher order. For each integer $k \geq 0$ (separately), let us introduce the following condition:

$\mathbf{Q(r)}$: For every $p, q \in (M, g)$ there exists a linear homothety $h \colon T_pM \to T_qM$ such that $h^*((\nabla^k R)_q) = (\nabla^k R)_p$.

A space (M, g) with the property that all the conditions $Q(0), \ldots, Q(r)$ are satisfied will be called *homothety curvature homogeneous up to order* r.

Note that the linear homotheties above might be completely independent for different k.

The condition $\mathbf{Q(r)}$ can be characterized in the following useful equivalent way (it is a generalization of Proposition 6 to higher orders; for the proof, see [15]):

Proposition 7 *The following conditions are equivalent for a smooth Riemannian manifold* (M, g)*:*

(a) (M, g) *satisfies the condition* $Q(k)$*, i.e., for every* $p, q \in (M, g)$ *there exists a linear homothety* $h \colon T_pM \to T_qM$ *such that* $h^*((\nabla^k R)_q) = (\nabla^k R)_p$.

(b) There is a smooth function φ *on* M *such that* $\varphi(p) = 0$ *for a fixed* $p \in M$ *and* $(\nabla^k \mathcal{R})_p = e^{(k+2)\varphi(q)} F^*((\nabla^k \mathcal{R})_q)$ *for each* $q \in M$*, where* $F \colon T_pM \to T_qM$ *is a linear isometry.*

Our aim is to compare the classical curvature homogeneity and the new concept of homothety curvature homogeneity. We verify by means of examples that the class of homothety curvature homogeneous (of order r) spaces is much wider than the class of curvature homogeneous (of order r) spaces. It is quite natural to start our exposition with the dimension $n = 3$.

4 METRICS WHICH ARE NOT LOCALLY HOMOGENEOUS

As a contrast to the above results concerning the conditions $P(0)$ and $P(1)$, Proposition 2, Proposition 3, we shall show that a 3-dimensional Riemannian manifold $(\mathbb{R}^3, \hat{g})$ of Sekigawa type given below, [21], [22], satisfies the conditions $Q(0)$ and $Q(1)$, therefore it is homothety curvature homogeneous up to order 1, but, yet, $(\mathbb{R}^3, \hat{g})$ is not locally homogeneous.

4.1 Original version of the example of K. Sekigawa.

In 1975, K. Sekigawa introduced [22], [21] Riemannian metrics (depending on two real parameters c_1, c_2 and a prescribed curvature $S < 0$) on $\mathbb{R}^3[u, u, w]$. His motivation was to find spaces that are not locally homogeneous. He used the idea of Bishop and O'Neil who constructed a wide class of Riemannian manifolds of negative curvature by warped product $B \times_f F$ using C^∞-function f on B, [1]. Namely, Sekigawa considered $\mathbb{R}^2 \times_f \mathbb{R}^1$ where $f = c_1 e^{t\sqrt{-S/2}} + c_2 e^{-t\sqrt{-S/2}}$, c_1, c_2, S are certain real numbers, c_1, c_2 are positive, $S < 0$, and $t = u \cos w - v \sin w$ where v, w are coordinates on $\mathbb{R}^2$. The Riemannian manifold is complete, irreducible, with the prescribed negative constant scalar curvature S. It was checked later that the constant principal Ricci eigenvalues of the metric are $\varrho_1 = \varrho_2 = -1$, $\varrho_3 = 0$, and the corresponding model space is $\mathbb{H}^2(-1) \times \mathbb{R}$, [11].

4.2 Properties of 3-dimensional Sekigawa-like metrics.

Let us show how the original construction was modified and extended step by step, and finally used for other purposes.

First let us describe here a version of Sekigawa's construction in a new notation, more suitable for further generalization. Let $f(x) = ae^x + be^{-x}$ be a function of one variable where a, b are positive constants. If we consider a metric $\hat{g}$ on $\mathbb{R}^3[w, x, y]$ given by the formula $\hat{g} = \hat{g}(a, b) = \sum_{i=0}^{2}(\omega^i)^2$ with respect to the orthonormal coframe consisting of one-forms $\omega^0 = f(x)dw$, $\omega^1 = dx - ydw$, $\omega^2 = dy + xdw$ then it can be checked that the 3-dimensional space $(\mathbb{R}^3, \hat{g})$ is irreducible, simply connected, complete, semi-symmetric, that is, it satisfies the algebraic identity $R(X, Y) \cdot R = 0$. It also satisfies the condition $P(0)$. Yet, it is not locally homogeneous, therefore it cannot satisfy $P(1)$.

To examine further properties of such metrics let us examine covariant derivatives of the curvature tensor with respect to the orthonormal moving frame $\langle E_0, E_1, E_2\rangle$ dual to the co-frame $\langle\omega^0, \omega^1, \omega^2\rangle$ introduced above. We have

$$E_0 = f^{-1}\left(\frac{\partial}{\partial w} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

A standard evaluation shows that the Riemannian (0,4)-curvature tensor is given by the formula

$$\mathcal{R} = 4f^{-1}f''\,\omega^0 \wedge \omega^1 \otimes \omega^0 \wedge \omega^1,$$

and the second order differential equation

$$f^{-1}f'' = 1 \tag{3}$$

is satisfied. Hence the only nonvanishing components of the curvature tensor are

$$R(E_0, E_1)E_1 = -E_0, \quad R(E_0, E_1)E_0 = E_1$$

and those enforced by the skew-symmetry of the curvature tensor.

As far as the sign convention is concerned, we follow the traditional convention as in [8] or [6] (the signs in some formulas in [11] are opposite to ours). It can be checked that

$$(\nabla_{E_0}\mathcal{R})(E_0, E_1, E_2, E_0) = -f^{-1}$$

and all other components of the first covariant derivative $\nabla\mathcal{R}$, up to natural permutations of the four inner arguments, vanish. Let us make a choice $a + b = 1$, and define a function $\varphi(q)$ on $\mathbb{R}^3$ by $e^{3\varphi(q)} = f(x) = ae^x + be^{-x}$ where $x = x(q)$. Then the condition (b) of Proposition 7 for the origin $p = [0, 0, 0]$ is satisfied in the case $k = 1$. Therefore the condition $Q(1)$ is satisfied, as well as $P(0)$, and hence $Q(0)$ holds. Let us also examine the condition $Q(2)$. We calculate the following components of the second covariant derivative:

$$(\nabla^2_{E_0E_0}\mathcal{R})(E_0, E_1, E_1, E_0) = 2f^{-2}$$

and

$$(\nabla^2_{E_0E_0}\mathcal{R})(E_0, E_1, E_2, E_0) = yf^{-3}f'.$$

Since the last two components are never equal, there does not exist a function $\varphi(q)$ satisfying the condition (b) of Proposition 7 for $k = 2$. As a conclusion we get that the condition $Q(2)$ is not fulfilled.

So as a contrast to the above results on curvature homogeneity, we proved that there exists a 3-dimensional Riemannian manifold $(M, \hat{g})$ of Sekigawa type satisfying the conditions $Q(0)$, $Q(1)$ but not $Q(2)$. Another speaking, it means that $(M, \hat{g})$ is homothety curvature homogeneous up to order 1, although it is not curvature homogeneous up to order one, but $(M, \hat{g})$ is not homothety curvature homogeneous up to order 2.

4.3 Further generalization in dim 3.

The above three-dimensional example by Sekigawa was extended by F. Tricerri, L. Vanhecke and O. Kowalski in [10] in dimension 3 for more general functions f, and further generalization to higher (arbitrary) dimension is also possible as we show below. Our main aim is to examine the arising new class of examples from the view-point of homothety curvature homogeneity.

Consider the real 3-dimensional space $R^3[w, x, y]$ with coordinates denoted by w, x, y and let $f(w, x)$ be a smooth function of two variables. In a domain $U \subset \mathbb{R}^3[w, x, y]$ introduce the Sekigawa-type Riemannian metric $g_f = \sum_{i=0}^{2}(\omega^i)^2$ by means of exterior differential forms

$$\omega^0 = f(w, x)dw, \quad \omega^1 = dx - ydw, \quad \omega^2 = dy + xdw$$

which constitue an orthonormal coframe as it can be checked. If $a(w)$, $b(w)$ are arbitrary functions of one variable and k is a real constant then the following can be verified:

(a) If $f(w, x) = a(w)e^{kx} + b(w)e^{-kx}$, then the corresponding principal Ricci curvatures are $\varrho_1 = \varrho_2 = -k^2$, $\varrho_3 = 0$.

(b) If $f(w, x) = a(w)\cos kx + b(w)\sin kx$, then the corresponding principal Ricci curvatures are $\varrho_1 = \varrho_2 = k^2$, $\varrho_3 = 0$.

Such metrics are always locally irreducible and are not locally homogeneous. We bring here the following generalization.

4.4 Extended class of Sekigawa type examples in any dimension.

In arbitrary dimension $m = n + 1$, we are able to present examples that we call here "examples of Sekigawa type", distinguish a subclass of curvature homogeneous spaces, and construct homothety curvature homogeneous spaces of Sekigawa type which are not curvature homogeneous.

Consider $\mathbb{R}^{n+1}$ with standard coordinates $(w, x^1, \ldots, x^n)$. Take an open subset U of $\mathbb{R}^2[w, x^1]$, a non-vanishing smooth function $f\colon U \to \mathbb{R}$ on U, and a skew-symmetric smooth $(n \times n)$-matrix function $A(w) = (A^i_j(w))$ in one variable. On U introduce the metric $g_{f,A(w)} = \sum_{j=0}^{n+1} \omega^j \otimes \omega^j$ by an orthogonal coframe

$$\omega^0 = f(w, x^1)dw, \qquad \omega^i = dx^i + \sum_{j=1}^{n} A^i_j(w)x^j dw, \quad i = 1, \ldots, n;$$

denote by $\langle E_0, E_1, \ldots, E_n\rangle$ be the corresponding orthonormal basis of vector fields.

An evaluation shows that the Riemannian (0,4)-curvature tensor is given by the formula ([3])

$$\mathcal{R} = 4f^{-1}f''_{x^1x^1}\, \omega^0 \wedge \omega^1 \otimes \omega^0 \wedge \omega^1. \tag{4}$$

We obtain that the non-zero components are just $R_{0101} = R_{1010} = -R_{1001} = -R_{0110} = -f^{-1}f''_{x^1x^1}$, and all other components R_{ijkl} vanish.

Now, the metric $g_{f,A(w)}$ introduced above is curvature homogeneous if and only if there exists a constant k such that $f^{-1}f''_{x^1x^1} = k$ which can be written equivalently as the second order differential equation

$$f''_{x^1x^1} = kf. \tag{5}$$

Any solution f of (5) takes one of the following forms:

$$\begin{aligned} f(w,x^1) &= a(w)\exp(\sqrt{k}x^1) + b(w)\exp(-\sqrt{k}x^1) \text{ if } k > 0, \\ f(w,x^1) &= a(w)\cos(\sqrt{-k}x^1) + b(w)\sin(\sqrt{-k}x^1) \text{ if } k < 0, \\ f(w,x^1) &= a(w)x^1 + b(w) \text{ if } k = 0 \end{aligned} \tag{6}$$

where $a(w)$ and $b(w)$ are differentiable functions such that $f(w,x^1) > 0$ in U. Moreover, $g_{f,A(w)}$ is non-flat if and only if $k \neq 0$, so that the last case can be omitted in what follows. Remark that U can be the whole plane in the case $k > 0$ and an open strip in the plane for $k < 0$. Recall that this class of spaces is remarkable because it includes all irreducible curvature homogeneous spaces which are not locally homogeneous and whose curvature tensor $\mathcal{R}$ "is the same" as that of a Riemannian symmetric space (so-called "non-homogeneous relatives of symmetric spaces", see [2]).

We shall need the following (see [29] for the proof)

Proposition 8 *Let (M,g) be a Riemannian manifold and let $\langle E_1,\dots,E_n\rangle$ be an orthonormal moving frame on a domain $U \subset M$. Fix a point $p \in U$. Suppose that, with respect to this moving frame, $R_{ijk\ell}(q) = \phi(q)R_{ijk\ell}(p)$ for each point $q \in U$ and for all choices of indices, where $\phi(q)$ is a smooth and positive function on U. Then there is a smooth function $\varphi(q)$ such that $\varphi(p) = 0$ and, for each point q, $\mathcal{R}_p = e^{2\varphi(q)}F_q^*(\mathcal{R}_q)$ where $F_q\colon T_pM \to T_qM$ is a linear isometry.*

Proposition 9 *There exist Sekigawa type homothety curvature homogeneous spaces which are not curvature homogeneous.*

To give the proof, we wish to construct "proper" homothety curvature homogeneous spaces. So let f be now an arbitrary smooth function on $\mathbb{R}^2$ such that f and $f^{-1}f''_{x^1x^1}$ are nonzero at all points and such that $f''_{x^1x^1}/f$ is never a constant in an open domain of $\mathbb{R}^2$. Then the corresponding metric $\overline{g} = g_{f,A(w)}$ defined in (a neighborhood of) $\mathbb{R}^{n+1}$ has the curvature components as in the formula (4). As we can see these curvature components $\overline{R}_{ijk\ell}$ satisfy

$$\overline{R}_{ijk\ell}(q) = (f^{-1}(q)f''_{x^1x^1}(q))/(f^{-1}(p)f''_{x^1x^1}(p))\overline{R}_{ijk\ell}(p)$$

for any pair of points $p,q \in \mathbb{R}^{n+1}$ and all indices i,j,k,ℓ. Fix a point p. Then the assumptions of the Proposition 8 are satisfied with the (positive) function $\phi(q)$ defined by

$$\phi(q) = f^{-1}(q)f''_{x^1x^1}(q)/(f^{-1}(p)f''_{x^1x^1}(p)).$$

By Propositions 8, 6 and our special additional assumptions on f we get: $(\mathbb{R}^{n+1},\overline{g})$ is a homothety curvature homogeneous space, but it is not curvature homogeneous (due to the above assumptions on f).

Remark 1 *Particularly if we take $n = 2$, $A = (a_{ij})$ with $a_{11} = a_{22} = 0$, $a_{12} = -a_{21} = -1$, $f(w,x^1) = a\exp(\lambda x^1) + b\exp(-\lambda x^1)$, where a,b are non-negative integers, we can see that the above metric is in fact a generalization of Sekigawa type example from 4.2.*

CONCLUSION

First we distinguish between curvature homogeneity and homothety curvature homogeneity, give a coordinate proof of Proposition 4, show why the converse does not hold, and notice that in low dimensions 3 and 4, curvature homogeneous spaces up to order one are locally homogeneous (Propositions 2, 3). Although some of the results were announced already in [29] we bring new facts here. In part 4 we start with one example of K. Sekigawa from 1975, of a Riemannian manifold given as a warped product $\mathbb{R}^2 \times_f \mathbb{R}^1$ described in 4.1., which served originally a counterexample to local homogeneity but was an inspiration for further investigations. We give here evaluations concerning Example 1 from [29]. We calculate covariant derivatives of the curvature tensor up to order two in this 3-dimensional case and conclude: there exists a 3-dimensional Riemannian manifold which is homothety curvature homogeneous up to order 1 (although it is not curvature homogeneous up to order one), but it is not homothety curvature homogeneous up to order 2, and is not locally homogeneous.

We show how the original construction can be modified and extended step by step. Finally we use it for other purposes. We generalize this 3-dimensional example to a huge class of spaces that we can construct in arbitrary dimension. Riemannian metrics of our examples, given in the space $\mathbb{R}^{n+1}$ with standard coordinates $(w, x^1, \dots, x^n)$, are of the form $g_{f,A(w)}$ from 4.3, depend on one smooth ("generating") function f of two variables and a finite number of functions of one variable (components of the functional matrix $A(w)$). We show that the original example of Sekigawa is a particular case of the more general construction (cf. Remark), endeed. Our main aim is to examine the arising class of examples (which was already mentioned in [29]) from the new view-point, namely under which conditions they are curvature homogeneous, or when they are "proper" homothety curvature homogeneous spaces. First we distinguish the subclass of curvature homogeneous spaces: this geometric property is formulated as a second order differential equation (5) for the generating function f, and all possible solutions of the equation are listed in (6).

Now if we take a non-vanishing function f for which the crutial differential equation (5) fails we are sure that the corresponding space is not curvature homogeneous. Due to calculations of the curvature we can see that the assumptions of Proposition 8 are satisfied, and due to Proposition 6 we are able to confirm that the space is homothety curvature homogeneous, Proposition 9 holds.

References

[1] Bishop, R.L., and O'Neil, B. Manifolds of negative curvature. *Trans. Amer. Math. Soc.*, 145, 1969, p. 1-49.

[2] Boeckx, E., Kowalski, O., Vanhecke, L. Non-homogeneous relatives of symmetric spaces. *Diff. Geom. and Appl.* 4, 1994, p. 45-69.

[3] Boeckx, E., Kowalski, O., Vanhecke, L. *Riemannian Manifolds of Conullity two.* World Scientific, 1996.

[4] Eisenhart, L.P. *Riemannian Geometry.* Princeton, 1949.

[5] Gilkey, P.J. The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds. In: *ICP Advanced Texts in Mathematics*, Vol. 2. Imperial College Press, 2007.

[6] Helgason, S. *Differential geometry and symmetric spaces.* Academic Press, 1962.

[7] Favard, J. *Cours de Géométrie Différentielle Locale.* Gauthier-Villars, Paris 1957.

[8] Kobayashi, S., Nomizu, K. *Foundations of Differential Geometry I.* Wiley-Intersc. Publ., New York, Chichester, Brisbane, Toronto, Singapore, 1991.

[9] Kowalski, O. A note to a theorem by K. Sekigawa, *Comment. Math. Univ. Carolinae*, 30, 1989, p. 85-88.

[10] Kowalski, O., Tricerri, F., Vanhecke, L. New examples of non-homogeneous Riemannian manifolds whose cubature tensor is that of a Riemannian symmetric space. *C. R. Acad. Sci. Paris*, 311, Série I, 1990, p. 355-360.

[11] Kowalski, O., Tricerri, F., Vanhecke, L. Curvature homogeneous Riemannian manifolds. *J. Math. Pures Appl.*, 71, 1992, p. 471-501.

[12] Kowalski, O. A classification of Riemannian 3-manifolds with constant principal Ricci curvatures $\varrho_1 = \varrho_2 \neq \varrho_3$. *Nagoya Math. J.*, 132, 1993, p. 1-36.

[13] Kowalski, O., Prüfer, F. On Riemannian 3-manifolds with distinct constant Ricci eigevalues. *Math. Ann.*, 300, 1994, p. 17-28.

[14] Kowalski, O., Ž. Nikčević, S. On Ricci eigenvalues of locally homogeneous Riemannan manifolds. *Geometriae Dedicata*, 62, 1996, p. 65-72.

[15] Kowalski O., Vanžurová, A. On curvature-homogeneous spaces of type (1,3). *Math. Nachr.* 284, No. 17-18, 2011, p. 2127-2132.

[16] Kowalski O., Vanžurová, A.: *On a Generalization of Curvature Homogeneous Spaces.* Results in Mathematics, 63, No. 1, 2013, p. 129-134.

[17] O. Kowalski, Z. Vlášek. Classification of Riemannian 3-manifolds with distinct constant principal Ricci curvatures. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 5, 1998, p. 59-68.

[18] Mikeš, J., Stepanova, E., Vanžurová, A. *Differential Geometry of Special Mappings.* Palacký University Olomouc, Fac. Sci, Olomouc 2015. ISBN 978-80-244-4671-4.

[19] Nicolodi, L. and Tricerri, F. On two theorems of I.M. Singer about homogeneous spaces. *Ann. Global Anal. Geom.*, 8, 1990, p. 193-209.

[20] Opozda, B. On curvarure homogeneous and locally homogeneous affine connections. *Proceedings of the AMS*, 134, No. 6, 1966, p. 1889-1893.

[21] Sekigawa, K. On some 3-dimensional Riemannian manifolds. *Hokkaido Math. J.*, 2, 1973, p. 259-270.

[22] Sekigawa, K. On the Riemannian manifolds of the form $B \times_f F$. *Kodai Math. Sem. Rep.*, 26, 1975, p. 343-347.

[23] Sekigawa, K. On some 3-dimensional curvature homogeneous spaces. *Tensor*, 31, 1977, p. 87-97.

[24] Singer, I.M. Infinitesimally homogeneous spaces. *Comm. Pure Appl. Math.*, 13, 1960, p. 685-697.

[25] Weyl, H. Reine Infinitesimalgeometrie. *Math. Zeitschrift*, 2, 1918, p. 384-411.

[26] Yamato, K. A characterization of locally homogeneous Riemannian manifolds of dimension 3. *Nagoya Math. J.*, 123, 1993, p. 77-99.

[27] K. Yano. *Integral Formulas in Riemannian Geometry.* Marcel Dekker, Inc., 1970.

[28] Tricerri, F. Locally homogeneous Riemannian manifolds. *Rend. Sem. Mat. Univ. Pol. Torino*, 50, No. 4, 1992, Differential Geometry, p. 411-426.

[29] Vanžurová, A. Homothety Curvature Homogeneity. In: *Mathematics, Information Technologies and Applied Sciences 2017, post-conference proceedings of extended versions of selected papers.* Brno: University of Defence, 2017, p. 248-255. [Online]. Available at: <http://mitav.unob.cz/data/MITAV2017Proceedings.pdf>. ISBN: 978-80-7582-026-6.

[30] Vanžurová, A. Generalization of Curvature Homogeneous Spaces. In: *Matematika, informační technologie a aplikované vědy.* Brno: Univerzita obrany, 2017, p. 1-4. ISBN: 978-80-7231-417-1.

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