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Aims and target group of the conference:

The conference **MITAV 2023** is the tenth annual MITAV conference. It should attract in particular teachers of all types of schools and is devoted to the most recent discoveries in mathematics, informatics, and other sciences as well as to the teaching of these branches at all kinds of schools for any age groups, including e-learning and other applications of information technologies in education. The organizers wish to pay attention especially to the education in the areas that are indispensable and highly demanded in contemporary society. The goal of the conference is to create space for the presentation of results achieved in various branches of science and at the same time provide the possibility for meeting and mutual discussions of teachers from different kinds of schools and orientation. We also welcome presentations by (diploma and doctoral) students and teachers who are just beginning their careers, as their novel views and approaches are often interesting and stimulating for other participants.

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Faculty of Education and Faculty of Economics and Administration, Masaryk University in Brno,
Faculty of Electrical Engineering and Communication, Brno University of Technology.

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Each MITAV 2023 participant received printed collection of abstracts **MITAV 2023** with ISBN 978-80-7582-245-1. CD supplement of this printed volume contains all the accepted contributions of the conference.

Now, in autumn 2023, this **post-conference proceedings** were published, containing extended versions of selected MITAV 2023 contributions. The proceedings are published in English and contain extended versions of 9 selected conference papers. Published articles have been chosen from 12 conference papers and every article was once more reviewed.

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ON THE REGULAR MATRIX METHOD OF SUMMABILITY AND $\mathcal{I}_c^g$-CONVERGENCE

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Abstract: *Recently we introduced $\mathcal{I}_c^g$ ideals generated by a special real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$. We investigated $\mathcal{I}_c^g$ convergence and $M_{p,g}$-summable method and studied their properties. In this paper we will study further properties of $\mathcal{I}_c^g$-convergence, $M_{p,g}$-summability and Riesz matrix summable method. For bounded sequences we show a connection between $\mathcal{I}_c^g$-convergence and regular matrix method of summability.*

Keywords: sequences of real numbers, convergence, ideal, summability.

INTRODUCTION

We recall the basic definitions and notations that will be used throughout the paper. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^+$ be the set of all positive real numbers. A system $\mathcal{I}$, $\emptyset \neq \mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal, provided that $\mathcal{I}$ is additive ($A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$) and hereditary ($A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$). The ideal is called nontrivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. If $\mathcal{I}$ is a nontrivial ideal, then $\mathcal{I}$ is called admissible if it contains the singletons ($\{n\} \in \mathcal{I}$ for every $n \in \mathbb{N}$). The fundamental notation which we shall use is $\mathcal{I}$-convergence introduced in the paper [11] (see also [5] where $\mathcal{I}$-convergence is defined by means of the dual notion of an ideal so-called filter). The notion $\mathcal{I}$-convergence corresponds to the natural generalization of the notion of statistical convergence (see [2], [4], [7], [8] and [13]).

Definition 1. *Let $x = (x_n)$ be a sequence of real (complex) numbers. We say that the sequence $\mathcal{I}$-converges to a number L, and write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.*

In the following, we suppose that $\mathcal{I}$ is an admissible ideal. Then for every sequence (x_n) we immediately have that $\lim_{n\to\infty} x_n = L$ (classic limit) implies that (x_n) also $\mathcal{I}$-converges to the same number L but the opposite is not true. In other words, for an admissible ideal $\mathcal{I}$ we have $\mathcal{I}_{fin} \subseteq \mathcal{I}$, where $\mathcal{I}_{fin}$ is the ideal of all finite subsets of $\mathbb{N}$ and $\mathcal{I}_{fin}$ convergence coincides with the usual convergence. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ then the statement $\mathcal{I}_1 - \lim x_n = L$ implies $\mathcal{I}_2 - \lim x_n = L$ (see [11]).

Let $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the asymptotic density of $A \subseteq \mathbb{N}$. The numbers $\underline{d}(A) = \liminf_{n\to\infty} \frac{\#\{a \leq n \,:\, a \in A\}}{n}$ and $\overline{d}(A) = \limsup_{n\to\infty} \frac{\#\{a \leq n \,:\, a \in A\}}{n}$ are called the lower and upper asymptotic density of the set A, respectively ($\#M$ denotes the cardinality of the set M). If $\underline{d}(A) = \overline{d}(A) = d(A)$ then $d(A)$ is said to be the asymptotic density of A. The usual $\mathcal{I}_d$-convergence is called statistical convergence (see [3], [4], [7] and [13]). For $0 < q \leq 1$ the ideal $\mathcal{I}_c^{(q)} = \left\{A \subset \mathbb{N} : \sum_{a\in A} a^{-q} < \infty\right\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \left\{A \subset \mathbb{N} : \sum_{a\in A} \frac{1}{a} < \infty\right\}$ is usually denote by $\mathcal{I}_c$ (see [4] and [10]). The class of all $\mathcal{I}$-convergent sequences is a linear space (see [11]).

1 DEFINITIONS AND NOTIONS

Definition 2. *Let* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a real function such that* $\sum_{n=1}^{\infty} \frac{1}{g(n)} = +\infty$. *Then we can define an ideal* $\mathcal{I}_c^g = \left\{ A \subset \mathbb{N} \,:\, \sum_{n \in A} \frac{1}{g(n)} < +\infty \right\}$. *The ideal* $\mathcal{I}_c^g$ *is an admissible ideal.*

The notion $\mathcal{I}_c^g$ is an union view for many famous ideals.

If $g(x) = k$, where $k \in \mathbb{R}$ then the ideal $\mathcal{I}_c^g$ contains only finite sets, hence $\mathcal{I}_c^g = \mathcal{I}_{fin}$. Next if $g(x) = x$, then $\mathcal{I}_c^g = \mathcal{I}_c$ and finally if we take $g(x) = x^q,\ q \in (0, 1\rangle$ then the ideal $\mathcal{I}_c^g = \mathcal{I}_c^{(q)}$.

In [3] we introduced a certain summable method so-called $M_{p,g}$ which can be defined as follows.

Definition 3. *Let* $p > 0$ *and* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$. *We say that the bounded sequence* $x = (x_k)$ *is* $M_{p,g}$*-summable to the real number* L *(and write* $M_{p,g} - \lim x_k = L$*) if*

$$K = \sum_{k=1}^{\infty} \frac{|x_k - L|^p}{g(k)} < +\infty.$$

It is easy to show that the summable method $M_{p,g}$ for $p > 0$ and a real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n \in \mathbb{N}} \frac{1}{g(n)} = +\infty$ implies $\mathcal{I}_c^g$-convergence to the same real number and the opposite is not true (see [3]).

Theorem 4 (Theorem 5 in [3])**.** *Let* $p > 0$ *and* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\sum_{n=1}^{\infty} \frac{1}{g(n)} = +\infty$. *If the sequence* $x = (x_k)$ *is* $M_{p,g}$*-summable to* $L \in \mathbb{R}$*, then* $\mathcal{I}_c^g - \lim x_k = L$.

The $\mathcal{I}_c^g$-convergence is an example of a linear functional defined on a subspace of the space of all bounded sequences of real numbers. Another important family of such functionals are so called matrix summability methods inspired by [9].

We will study connections between $\mathcal{I}_c^g$-convergence and one class of matrix summability methods. Let us start by introducing a notion of regular matrix transformation (see. [6])

Let $A = (a_{nk})$ $(n, k \in \mathbb{N})$ be an infinite matrix of real numbers. The sequence $t = (t_n)$ of real numbers is said to be A-limitable to the number s if $\lim_{n\to\infty} s_n = s$, where

$$s_n = \sum_{k=1}^{\infty} a_{nk} t_k \quad (n = 1, 2, \dots).$$

If $t = (t_n)$ is A-limitable to the number s, we write $A - \lim_{n\to\infty} t_n = s$.

We denote by $\mathcal{F}(A)$ the set of all A-limitable sequences. The set $\mathcal{F}(A)$ is called the convergence field. The method defined by the matrix A is said to be *regular* provided that $\mathcal{F}(A)$ contains all convergent sequences $t = (t_n)$ and $\lim_{n\to\infty} t_n = s$ implies $A - \lim_{n\to\infty} t_n = s$. Then A is called a *regular matrix.*

It is well-known that the matrix A is regular if and only if it satisfies the following three conditions (see [6] and [12]):

(i) there exists $K > 0$ such that for every $n \in \mathbb{N}$, $\sum_{k=1}^{\infty} |a_{nk}| \le K$,

(ii) for every $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{nk} = 0$,

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

It is well known that a bounded sequence $x = (x_n)$ of real numbers $\mathcal{I}_d$-converges to real number L if and only if the sequence is strongly summable to L in Cesàro sense. The complete characterization of statistical convergence ($\mathcal{I}_d$-convergence) is described by J. A. Fridy and H. I. Miller in the paper [9]. They defined a class $\mathcal{T}$ of lower triangular matrices A with properties:

(1) for every $n \in \mathbb{N}$, $\sum_{k=1}^{\infty} a_{nk} = 1$,

(2) if $C \subseteq \mathbb{N}$ such that $d(C) = 0$, then $\lim_{n\to\infty} \sum_{k\in C} a_{nk} = 0$.

They proved the following assertion:

Theorem 5 (Theorem 1 in [9]). *The bounded sequence $x = (x_n)$ is statistically convergent to L if and only if $x = (x_n)$ is A-summable to L for every $A \in \mathcal{T}$.*

In the paper [10] is proved analogous result for $\mathcal{I}_c^{(q)}$-convergence, which is a special type of $\mathcal{I}_c^g$-convergence.

2 MAIN RESULTS

Here we prove analogous result to the theorem above for $\mathcal{I}_c^g$-convergence. First we define the class $\mathcal{T}_g$ lower triangular nonnegative matrices as follows:

Definition 6. *Matrix $A = (a_{nk})$ belongs to the class $\mathcal{T}_g$ for a positive real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{k\in\mathbb{N}} \frac{1}{g(k)} = +\infty$ if and only if it satisfy the following conditions:*

(1') $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$,

(2') if $C \subseteq \mathbb{N}$ and $C \in \mathcal{I}_c^g$ then $\lim_{n\to\infty} \sum_{k\in C} a_{nk} = 0$.

It is easy to see that every matrix of class $\mathcal{T}_g$ is regular. Example 4.2. in [10] shows that the converse does not hold.

Further we need the next lemma.

Lemma 7 (Lemma 4.3. in [10]). *If the bounded sequence $x = (x_n)$ is not $\mathcal{I}$-convergent then there exist numbers $\lambda < \mu$ such that neither the set $\{n \in \mathbb{N} : x_n < \lambda\}$ nor the set $\{n \in \mathbb{N} : x_n > \mu\}$ belongs to the ideal $\mathcal{I}$.*

As the proof is the same as the proof of Lemma in [9] we will omit it.

Next theorem shows connection between $\mathcal{I}_c^g$-convergence of bounded sequence of real numbers and A-summability of this sequence for matrices A from the class $\mathcal{T}_g$. It is a slightly generalization of results Theorem 1 in [9] and Theorem 4.4 in [10].

Theorem 8. *A bounded sequence $x = (x_n)$ of real numbers $\mathcal{I}_c^g$-converges to $L \in \mathbb{R}$ for a positive real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{k\in\mathbb{N}} \frac{1}{g(k)} = +\infty$ if and only if it is A-summable to L for every matrix $A \in \mathcal{T}_g$.*

Proof. Let $x = (x_n)$ be bounded sequence of real numbers such that $\mathcal{I}_c^g - \lim x_n = L$ and $A \in \mathcal{T}_g$. As A is regular there exists a $K \in \mathbb{R}$ such that for each $n = 1, 2, \dots$ we have $\sum_{k=1}^{\infty} |a_{nk}| \leq K$.

It is sufficient to show that $\lim_{n\to\infty} b_n = 0$, where $b_n = \sum_{k=1}^{\infty} a_{nk}(x_k - L)$. For $\varepsilon > 0$ put $B_\varepsilon = \{k \in \mathbb{N} \,:\, |x_k - L| \geq \varepsilon\}$. By assumption we have $B_\varepsilon \in \mathcal{I}_c^g$. By the condition (2') from Definition 6 we have

$$\lim_{n\to\infty} \sum_{k \in B_\varepsilon} |a_{nk}| = 0. \tag{1}$$

As the sequence $x = (x_n)$ is bounded, there exists $M > 0$ such that for every $k \in \mathbb{N}$

$$|x_k - L| \leq M. \tag{2}$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} |b_n| &\leq \sum_{k \in B_{\frac{\varepsilon}{2K}}} |a_{nk}|\,|x_k - L| + \sum_{k \notin B_{\frac{\varepsilon}{2K}}} |a_{nk}|\,|x_k - L| \\ &\leq M \sum_{k \in B_{\frac{\varepsilon}{2K}}} |a_{nk}| + \frac{\varepsilon}{2K} \sum_{k \notin B_{\frac{\varepsilon}{2K}}} |a_{nk}| \\ &\leq M \sum_{k \in B_{\frac{\varepsilon}{2K}}} |a_{nk}| + \frac{\varepsilon}{2}. \end{aligned} \tag{3}$$

By part (2') of Definition 6 there exists an n_0 such that for all $n > n_0$

$$\sum_{k \in B_{\frac{\varepsilon}{2K}}} |a_{nk}| < \frac{\varepsilon}{2M}.$$

Together by (3) we obtain $\lim_{n\to\infty} b_n = 0$.

Conversely, suppose that $\mathcal{I}_c^g - \lim_{n\to\infty} x_n = L$ does not hold. We show that there exists a matrix $A \in \mathcal{T}_g$ such that $A - \lim_{n\to\infty} x_n = L$ does not hold, too.

Without lost of generality we may assume that $x = (x_n)$ is not $\mathcal{I}_c^g$-convergent. In the opposite case $\mathcal{I}_c^g - \lim_{n\to\infty} x_n = T$ and $T \neq L$. From the first part of this proof we have that $A - \lim_{n\to\infty} x_n = T$ for any $A \in \mathcal{T}_g$.

By the Lemma 7 there exist λ and μ ($\lambda < \mu$), such that neither the set

$$U = \{k \in \mathbb{N} \,:\, x_k < \lambda\}$$

nor the set

$$V = \{k \in \mathbb{N} \,:\, x_k > \mu\}$$

belongs to the ideal $\mathcal{I}_c^g$. It is clear that $U \cap V = \emptyset$. If $U \notin \mathcal{I}_c^g$ and $V \notin \mathcal{I}_c^g$ then $\sum_{i\in U} \frac{1}{g(i)} = +\infty$ and $\sum_{i \in V} \frac{1}{g(i)} = +\infty$ respectively. Denote by $M_n = M \cap \{1, 2, \dots, n\}$ for a set $M \subseteq \mathbb{N}$. As

$U, V \notin \mathcal{I}_c^g$ we have $\lim_{n\to\infty} \sum_{i\in U_n} \frac{1}{g(i)} = +\infty$ and $\lim_{n\to\infty} \sum_{i\in V_n} \frac{1}{g(i)} = +\infty$. Define

$$a_{nk} = \begin{cases} \frac{\frac{1}{g(k)}}{\sum_{i\in U_n} \frac{1}{g(i)}} & \text{if } n \in U \text{ and } k \in U_n, \\ \frac{\frac{1}{g(k)}}{\sum_{i\in V_n} \frac{1}{g(i)}} & \text{if } n \in V \text{ and } k \in V_n, \\ \frac{\frac{1}{g(k)}}{\sum_{i=1}^{n} \frac{1}{g(i)}} & \text{if } n \notin U \cup V \text{ and } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let us check that $A \in \mathcal{T}_g$. Obviously A is a lower triangular nonnegative matrix. Condition (1') is clear from the definition of matrix A. Condition (2'): Let $B \in \mathcal{I}_c^g$ and $b = \sum_{k\in B} \frac{1}{g(k)} < +\infty$. Then

$$\sum_{k\in B} a_{nk} \leq \frac{1}{\sum_{i=1}^{n} \frac{1}{g(i)}} \sum_{k\in B_n} \frac{1}{g(k)} \chi_B(k) \leq \frac{b}{\sum_{i=1}^{n} \frac{1}{g(i)}} \to 0$$

for $n \to \infty$. Thus $A \in \mathcal{T}_g$. For $n \in U$

$$\sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{\sum_{i\in U_n} \frac{1}{g(i)}} \sum_{k=1}^{n} \frac{1}{g(k)} \chi_U(k) x_k < \frac{\lambda}{\sum_{i\in U_n} \frac{1}{g(i)}} \sum_{k=1}^{n} \frac{1}{g(k)} \chi_U(k) = \lambda,$$

on other hand for $n \in V$

$$\sum_{k=1}^{\infty} a_{nk} x_k = \frac{1}{\sum_{i\in V_n} \frac{1}{g(i)}} \sum_{k=1}^{n} \frac{1}{g(k)} \chi_V(k) x_k > \frac{\mu}{\sum_{i\in V_n} \frac{1}{g(i)}} \sum_{k=1}^{n} \frac{1}{g(k)} \chi_V(k) = \mu.$$

Therefore $A - \lim_{n\to\infty} x_n$ does not exist. □

Let us consider which matrices are members of $\mathcal{T}_g$. Most classical summability matrices are nonnegative and satisfy the property (1'), so we focus our attention on the property (2'). In [1] R. P. Agnew showed that

$$\lim_{n\to\infty} \left(\max_{k\in\mathbb{N}} |a_{nk}| \right) = 0 \tag{4}$$

implies that a regular matrix A is stronger than ordinary convergence. We show that the property (4) is implied by the property (2') and so (4) is necessary for matrix $\mathcal{T}_g$.

Theorem 9. *Let A be a nonnegative matrix having the property (2') then A has also the property (4) ergo* $\lim_{n\to\infty} (\max_{k\in\mathbb{N}} |a_{nk}|) = 0$.

Proof. Suppose that $\lim_{n\to\infty} (\max_{k\in\mathbb{N}} |a_{nk}|) \neq 0$ and choose a subsequence of rows (n_m) and columns (k_m) such that $K = \{k_m \ : \ m \in \mathbb{N}\}$ belongs to the ideal I_g and $a_{n_m k_m} \geq \varepsilon > 0$ for each $m \in \mathbb{N}$. Then

$$\sum_{k\in K} |a_{n_m k}| \geq |a_{n_m k}| \geq \varepsilon \quad \text{for each } m \in \mathbb{N}.$$

Hence A do not satisfy the property (2'). □

Further we show some type well known matrix so called Riesz matrix (see [12]) which fulfills condition (1'). Let $p = (p_j)$ be the sequence of positive real numbers. Put $P_n = p_1 + p_2 + \cdots + p_n$. Now we define Riesz matrix $A = (a_{nk})$ as follows:

$$a_{nk} = \begin{cases} \dfrac{p_k}{P_n} & k \le n, \\ 0 & k > n. \end{cases}$$

Especially we put $p_n = \frac{1}{g(n)}$ for a positive real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$. This special class of matrix we denote (R, g).

It is clear that this matrix fulfills conditions (1') and (2'). Moreover (R, g) matrix is regular if and only if $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{g(k)} = +\infty$ (see [12]). For this class of matrix is true the following assertion.

Theorem 10. *Let $x = (x_n)$ be a bounded sequence of real numbers. For a positive real function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{k\in\mathbb{N}} \frac{1}{g(k)} = +\infty$, $\mathcal{I}_c^g$-convergence implies (R, g) summability of $x = (x_n)$.*

Proof. See the first part of the proof of Theorem 8. □

Converse does not hold. For this is sufficient to find a bounded sequence $x = (x_k)$ such that $(R, g) - \lim_{k\to\infty} x_k$ exists, but the sequence $x = (x_k)$ is not $\mathcal{I}_c^g$-convergent. For a function $g(x) = x^\alpha$ for any $\alpha \in (0, 1)$ can be find such example in [10].

Corollary 11. *Let $p > 0$. For a function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sum_{n\in\mathbb{N}} \frac{1}{g(n)} = +\infty$ we have that $M_{p,g}$-summability implies (R, g)-summability for bounded sequences.*

Proof. The proof immediately immplies from Theorem 5 in [3] and the proof of the sufficient condition of Theorem 8. □

Problem 12. *If we take an admissible ideal $\mathcal{I}$ and define the class $\mathcal{T}_{\mathcal{I}}$ of matrices by replacing the condition (2') in Definition 6 by the following condition (2"):*

(2") If $C \subseteq \mathbb{N}$ and $C \in \mathcal{I}$ (an admissible ideal on $\mathbb{N}$) then $\lim_{n\to\infty} \sum_{k\in C} |a_{nk}| = 0$.

Then it is easy to see that the sufficient condition of Theorem 8 holds for $\mathcal{I}$ too. The question is what about the necessary condition.

CONCLUSION

It turns out that the study of $\mathcal{I}$-convergence of sequences (namely sequences related to arithmetical functions) for different kinds of ideals $\mathcal{I}$ (see [4]) gives a deeper insight into the behaviour and properties of these arithmetical functions. Algebraic Number Theory (ANT) has many deep applications in cryptology. Many basic algorithms, which are widely used, have their security due to ANT.

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SYMMETRIC POLYNOMIALS IN SOLVING ELEMENTARY GEOMETRY PROBLEMS

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Abstract: *The article offers ideas for problem teaching in mathematics at secondary schools and universities. There is solved a problem how to calculate the dimensions of a triangle and a cuboid from the given general parameters (e.g. for a triangle its circumference and area are given). While solving there are used symmetric polynomials and the relations between roots and coefficients of an algebraic equation, but also the knowledge of the behaviour of a cubic function.*

Keywords: Algebraic equation, cubic function, symmetric polynomial, triangle, cuboid.

INTRODUCTION

Cultivating students creative mathematical thinking and their interest in mathematics is the substantial part of teaching mathematics at any type or grade of school. While engaging students, any example of possible utilisation of mathematics theories are of great importance. This article focuses on the use of elementary symmetric polynomials in geometric calculations.

The statement that every symmetric polynomial can be represented uniquely with the help of elementary symmetric polynomials is commonly presented and proved, and the practical transformations are practised. However, only rarely can students see the use of this topic at secondary school mathematics (tasks like: Finding the sum of third powers of the cubic function roots without solving it; or solving systems of equations of higher orders).

In this article we will show how it is possible to use elementary polynomials while solving practical exercises from elementary geometry. It is interesting that even though the assignment suggests that these are tasks for a basic school pupil, while solving it we need the knowledge of at least a secondary school pupil.

To conclude this part, let us remind the necessary theoretical information (See e.g. [1], [4], [5], [6], [8], [9]). The symmetric polynomial of n variables is such polynomial which does not change at any permutation of its variables. There applies generally that every symmetric polynomial of n variables can be formulated uniquely with the help of elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$. For $n = 2$ there holds $\sigma_1 = x + y$, $\sigma_2 = xy$, for $n = 3$ there applies $\sigma_1 = x + y + z, \sigma_2 = xy + xz + yz$, $\sigma_3 = xyz$. When we want to represent the symmetric polynomial, for $n = 2$ there holds $x^2 + y^2 = \sigma_1^2 - 2\sigma_2$, $x^3 + y^3 = \sigma_1^3 - 3\sigma_1\sigma_2$, for $n = 3$ we get $x^2 + y^2 + z^2 = \sigma_1^2 - 2\sigma_2$, $x^3 + y^3 + z^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$.

1 THE RECTANGULAR CUBOID PROBLEM

Problem 1 (See [10]): Let $m \geq n \geq p$ be the lengths of face diagonals of a cuboid. Calculate the lengths of cuboid´s edges a, b, c.

Solution: This problem is quite easy, and it can be solved using formulas from elementary geometry. With the "usual" notation of cuboid´s vertices $ABCDA`B`C`D`$, let us denote $a = AB$, $b = AD$, $c = AA`$, $m = AD`$, $n = AB`$, $p = AC$. Then our task is to calculate the cuboid´s dimensions a, b, c.
From right-angled triangles $ADD`$, $ABB`$, ABC which lie in cuboid´s faces there holds according to Pythagoras´ theorem

$$m^2 = b^2 + c^2, \qquad (i)$$
$$n^2 = a^2 + c^2, \qquad (ii)$$
$$p^2 = a^2 + b^2. \qquad (iii)$$

Without detriment to generality, let us suppose according to the assignment $m \geq n \geq p$. We will sum the equation (ii) and (iii), and from the sum we will distract the equation *(i)*. We will get $-m^2 + n^2 + p^2 = 2a^2$ and from that after modification

$$a^2 = \frac{1}{2}(-m^2 + n^2 + p^2).$$

Provided that $-m^2 + n^2 + p^2 > 0$, after simplification we will get the formula for the calculation of the length of the edge a:

$$a = \sqrt{\frac{1}{2}(n^2 + p^2 - m^2)}.$$

Similarly, we can find the two remaining cuboid´s edges:

$$b = \sqrt{\frac{1}{2}(m^2 + p^2 - n^2)},$$
$$c = \sqrt{\frac{1}{2}(m^2 + n^2 - p^2)}.$$

For the problem to have solutions, the numbers under the root in all three formulas have to be positive numbers. Therefore numbers m^2, n^2, p^2 have to satisfy the triangle inequality. Under such condition, problem 1 has a solution and the lengths of edges a, b, c are given by the above-mentioned formulas.

Exercise: Let $p = 4$, $n = 5$, $m = 6$. The powers of these numbers $16, 25, 36$ satisfy triangle inequality. Therefore from the above-mentioned formulas $a = \sqrt{\frac{5}{2}}$, $b = \sqrt{\frac{27}{2}}$, $c = \sqrt{\frac{45}{2}}$, after calculation the approximate values are $a = 1{,}58$, $b = 3{,}67$, $c = 4{,}74$.

Problem 2: This problem is slightly complicated. Let us denote o as the sum of lengths of all cuboid´s edges, S as its surface. Prove that there exists a rectangular cuboid for which there applies $o = 24$, $S = 18$. Calculate the volume of a cube with the same length of a solid diagonal as of this cuboid.

Solution: We will solve the first part of the task without the theory of symmetric polynomials using the knowledge of a secondary school pupil. The symmetric polynomials will be used in the second part.

We will denote the lengths of the cuboid´s edges as a, b, c. According to the assignment there holds $4a+4b+4c=24$, therefore $a+b+c= 6$. Then from the assignment $2ab + 2ac + 2bc = 18$ and then $ab + ac + bc = 9$. The task of the first part is to prove that there exist three positive real numbers a, b, c with properties:

$$\begin{aligned} a + b + c &= 6 \\ ab + ac + bc &= 9. \end{aligned} \tag{1}$$

Let us proceed from the equation $a+ b+ c= 6$, and let us substitute $a + b= 6- c$ to the equation $ab + ac + bc = 9$. After modification we will get

$$c^2 - 6c - ab + 9 = 0.$$

Let us substitute $b = 6 - c - a$ (*) and after a simple modification we will get a quadratic equation with a parameter c:

$$a^2 + a(c - 6) + (c - 3)^2 = 0.$$

When we solve it and substitute back for a to (*), we will get formulas for a, b dependent on the parameter c:

$$a = \frac{6 - c + \sqrt{12c - 3c^2}}{2}, \quad b = \frac{6 - c - \sqrt{12c - 3c^2}}{2}\ . \tag{2}$$

Now we have to restrict the choice of the parameter c. From the condition $12c - 3c^2 \geq 0$ there holds $0 < c < 4$. At the same time, both values a, b have to be positive. Evidently it is sufficient if we focus on the numerator from the formula for b. From the condition $6 - c - \sqrt{12c - 3c^2} > 0$ we get $(2c - 6)^2 > 0$, the inequality holds for all real numbers with the exception $c = 3$. The conclusion for the first part is the following: with any choice of the real number c from the union of intervals $(0, 3) \cup (3, 4\rangle$ and calculation of numbers a, b according to (2), we will get three numbers a, b, c satisfying (1); then they are the lengths of the cuboid with the properties required by the assignment. Now it is necessary to perform the verification if the above given three numbers a, b, c really satisfy the relations (1). Due to its extent, we will not present here (it is only modification of algebraic relations), let us only state that the correctness will be verified. Let us note that one possible solution (even the integer one) is je $a = 1, b = 1, c = 4$.

Now let us focus on the calculation of the length of the solid diagonal of the desired cube. This cube has the same length of a diagonal as the cuboid. We know that the length of the cuboid equals $l = \sqrt{a^2 + b^2 + c^2}$. If we use elementary symmetric polynomials ($\sigma_1 = a + b + c = 6$, $\sigma_2 = ab + ac + bc = 9$) for expressing the formula below the root, we will get $l = \sqrt{\sigma_1^2 - 2\sigma_2} = 3\sqrt{2}$. From here we get an interesting statement that all cuboids, whose lengths of edges satisfy (1), have the same lengths of their solid diagonals. The assignment of the second part is therefore correct, the volume of the cube will be determined unambiguously.

If we denote the length of the cube´s edge x, the length of the solid diagonal equals $\sqrt{3}\,x$. However, there holds $\sqrt{3}\,x = 3\sqrt{2}$, from here $x = \sqrt{6}$. The desired volume of the cube equals the value $6\sqrt{6}$. Let us consider another interesting feature which is possible by using symmetric polynomials: While solving, we got an expression $\sqrt{a^2 + b^2 + c^2}$, without knowing the values of a, b, c. Therefore this assignment can be used as a motivation for the study of symmetric polynomials.

Now, we will deal with the generalization of the first part of the given assignment, i.e. the question of the general existence of cuboids from the given parameters. Let there be two arbitrary positive real numbers o, S; the assignment is to find out if there exists the cuboid with the lengths of edges $a, b, c,$ for which there applies

$$4a + 4b + 4c = o, 2ab + 2ac + 2bc = S.$$

For the simplification, in the part dealing with the cuboid, we will introduce the notation $\frac{o}{4} = k,\ \frac{S}{2} = l$. For didactic reasons we will not use symbols σ_1, σ_2, even though numbers k, l denote elementary symmetric polynomials (there holds $\sigma_1 = a + b + c = k$, $\sigma_2 = ab + ac + bc = l$) . Even if the existence of the cuboid is not guaranteed for the given numbers o, S, we will denote m as its volume in case that it exists. Then there holds $\sigma_3 = abc = m$. From algebra, there are known relations between coefficients and the roots of the polynomial, which can be used for representation of the polynomial´s coefficients with the help of elementary symmetric polynomials of its roots (see e.g. [5], [6], [9]). From these relations now we get that the desired cuboid exists if and only if the algebraic equation

$$x^3 - k x^2 + l x - m = 0 \qquad (3)$$

has just three positive real solutions a, b, c. In the equation (3), numbers k, l are set by the assigned values o, S, number m is a parameter. We will denote the left side of the equation (3) as $f(x)$. This function is a cubic one and its graph is a cubical parabola. It always has two local extremes x_1, x_2; with respect to the leading coefficient +1 this function is increasing in the interval $(-\infty, x_1)$, decreasing in the interval (x_1, x_2) and increasing in the interval (x_2, ∞). For the equation (3) to have three positive real solutions (for now let us suppose that they are mutually unlike), there have to apply the following conditions: $x_1 < x_2$, $x_1 > 0$, $x_2 > 0$, $f(0) < 0$, $f(x_1) > 0$, $f(x_2) < 0$.

If the cuboid exists, its volume m is always a positive number. Provided that $m > 0$ there always holds that $f(0) < 0$. Let us find the coordinates of the extremes x_1, x_2 : There holds $f'(x) = 3x^2 - 2kx + l$. we will find the stationary points, i.e. we will solve the quadratic equation $3x^2 - 2kx + l = 0$. Its solution is as follows:

$$x_1 = \frac{k - \sqrt{k^2 - 3l}}{3}, \qquad x_2 = \frac{k + \sqrt{k^2 - 3l}}{3}.$$

Now it is necessary to ensure that both two roots of the quadratic function are real and positive (the inequality $x_1 < x_2$ is evident). Both roots are real numbers provided that $k^2 - 3l \geq 0$, which is a necessary condition for the existence of the cuboid of the given parameters (because $\frac{o}{4} = k,\ \frac{S}{2} = l$). Now it is sufficient if we verify the inequality $x_1 > 0$. After substitution, we get the inequation $\frac{k - \sqrt{k^2 - 3l}}{3} > 0$, therefore $k > \sqrt{k^2 - 3l}$. Provided that $k^2 - 3l \geq 0$, we can raise this inequation to a power (both sides are nonnegative numbers). After this step and simplification, we will get the inequation $0 > -3l$, which applies for any value of the variable l (according to the assignment is always positive). Provided that $k^2 - 3l \geq 0$, all conditions $x_1 < x_2$, $x_1 > 0$, $x_2 > 0$, $f(0) < 0$ are satisfied. To conclude the proof of the existence of the cuboid with the given parameters o, S, it remains to ensure the conditions $f(x_1) > 0$, $f(x_2) < 0$. They will also provide if for the given o, S (and therefore for k, l) it is possible to assign such parameter m that the equation (3) has three real solutions. Thus we will define the real interval containing all possible values of the cuboid´s volumes with parameters o, S.

Firstly, let us find $f(x_1)$. We will substitute for x the expression $\frac{k-\sqrt{k^2-3l}}{3}$ to the equation (3) and simplify it. After simplification we will get the relation

$$f(x_1) = -\frac{2}{27}k^3 + \frac{2}{27}k^2\sqrt{k^2-3l} - \frac{6}{27}l\sqrt{k^2-3l} + \frac{9}{27}kl - m \ .$$

We have already derived the condition $k^2 - 3l \geq 0$, values k, l are given from the assignment. Now it is necessary to set the parameter m , so for the inequality $f(x_1) > 0$ to be satisfied, it is necessary to hold

$$m < -\frac{2}{27}k^3 + \frac{1}{3}kl + \frac{2}{27}k^2\sqrt{k^2-3l} - \frac{2}{9}l\sqrt{k^2-3l} \,.$$

Similarly, from the condition $f(x_2) < 0$, we will derive the inequality

$$m > -\frac{2}{27}k^3 + \frac{1}{3}kl - \frac{2}{27}k^2\sqrt{k^2-3l} + \frac{2}{9}l\sqrt{k^2-3l} \,.$$

Joining both conditions, we will get the interval for m: (4)

$$m \in \left(-\frac{2}{27}k^3 + \frac{1}{3}kl - \frac{2}{27}k^2\sqrt{k^2-3l} + \frac{2}{9}l\sqrt{k^2-3l}\,,\ -\frac{2}{27}k^3 + \frac{1}{3}kl + \frac{2}{27}k^2\sqrt{k^2-3l} - \frac{2}{9}l\sqrt{k^2-3l}\right)$$

In order for the last condition for m to be correct, it is necessary to verify the inequality between the borders of this interval. After substituting of the calculated values and further simplification we will get the inequation

$$\frac{4}{27}k^2\sqrt{k^2-3l} \geq \frac{4}{9}l\sqrt{k^2-3l} \,,$$

Which is satisfied if and only if there applies $k^2 - 3l \geq 0$. Thus we have proved that if the numbers k, l satisfy the inequality

$k^2 - 3l \geq 0$ (i.e.. $o^2 - 24\,S \geq 0$),

then the given cuboid always exists, and all its possible volumes are determined by numbers from the interval (4). The condition $k^2 - 3l \geq 0$ is also sufficient. With the given o, S satisfying $o^2 - 24\,S \geq 0$, we will choose m from the interval (4) and solve the equation (3). Its solution are real numbers which denote the lengths of the demanded cuboid´s edges.

Let us note that if the values satisfy $o^2 - 24\,S = 0$, the expressions under the radical sign always equal zero. After calculation we get the following values: The only possibility is $m = \frac{k^3}{27}$. Further $x_1 = x_2 = \frac{k}{3}$, $f(x_1) = f(x_2) = 0$. After substitution $m = \frac{k^3}{27}$, $l = \frac{k^2}{3}$ (from $k^2 - 3l = 0$), we can rearrange the equation (3) to the form $(x - \frac{k}{3})^3 = 0$. From here there follows that the desired cuboid is a cube with the length of the edge $a = \frac{k}{3} = \frac{o}{12}$.

Exercise: a) Let $o = 24, S = 22$. Then $k = 6, l = 11$. The condition $k^2 - 3l \geq 0$ is satisfied, so the cuboid with these parameters exists. However, it is not unambiguous, we have to choose the parameter m. While solving, we will find out that $m \in \langle 6 - \frac{2}{9}\sqrt{3}, 6 + \frac{2}{9}\sqrt{3} \rangle$. If we choose $m = 6$, then the cubic equation $x^3 - 6x^2 + 11x - 6 = 0$ has three real solutions $a = 1, b = 2, c = 3$ and it is possible to solve it with the means of the school mathematics. The desired cuboid has dimensions 1, 2, 3. If we choose differently, e. g. $m = 6{,}1$, the equation $x^3 - 6x^2 + 11x -$

$6{,}1 = 0$ is best solvable with the help of computer technology; the roots (and thus the cuboid´s dimensions) are after rounding equal $a = 1{,}054$, $b = 1{,}899$, $c = 3{,}047$.

b) Let $o = 24$, $S = 24$, then $k = 6$, $l = 12$. There holds $k^2 = 3l$. Then necessarily there must be $m = 8$, the equation (3) is $(x-2)^3 = 0$, and therefore $a = b = c = 2$.

After solving the first problem of existence of the cuboid with the given parameters *o, S*, there appear other possibilities how to extend this topic. Such "grapes of problems" are dealt with in the book by J. Kopka [7]. Using the above given notation for the cuboid, we can set some of the three values *k, l, m* and solve the question of its existence and the lengths of its edges. Contrary to the already solved problem, other variants are much more difficult to calculate. In the described case (*o, S* are given), the parameter *m* does not appear in the first derivative of the left side of the equation (3), so we will use it while calculating the values $f(x_1)$, $f(x_2)$. Predominantly, the given values *k, l* appear in calculations. If one of these values *k, l* was not given, we can easily imagine the complexity of the given calculations. While "classical" method without using symmetric polynomials, and in this case as well, the process of solution exists. Let us outline briefly the solution of one of other variants of the cuboid´s problem:

Problem 3: Let there be given two arbitrary positive numbers *o, V*. The assignment is to find out if there exists a cuboid with the lengths of edges *a, b, c* such that $4a + 4b + 4c = o$, $abc = V$.

Using the above-mentioned notation, there are given numbers $k = a + b + c$, $m = abc$. The value *l* is not given, we have to determine it. We will choose the length of the edge *c* as the parameter.

$$l = ab + ac + bc = ab + \frac{m}{b} + \frac{m}{a} = \frac{a^2b^2 + am + bm}{ab} = \frac{\left(\frac{m}{c}\right)^2 + m(k-c)}{\frac{m}{c}} = \frac{m + c^2(k-c)}{c}.$$

As m, c, $k-c$ are positive numbers, the calculated value *l* is always positive. The obvious requirement for the choice of the parameter *c* is $c < k$.

The procedure while solving the question of the cuboid´s existence and finding its dimensions with the given *o, V* (i.e. *k, m*) is as follows: the choice of the number *c* , the calculation of the value *l* and the solution of the system of three equations with three unknown:

$$a + b + c = k$$
$$ab + ac + bc = l$$
$$abc = m\,.$$

The calculation is quite complicated and require great deal of patience. Let us give only the solution: For the chosen *c* ($0<c<k$) other dimensions of the cuboid are determined by relations

$$a = \frac{k - c + \sqrt{(k-c)^2 - \frac{4m}{c}}}{2}, \quad b = \frac{k - c - \sqrt{(k-c)^2 - \frac{4m}{c}}}{2}.$$

Let us note that both relations always determine the positive value ($\frac{m}{c} = ab$, $k - c = a + b$).

Exercise: $o = 64$, $V = 12$. Determine if the cuboid with these parameters exists. According to this assignment $k = 16$, $m = 12$. Let us choose $c = 4$. After calculation $l = 51$, $a = 6 + \sqrt{33}$, $b = 6 - \sqrt{33}$, which are the lengths of edges of one of the possible solutions.

2 THE TRIANGLE PROBLEM

Problem: Let be given two positive real numbers o, S. Decide if there exists a triangle with the lengths of sides a, b, c such that the number o expresses its circumference ($o = a + b + c$) and S expresses its area. If it exists, determine the lengths of sides of this triangle with the help of numbers o, S. We will always count on the fact that the lengths of sides a, b, c are positive non-zero real numbers satisfying the triangle inequality.

Obviously, there applies $o = a + b + c$. The area of a triangle can be expressed with the help of its sides using Heron´s formula $S = \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$, where $s = \frac{o}{2}$, so

$S = \sqrt{\frac{o}{2} \cdot \frac{o-2a}{2} \cdot \frac{o-2b}{2} \cdot \frac{o-2c}{2}}$. Apparently, none of the factors under the radical sign is greater than $\frac{o}{2}$; therefore, we get the first necessary condition for the existence of the triangle with the given parameters:

$S < \sqrt{\frac{o^4}{16}} = \frac{o^2}{4}$. Further, we will continue by the method of analysis. Let us write Heron´s formula in the form

$$S = \sqrt{\frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2}}\,.$$

We will rearrange this relation. Both sides are non-negative numbers, so we can exponentiate the inequality (it is an equivalent operation). After further rearrangement we will get

$$16S^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$$

Now let us use the elementary symmetric polynomials. With the above given denotation $\sigma_1 = a+b+c$, $\sigma_2 = ab+ac+ bc$, $\sigma_3 = abc$ we can get, using the theory of elementary symmetric polynomials (see e. g. [4]), get the relations:

$$a^4 + b^4 + c^4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3\,, \qquad a^2b^2 + b^2c^2 + a^2c^2 = \sigma_2^2 - 2\sigma_1\sigma_3\,.$$

After the substitution and rearrangement, we will get

$$-\sigma_1^4 + 4\sigma_1^2\sigma_2 - 8\sigma_1\sigma_3 = 16S^2.$$

Further, we will substitute $\sigma_1 = o$ to the last relation; the relation will change to

$$16S^2 + o^4 = 4o^2\sigma_2 - 8o\sigma_3\,. \tag{4}$$

In this relation, numbers o, S are given in the assignment, numbers σ_2 and σ_3 represent parameters dependent on positive real numbers o, S. As the expression $4o^2\sigma_2 - 8o\sigma_3$ has to be a positive number, both parameters are bound by the condition $o\sigma_2 > 2\sigma_3$ (*). The next possible procedure of solving the triangle is as follows: We will choose the value of the parameter σ_2 and with its help we will count the value of the parameter σ_3.. Knowing values $\sigma_1, \sigma_2, \sigma_3,$ we will again use the theory of symmetric polynomials, similarly to solving the

dimensions of the cuboid in the second part. Prior to the choice of the parameter σ_2 let us determine the restricting conditions.
We know that the relation $S = 0{,}5\ ab \sin\gamma$ holds (and evidently there hold also two further relations obtained by the cyclic permutation). From this relation there follows $\sin\gamma = \frac{2S}{ab}$ and from there the inequality $\frac{2S}{ab} \le 1$ (the inequality $\frac{2S}{ab} \ge 0$ is obvious). Therefore there holds the inequality $2S \le ab$, similarly through the cyclic permutation $2S \le ac$, $2S \le bc$. Counting up all three inequalities we will get the necessary condition for the choice of σ_2 and thus the next necessary condition for the existence of the given triangle $6S \le \sigma_2$. After the choice of σ_2, we will count from the relation (4) the value of σ_3:

$$\sigma_3 = \frac{4o^2\sigma_2 - 16S^2 - o^4}{8o}. \qquad (5)$$

The condition (*) can be verified easily. After substituting for σ_3, we will find out that this condition is always satisfied. There appears a more interesting situation for the numerator of the fraction in (5), which has to be a positive number as well. From here we will calculate the next necessary condition for the choice of σ_2, and therefore for the existence of the triangle: $\sigma_2 > \frac{16S^2 + o^4}{4o^2}$. Now, let us sum up the acquired conditions:

The condition for the assignment: $S < \frac{o^2}{4}$, the condition for the choice: $\frac{16S^2 + o^4}{4o^2} < \sigma_2$, $6S \le \sigma$. After the calculation σ_3, we will verify the condition $o\sigma_2 > 2\sigma_3$.

Now let us proceed to the question how to get the lengths of the triangular´s sides a, b, c knowing the values $\sigma_1, \sigma_2, \sigma_3$. Let us remark again that σ_1 is determined from the assignment as o, σ_2 will be chosen according to the above given conditions and σ_3 will be counted from the relation (5). Knowing the relations between roots and coefficients of a polynomial, there follows that a, b, c are the roots of a cubic equation

$$x^3 - o\,x^2 + \sigma_2\,x - \sigma_3 = 0 \qquad (6).$$

Now the question is if the cubic equation has three real roots. If the equation has them and these three numbers satisfy the triangle inequality, then these values are the desired lengths of triangle´s sides with parameters o, S. From these facts it is evident that the above given conditions are only necessary, but they are not sufficient. Everything depends on the solution of the equation (6). Further we have to deal with the next necessary condition, which we derive from the requirement for the equation (6) to have three real solutions. We will proceed similarly to the second part where we solved the same problem at the cuboid. The left side of the equation (6) will be denoted as $f(x)$. For $x = 0$ there always applies $f(x) < 0$ (because while satisfying the condition for the choice of σ_2 there applies that σ_3 is always a positive real number). Therefore, it suffices to secure that the first derivative of the function $f(x)$ has two real roots x_1, x_2, while $f(x_1) > 0$, $f(x_2) < 0$. For the first derivative there applies $f'(x) = 3x^2 - 2ox + \sigma_2$. This derivative has two stationary points $x_{1,2} = \frac{o \pm \sqrt{o^2 - 3\sigma_2}}{3}$,which are real numbers for $o^2 - 3\sigma_2 \ge 0$, so $\sigma_2 \le \frac{o^2}{3}$. Thus we get another condition restricting the choice of the parameter σ_2. The process of solving both inequalities $f(x_1) > 0$, $f(x_2) < 0$ after substituting the calculated values $x_{1,2}$ is rather complicated and impractical; therefore we will not present it. We prefer to deal with the equation (6) in practical examples.

In conclusion, let us sum up the results which we got while solving the triangle with the given parameters o, S: The given values have to satisfy $S < \frac{o^2}{4}$. Now let us choose the number σ_2 satisfying the following conditions:

$$\frac{16S^2 + o^4}{4o^2} < \sigma_2, \quad 6S \leq \sigma_2, \quad \sigma_2 \leq \frac{o^2}{3}.$$

According to the relation (5), we will calculate the value of number σ_3 and verify the condition $o\sigma_2 > 2\sigma_3$. Let us solve the equation (6). If this equation has three real solutions and these solutions satisfy the triangle inequality, we will get the lengths a, b, c of the sides of the triangle, whose circumference equals o and whose area equals S.

Examples: a) $o = 14$, $S = \sqrt{56}$. While calculating we will determine that we have to choose the parameter σ_2 from the interval $(50{,}1;\, 65{,}3\rangle$. Let us choose $\sigma_2 = 63$. After substituting to (5), we will determine $\sigma_3 = 90$. The equation (6) is in the form $x^3 - 14x^2 + 63x - 90 = 0$, its roots are real numbers 3, 5, 6. These three numbers satisfy the triangle inequality and are the solutions of the triangular problem. There applies $o = 14$, $S = \sqrt{7 \cdot 4 \cdot 2 \cdot 1} = \sqrt{56}$.

b) $o = 14$, $S = \sqrt{56}$. Let us choose $\sigma_2 = 60$, then let us calculate $\sigma_3 = 69$. The equation $x^3 - 14x^2 + 60x - 69 = 0$ has complex roots; for the given choice of parameters such triangle does not exist.

c) $o = 21$, $S = \frac{27\sqrt{7}}{4}$, which is approximately $17{,}86$. After substituting to the conditions of the choice, calculation and rounding, we will get the inequality which restricts the choice of σ_2.

There has to hold $113 < \sigma_2 < 147$. We will chose $\sigma_2 = 144$. Now while calculating according to (5), we will determine $\sigma_3 = 324$. We will compile the equation $x^3 - 21x^2 + 144x - 324 = 0$ and solve it. The solution are three positive numbers $6, 6, 9$, which satisfy the triangle inequality. The desired triangle is a isosceles triangle; the arms are of the length 6 and the base is of the length 9. The proof will show that there really holds $o = 21$,

$$S = \sqrt{\frac{21}{2} \cdot \frac{9}{2} \cdot \frac{9}{2} \cdot \frac{3}{2}} = \sqrt{\frac{5103}{16}} = \frac{27\sqrt{7}}{4}.$$

CONCLUSION

In the article we dealt with the problem of the existence and determination of dimensions of the cuboid and triangle with the given conditions. Although the results are not significant for geometry, for the students´ instructions they are suitable and useful. While solving the problems at lessons, the main concern does not consist in the originality, significance or importance of the derived mathematical formulas, but in the development of students´ knowledge and thinking in mathematics (thus acquiring the necessary competencies for studying and solving problems). Problem teaching has been dealt with in a lot of publications for years (see e.g. [2], [3]);

therefore, the topic from this article is suitable for this purpose. The topic can be extended to the existence of other geometric figures in the plane or space.

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MATHEMATICAL PARTY AND CHESS ALIAS APPLICATION OF ROOK POLYNOMIALS

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Abstract: *In the paper, there is shown one examples of application of the rock polynomials. That example concerns permutations with limited conditions. The theory can be suitably used in the teaching of combinatorics. Some parts of this text are written in the structural form.*

Keywords: rook polynomials, rook moves, permutations, structural text.

INTRODUCTION

It is often necessary to solve a combinatorial problem from practice, which has many limiting conditions. Classical methods of solutions tend to be long and complicated. For simplification, the theory of the rook polynomials is suitable, which is explained in this article using one example. For understanding the following method it is enough to know how to move the rook. This fact is well known even to students of secondary schools and universities, and so it is possible to use the following method in the various modifications of examples from life for the enriching of lessons. Solving problems using permutations with limited conditions using the property of rook polynomials expands students´ knowledge of the given problem. The article could therefore be of benefit and inspiration for students and teachers in mathematics classes when discussing this topic.
Hand in hand with the correct understanding of the mathematical text, the requirements for the intelligibility and suitability of learning material are increasing more and more. Therefore, the so-called structural form of the text is used in some parts of this paper.
This topic was particularly popular with mathematicians at the turn of the century (see for example [1] , [2] , [3] , [4] , [5], [6], [7].) , but it is still very relevant. For example, authors in the paper [7] study the relationship between the rook vector of a general board and the chromatic structure of an associated set of graphs. They give algebraic relations between the factorial polynomials of two boards and their union and sum, and the chromatic polynomials of two graphs and their union and sum.

1 ROOK POLYNOMIALS

The term **rook polynomial** was coined by John Francis Riordan (1903 – 1988), the American mathematician dealing with the combinatorial analysis. Despite derivation of the name from chess, the impetus for studying rook polynomials is their connection with counting (partial) permutations with restricted positions.
It is defined as a polynomial whose number of ways k non-attacking rooks can be arranged on $m \times n$ chessboard. It follows from it that no two rooks may be in the same row or columns. The rook endangers just fields of the chess-board, whose lie in the row and in the column, in their intersection the rook stands (see Fig. 1).

Definition: The **rook polynomial** $R_B(x)$ of a chessboard B is the generating function for the numbers of arrangements of non-attacking rooks

$$R_B(x) = \sum_{i=0}^{\min(m,n)} r_i(B)x^i, \tag{1}$$

where $r_i(B)$ is the number of ways to place k non-attacking rooks on the chessboard B. The rook polynomial of a chessboard of the type $m \times n$ is closely related to generalized **Laguerre polynomial** $L_n^\alpha(x)$ by the identity

$$R_{m,n}(x) = n!\, x^n L_n^{(m-n)}(-x^{-1}). \tag{2}$$

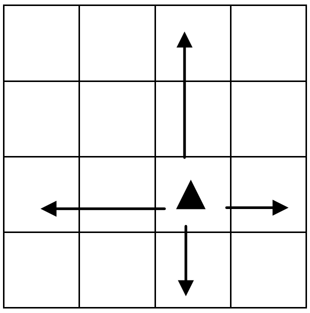
Solutions to the associated Laguerre differential equations with $\upsilon \neq 0$ and k an integer are called **the associated Laguerre polynomials** $L_n^\alpha(x)$ – see [8] and [9] .

The Laguerre polynomial (see [8]) is given by the relation

$$L_n(x) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{i!} x^i \tag{3}$$

We can look at the first few rook polynomials.

$$\begin{aligned} R_1(x) &= x + 1 \\ R_2(x) &= 2x^2 + 4x + 1 \\ R_3(x) &= 6x^3 + 18x^2 + 9x + 1 \\ R_4(x) &= 24x^4 + 96x^3 + 72x^2 + 16x + 1 \\ &\dots \end{aligned} \tag{4}$$



There can be a maximal number of non-attacking rooks on the chessboard. Indeed, there cannot be placed more rooks than the number of rows or columns on the chessboard - hence the limit min (m,n).

Let us denote:

A chessboard of the type $m \times n$ … $B_{m,n}$, $R_{B_{m,n}}(x) = R_{m,n}(x)$

A chessboard of the type $n \times n$ … B_n, $R_{B_n}(x) = R_n(x)$

Fig. 1. The rook moves
Source: own

Example 1: Let us illustrate the case of positions of rooks for $n = 2$ in Fig. 2. No rook – 1 possibility, 1 rook – 4 possibilities, 2 rooks – 2 possibilities, i.e. $R_2(x) = 2x^2 + 4x + 1$.

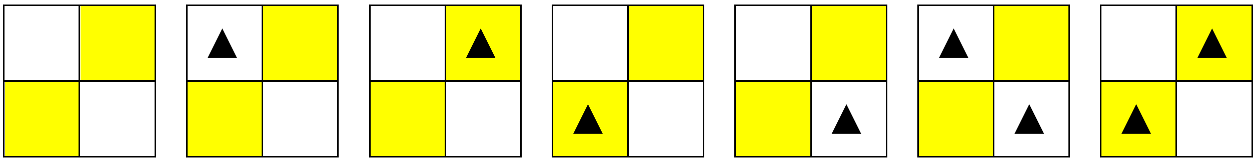


Fig. 2. The rook positions for $n = 2$
Source: own

2 MATH CELEBRATION

In the following example, we will explain the aforementioned method and show its use in practice.

Example 2: 6 different bottles with 6 different drinks are prepared at a known mathematician´s birthday party. In how many different ways can the 6 men celebrating this birthday choose a drink if we know the following facts?

Adam	does not like	**w**hisky
Bob		**b**randy and **w**hisky
Danny		**w**ine
Ed		**f**ernet and **w**hite **w**ine
Fred		**r**um
Carl	drinks all	

Tab. 1. The party and drinks
Source: own

Let us indicate:
B … brandy, **W** … whisky, **F** … fernet, **R** … rum, **RW** … red wine, **WW** … white wine

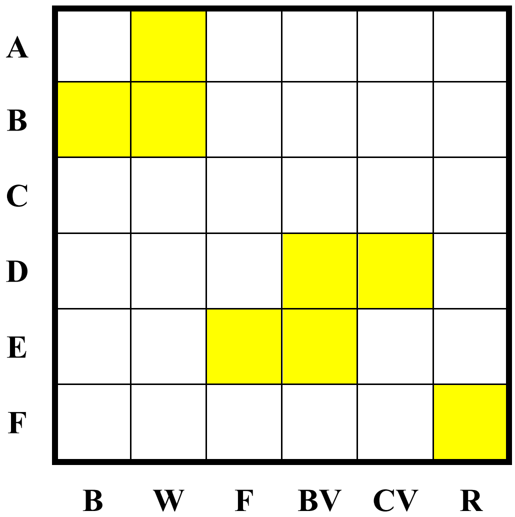


Fig. 3. The party situation
Source: own

This situation can be illustrated in a 6 × 6 gird. The men being honour are arrangement in rows, their drinks in a column. The coloured squares represent all prohibited possibilities. For example, conditions for Danny mean that the squares D4 and D5 will be coloured (Fig. 3).
The given example is one of typical examples being solved by permutations. However, there exists a lot of bounded conditions complicated the given problem very well in this example. If we use the students´ knowledge of the rook moves, we can show them the different possibility of solving. Students can investigate the continuity between an algebraic theory (polynomials) and combinatorics (permutations).

Considering that the inclusion and exclusion principle for a finite set M and its subsets M_1, M_2, …, M_n, where $|M|$ is a number of elements of M and M_i^c is the complement of M, according to M, is held, we can write

$$|M_1^c \cap M_2^c \cap \cdots \cap M_n^c| = |M| + \sum_{i=1}^{n} (-1)^i P_i, \qquad (5)$$

where P_i , $i = 1, \ldots, n$, are numbers of permutations. These numbers are determined by conditions (see for example [2])

(6)

$$P_1 = \sum_{i=1}^{n} |M_i|,$$

$$P_2 = \sum_{i,j=1, i<j}^{n} |M_i \cap M_j|,$$

$$P_3 = \sum_{i,j,k=1, i<j<k}^{n} |M_i \cap M_j \cap M_k|,$$

$$\ldots$$

$$P_n = \sum_{i_1, \cdots i_n, =1, \, i_1 < \cdots < i_n}^{n} |M_{i_1} \cap M_i \cap \cdots \cap M_{i_n}| = |M_1 \cap M_2 \cap \cdots \cap M_n|.$$

Thus, the total number of permutations is

(7)

$$p = |M| \sum_{i=1}^{n} (-1)^i P^i$$

However, the method of calculating all permutations and their summation is rather lengthy, therefore the summaries P_i, which appear in the principle of inclusion and exclusion, will be determined by a different way.

Let us imagine the given situation in Fig.3 and try to interpret that problem as a special rook problem and combine with the theory. Looking at Fig. 3 we can imagine that the selection of the *i* squares where each 2 squares do not lie in the same row and column is equivalent to the *i* non-endanger rooks each other (the sets M_1, M_2, …, M_n). If we can divide the chessboard *B* into 2 **disjunct sets** B_1, and B_2 then the final rook polynomial associated to the chessboard *B* is a product of rook polynomials of B_1, and B_2. The case, where the chessboard cannot be divided into final number of disjunct sets is solved for example in [3].

In our case, let us denote $B_1 = \{AW, BB, BW\}$, $B_2 = \{DBV, DCV, EF, EBV, FR\}$. These sets are evidently finite and disjunctive. It means, that the set *B* is the set of all coloured squares (disabled options), B_1 and B_2 consist of the different rows and columns.

Let $r_i(B), i \geq 1$ be the number of possibilities, how to place the *i* non-endanger rooks each other to the grid *B*, similarly $r_i(B_1)$, resp. $r_i(B_2), i \geq 1$, for the grid B_1 and B_2. We also define

$$r_0(B_1) = r_0(B_2) = r_0(B) = 1. \quad (8)$$

Looking at Fig. 3 and knowing the rook moves, the numbers $r_i(B_i), i \geq 1$ can be found very easily. There are

$$r_1(B_1) = 3, r_2(B_1) = 1, \qquad r_i(B_1) = 0 \ \ \forall\, i \geq 3 \tag{9}$$

$$r_1(B_2) = 5, r_2(B_2) = 7, r_3(B_2) = 3 \qquad r_i(B_2) = 0 \ \ \forall\, i \geq 4$$

in our specific example. We will combine these numbers:

$$r_i(B) = r_0(B_1) \cdot r_i(B_2) + r_1(B_1) \cdot r_{i-1}(B_2) + \cdots + r_i(B_1) \cdot r_0(B_2), i \geq 1. \tag{10}$$

Then

$$r_1(B) = 8, r_2(B) = 23, r_3(B) = 29, r_4(B) = 16, r_5(B) = 3, \tag{11}$$

$$r_i(B) = 0 \ \ \forall\, i \geq 6.$$

The coefficients $r_i(B), i \geq 1$, can be considered by the coefficients

$$c_i = a_0 b_i + a_1 b_{i-1} + \cdots + a_i b_0 \tag{12}$$

of a polynomial $\sum_{i=0}^{m} c_i x^i$, that was determined by a product of two polynomials

$$\sum_{i=0}^{n} a_i x^i, \ \ \sum_{i=0}^{n} b_i x^i, \ \ r_1(B_1) = a_{i,}\ r_i(B_2) = b_{i,} \tag{13}$$

$$a_i = 0 \ \ \forall\, i > m_1, b_i = 0 \ \ \forall\, i > m_2, \ \ m_1, m_2 \in \mathbf{N}.$$

Thus

$$r_{B_1}(x) = r_0(B_1) + r_1(B_1)x + \cdots r_{m_i}(B_1)x^{m_i}, \tag{14}$$

$$r_{B_2}(x) = r_0(B_2) + r_1(B_2)x + \cdots r_{m_i}(B_2)x^{m_i},$$

The polynomial

$$R_B(x) = r_0(B) + r_1(B)x + \cdots r_{m_i}(B)x^m \tag{15}$$

corresponds to the definition **of the rook polynomial** (1).

According to the inclusion and exclusion principle, we can prove the following theorem (for example [2]).

Theorem: The total number of permutations with limited conditions from *n* different objects is equal to the sum

$$p = \sum_{i=0}^{n} (-1)^i r_{\mathrm{i}}(B)(n-i)!, \tag{16}$$

where $r_{\mathrm{i}}(B)$ are coefficients of the rook polynomial $R_B(x)$ of the chess board B illustrating the limited conditions.

Going back to the example above, we have got: (17)

$$R_B(x) = (1+3x+x^2)(1+5x+7x^2+3x^3) = 1+8x+23x^2+29x^3+16x^4+3x^5$$

The **total different ways** of choosing drinks is equal to

$$p = \sum_{i=0}^{6} (-1)^i r_{\mathrm{i}}(B)(6-i)! = 1 \cdot 6! - 8 \cdot 5! + 23 \cdot 4! - 29 \cdot 3! + 16 \cdot 2! - 3 \cdot 1! = 167. \tag{18}$$

CONCLUSION

The theory of rook polynomials can be interesting for students and can add variety to combinatorics classes. In many cases, there are a lot of limiting conditions that can complicate the classical solution using permutations. The use of graph showing the moves of the rook on the chessboard makes the situation clearer and the solution to the problem can be more understandable for students. Then their solutions makes also simpler and shorter.

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FINDING AND FIXING ERRORS IN MATHEMATICAL EXAMPLES: CAN THIS FOSTER MATH LEARNING OUTCOMES?

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Abstract: *In our study we want to point out the difference in mathematical success when error analysis is included in teaching mathematics, compared to the traditional teaching approach only presenting the right solutions. We believe that explaining and justifying correct and incorrect solutions to problems is more beneficial for achieving better results in mathematics education than justifying the right solutions. Such a teaching process can lead to a more informal and better understanding of mathematical concepts. Our pedagogical experiment findings confirmed that the presence of cognitive conflict through exercises solved with errors, followed by reflection and explanation of errors, leads to a deeper understanding of mathematical concepts. Error analysis can support a deeper and more comprehensive understanding of mathematical content as well as the essence of mathematics itself.*

Keywords: mathematical education, potential of the errors, common errors in mathematics, pedagogical experiment

INTRODUCTION

The error plays an important, sometimes even essential role in the student's life and each person. We also understand it as a specific cultural and social value. Therefore, it is necessary to think about, describe, and identify the place and role of error in learning theory. The emotional perception of error in the Christian tradition opposes the rational perception of error in ancient culture – here, the error is perceived as a means for a more correct, consistent, and more profound knowledge of reality.

In our schools, the mistake or error is often perceived as an undesirable phenomenon, as something to be avoided, as something that both the teacher and the student are afraid of. However, the error understood in this way de–motivates (deactivates). Every failure or error in the teaching process can be productive for a person; it depends on the attitude taken in this situation. If mathematics teaching is understood only as the transfer of knowledge in the form of an explanation or lecture, the teacher must avoid any mistake - not sharing incorrect information (Kuřina, 2017). Any student's lack or error must be punished in such a case because he "failed to master the subject".

If we use a creative, interactive, constructive teaching process, errors are like milestones along the way. They point in the right direction when looking for solutions and provide us with the option to find the right results. Teaching is thus realized between two poles: Error cursed - error praised (Kuřina, 2017). In the introduction of our paper, we discuss how different teaching theories in the past understood errors in the learning process. Above all, we were interested in accepting the error as a positive, as a "potential for the student" in the future. We analyse different approaches of the teacher to the errors done by students.

1 LITERATURE REVIEW

In the professional literature, we find several studies on the use of error analysis in mathematics (Adamas, 2014; McLaren 2015). The study carried out for this article differs from previous studies in mathematical content, the number of teachers and students involved in the study, and online teaching. Loibl and Rummel (Loibl and Rummel, 2014) found that secondary school students became more aware of their knowledge gaps when analysing exercises with errors. Demonstrative comparisons of wrong-done tasks with correctly calculated tasks have filled learning gaps. Gadgil et al. (2012) conducted a study in which students who compared incorrectly solved tasks with correctly solved tasks gained a more remarkable ability to correct their errors than students who only received the correct procedures and problem-solving. This conclusion was subsequently supported by other researchers (Durkin et al., 2012; Kawasaki, 2010; Stark et al., 2011). Each of these researchers found students at all levels of mathematics education, from elementary school to secondary school students, who learned more than students who only faced the correct solutions of the task when analysing them and at the same time incorrect solutions to the task. This was particularly the case when the tasks with errors done were like the errors they made (Kawasaki, 2010; Stark et al. 2011) added that it is essential for students to be given sufficient explanation in well-designed examples before and in addition to erroneous tasks with errors.

Hejný (Hejný et al, 2004) perceives error as an element of the teacher's educational strategy and emphasizes the requirement to suppress the student's unwanted fear of error, requiring the teacher not to perceive error as an undesirable phenomenon. The error detection and process to solve it is divided into six phases:

- identification (error presence noted),
- error localization,
- factual analysis of the error (why the given idea is incorrect, or what is this wrong idea related to and with which other mathematical concepts it is connected),
- elimination of the error
- process analysis of the error (how this error occurred),
- forming the conclusion.

Research by Brown et al. (Brown et al., 2016) shows how analyzing student errors can help a teacher design an effective teaching approach to address student misconceptions and determine the correct concept, strategy, or procedure. Steps to perform error analysis:

- Data collection
- Identify error patterns
- Determine the causes of errors
- Choosing an educational strategy.

The study by Sovia and Herman (Sovia and Herman, 2019) also focused on slower learners and specifically on the analysis of arithmetic solution errors in elementary schools in Indonesia. The results of the study showed that four types of errors: comprehension errors (50%), transformational errors (8%), procedural skill errors (17%) and coding errors (25%). This study was intended to provide mathematics teachers with material for teaching practice for slow learning students in the study of mathematics. Research by Kyaruzi et al (Kyaruzi et al , 2020) investigated the impact of short-term teacher training on students' perceptions of their mathematics teacher's support for error situations as part of instruction, students' perceptions of error situations during learning, and mathematics teacher's actual error management practices in secondary schools in Tanzania . The results showed that mathematics teachers who received short-term professional development training appeared more error-friendly and used errors in teaching.

Research by Khasawneh et al. (Khasawneh et al., 2022) examined the effect of learning based on mathematical error analysis on the proportional reasoning ability of elementary school students in Jordan. The results of the study showed that learning based on error analysis led to a significant improvement in proportional reasoning and contributed to providing students with a positive experience in learning mathematics. In light of these results, a set of recommendations for educational researchers, mathematics curriculum developers, and mathematics teachers was presented. Zhao et al.'s (Zhao et al., 2022) work investigated the effect of high school mathematics teachers' error orientations on their emotions and how teachers' error orientations and emotions were related to students' mathematics learning strategies in China. The findings highlight the importance of teachers' positive error orientation and positive emotions for students' mathematics learning.

2 THE MOST COMMON ERRORS IN MATHEMATICS

This section describes the errors that we have frequently seen in undergraduate mathematics. At the beginning of each semester, we notify students of these "chronically recurring" errors. Unfortunately, we must state that the situation is not improving; on the contrary, it is getting worse. In addition, the last two years affected by the corona crisis have worsened the situation as well. In carrying out our experiment, we therefore began by identifying the most frequently recurring mathematical errors of secondary school graduates’ students. We were also interested in other countries' situations and processed information from Eric Schechter's website. (More than 500 teachers from different countries published their observations on errors in the subject of mathematics in school), Paul Cox's website, as well as publications by Bradis, Minkovsky and E.A. Maxwell. We divided errors made by students into several categories.

2.1 Communication errors

These negative aspects can be relatively quickly eliminated by the teacher with sufficient supervision and thus improve the quality of work. We register them in the teacher-student relationship (or vice versa, student-teacher). The teacher often perceives the student as the enemy, is not open to students' questions, and is more focused on mathematics than on the student (whether and how the student understands the explained subject matter). The hidden negative attitude of the teacher implies the student´s fear, their inability to ask questions, engage in fruitful discussion, and be an active member of the teaching process. The teacher is often tempted to communicate more with gifted or active students. Nevertheless, these are exactly the slower ones in need of our help. If we focus on students' facial expressions while teaching them, it is relatively easy to grasp their understanding (or misunderstanding) – from their facial expressions.

Many problems in teaching mathematics are also related to students' poor reading comprehension skills. In Slovakia, we have registered a significant reduction in pupils' and students' level of language culture in recent years (as evidenced by several researches within the OECD countries – PISA). Students often do not understand the context or do not read the tasks to the very end or are distracted and inconsistent when reading them. In addition, the language of mathematics uses, in addition to the general language, specific terminology, the language of formulas, algebra and requires an understanding of nonverbal expression using diagrams, graphs and figures.

In the category of communication errors it is also necessary to include problems related to the student's unreadable handwriting (the student does not understand, misread, own illegible notes or the teacher cannot read the content of the student's work).

2.2 Algebra errors

We can conclude that we register each of the errors we mention in this paragraph at all levels of mathematics education. Many of them are caused by the usual lack of attention or poor concentration of students at work. Sometimes it would be enough to count slower, with more focus on the task. Many errors could be avoided in this way. In general, we could divide these errors into errors at the primary level and errors caused by a lack of more profound theoretical knowledge. The errors are listed in a clear table indicating the source (cause) of the errors.

1	$\boldsymbol{-7x.(2x-4)=-14x^2-28x}$ $\boldsymbol{5(3x^2-20)=15x^2-20}$ $\boldsymbol{5x+15=5(x+15)}$ $\boldsymbol{\frac{1}{7x}=7x^{-1}}$	**The errors due to inattention** When expanding brackets: a standard error made is to multiply only out part of the brackets, error in sign, bad extraction before parenthesis.
2	$\boldsymbol{x^5.x^{-5}=0}$ $\boldsymbol{x+5x=6x^2}$ $\boldsymbol{\frac{25y^3}{5y^{-3}}=5y^{-6}}$ $\boldsymbol{(2x^4y^3)^5=32x^9y^8}$ $\boldsymbol{\sqrt{36x^9}=6x^3}$ $\boldsymbol{-4^2=16}$ $\boldsymbol{-(-x)^2=x^2}$	**Inability to correctly apply formulas for working with powers to specific tasks - formal knowledge.** The multiplying of the indices that should have been added and to divide indices that should have been subtracted, the subtracting the indices in the wrong order, the problems using integer and rational exponents.
3	$\boldsymbol{(3+x)^2=9+x^2}$ $\boldsymbol{\sqrt{4+x^2}=2+x^2}$ $\boldsymbol{3\cdot(4x-5)^2=(12x-15)^2}$ $\boldsymbol{x^2-16=(x-8)(x+8)}$ $\boldsymbol{x^2-16=x.(x-16)}$ $\boldsymbol{x^2-16=(x-4)^2}$ $\boldsymbol{\frac{2x^2+5x-3}{x+3}=\frac{(x+3).\left(x-\frac{1}{2}\right)}{x+3}}$	**The problems when calculating with formulas $\boldsymbol{(a\mp b)^2}$, $\boldsymbol{a^2-b^2}$** The formal perception of both formulas leads to an incorrect decomposition into the product, or to incorrect modifications within more extensive algebraic expressions, the errors in determining the roots of a quadratic equation.
4	$\boldsymbol{\frac{5x^4-x}{x}=5x^4-1}$ $\boldsymbol{\frac{2x\cdot(x+2)-(x^2+1)}{(x+2)}=2x-(x^2+1)}$	**Incorrect the abbreviation of the mathematical expressions** Here is an error that we have seen often,

	$\frac{1}{2}.(x+5)^{\frac{1}{2}} = x+5$ $\frac{x^3}{3} = x$	but we do not have a clear idea why students make it. We conclude that the student does not have sufficient practice in editing expressions, a formal procedure prevails (without a deeper understanding, a mechanically "learned" procedure)
5	$(x+y)^2 = x^2 + y^2$ $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ $\frac{1}{x^3+y^3} = \frac{1}{x^3} + \frac{1}{y^3}$ $sin(x+y) = sinx + siny$ $(f \cdot g)' = f' + g'$ $e^{x+y} = e^x + e^y$ $\int (f \cdot g) dx = \int f dx \cdot \int g dx$	**Formal generalization of the additive properties also to the expressions, that do not have this property** A formula or notation may work properly in one context, but some students try to apply it in the broader context, where it may not work correctly at all. Robin Chapman also calls this type of error "crass formalism." Here are some examples that from our own teaching experience.
	$sin^3 x = sinx^3$ $tg^{-1}x = \frac{1}{tgx}$ $tg^{-1}x = arctgx$ $(3^{sinx})' = sinx.(3^{sinx-1})$ $\int (sinx)^3 dx = \frac{(sinx)^4}{4} + C$ $\int \frac{1}{x} dx = \int x^{-1} dx = x^0 + C$ $(lncosx)' = -\frac{1}{sinx}$ $x^2 > 16 \Leftrightarrow x > 4$	**The various errors, the source of which is insufficient knowledge and understanding of the mathematical theory so necessary for the correct calculation of examples.** Here we list some of the most common mistakes that students made on the Mathematical Analysis exam in the 1st year of bachelor's studies. We see that their source is clearly ignorance of mathematical theory.

Table. 1. The most common errors occurring in mathematical calculations

2.3 The most common mistakes in solving quadratic equations

Too many students get used to just canceling out(i.e., simplifying) things to make their life easier. So, the biggest mistake in solving this kind of equation

$$3x^2 = 6x$$

is to cancel out an x from both sides to get

$$3x = 6$$
$$x = 2.$$

We missed the x = 0 in this attempt because we tried to make our life easier by "simplifying" the equation before solving it. While some simplification is a good and necessary thing, we should NEVER divide out a term as we did in the first attempt when solving. If we do this, we WILL lose solutions. The second chronically recurring error is the incorrect use of the formula $\sqrt{x^2} = |x|$. To solve an example $x^2 - 81 = 0$ we very often get the wrong solution $x = 9$. Again, one of the roots of the equation was lost.

3 MATERIALS AND METHODS

In the practical part of the article, we present the results of an experiment conducted among students of the 1st year of the The Faculty of Operation and Economics of Transport and Communications at the University of Žilina in 2022. A randomly selected group of 103 repondents from among 850 students of the 1st year of study were given a test aimed at identifying errors when solving problems from high school mathematics. Our goal was to find out the level of students' knowledge after the almost two-year absence of face-to-face teaching due to the Covid pandemic. We also wanted to map which areas of the mathematics curriculum are the most problematic for the students, and then target the tutoring course on these topics. We reveal the causes (roots) of students' formal and incorrect understanding of mathematical concepts. We present and analyse the results of the diagnostic test and look for ways to eliminate them.

The pedagogical experiment evidenced the participation of, in total, 103 students of specialization in the transport of the first year of the faculty PEDAS of ŽU in Žilina. In the first week of the semester, the students passed a diagnostic test. That consisted of 21 solved tasks of secondary school mathematics. Their task was to evaluate the correctness (incorrectness) of solving each of the twenty-one assigned tasks. They received 1 point for each correct statement. The maximum number of possible points was 21. The evaluation of the test reflects which errors are most common among students and which areas need to be deeper and more precisely focused on when teaching university mathematics.

We also analyzed the students' ability to correct the errors found. Therefore, we also used an extended evaluation of the test: In case the student "found an error", he had the task of writing the correct result of the assignment in the free column. The assessment was: x - incorrect answer, 0 - correct answer (the assignment was calculated correctly), 00 - correctly evaluated error in the solution and subsequently correctly corrected the assignment, 0x - correctly evaluated error in the solution and subsequently incorrectly corrected assignment.

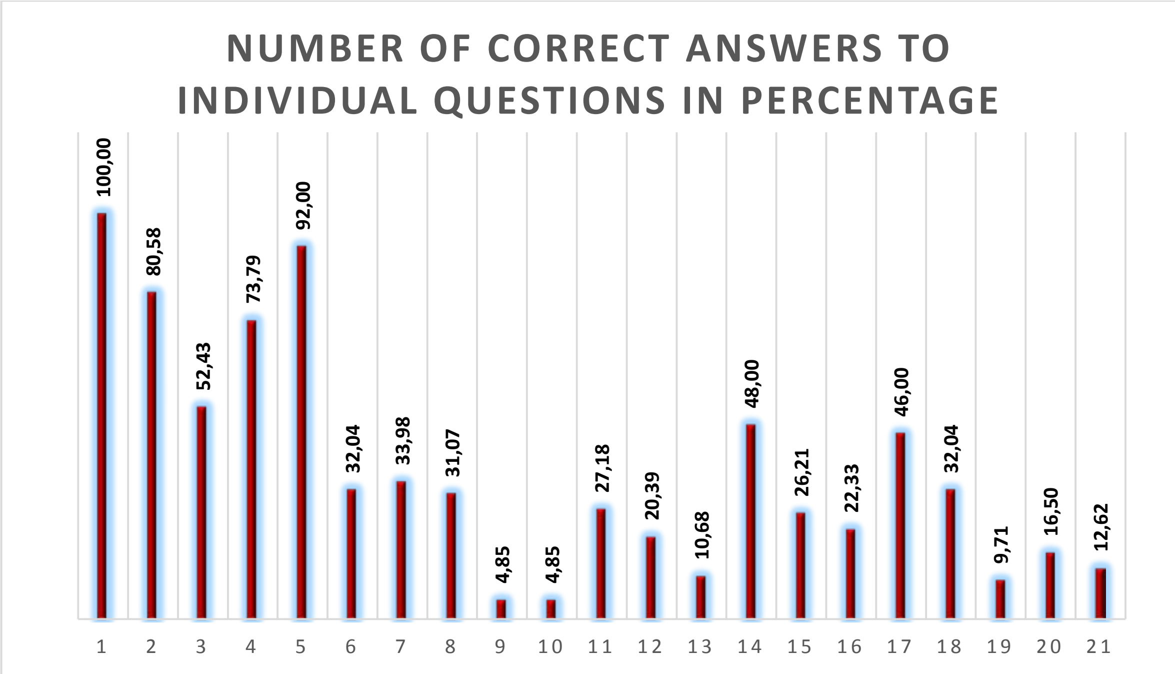
The most successful student received 21 points and the worst achieved score was 2 points. Figure 1 presents a blank test sheet. The average number of points obtained by individual students is 7,98 points.

	Example	Right OK Wrong X	fill in the correct answer
1	$4(2-5x)=8-20x$		
2	$-7x.(2x-4)=-14x^2-28x$		
3	$x^5.x^{-5}=0$		
4	$x+5x=6x^2$		
5	$4x^2.10x^3=40x^5$		
6	$\frac{25y^3}{5y^{-3}}=5y^{-6}$		
7	$(2x^4y^3)^5=32x^9y^8$		
8	$(3+x)^2=9+x^2$		
9	$\sqrt{(4+x^4)}=2+x^2$		
10	$\sqrt{36x^9}=6x^3$		
11	$-4^2=16$		
12	$\sqrt{11x^6}=11x^3$		
13	$\frac{1}{7x}=7x^{-1}$		
14	$\frac{3}{2x^5}=\frac{3}{2}x^{-5}$		
15	$3\cdot(4x-5)^2=(12x-15)^2$		
16	$\frac{1}{2}\cdot(x+5)^{\frac{1}{2}}=x+5$		
17	$\frac{4}{(x-5)^{-3}}=4(x-5)^3$		
18	$\frac{5x^4-x}{x}=5x^4-1$		
19	$\frac{(2x+3)\cdot(x+2)-(x^2+1)}{(x+2)}=(2x+3)-(x^2+1)$		
20	$x.e^x+5e^x-2e^{2x}=e^x(x+5-2e^2)$		
21	$\frac{x^3}{3}=x$		

Fig. 1. Diagnostic test
Source: own

4 FINDINGS AND DISCUSSION

We analyzed and evaluated the obtained results on two levels. On one hand, we evaluated the success of solving individual types of the tasks and on the other hand, the obtained scores of the students. Graph no. 1 presents the results with which the students solved the individual assignments. The success rate for each example is expressed in percentages.



Graph 1. Diagnostic test
Source: own

As we can see, tasks 9 and 10 turned out the worst. More than 95% of the students commit a very frequent error, in which they „tear the square root" ($\sqrt{x^2+4} = x+2$), and thus perform an unauthorized modification of the expression. It testifies to formal knowledge and ignorance of the rules for working with the powers and square roots. The errors of the type 13 follow $\left(\frac{1}{7x} = 7.x^{-1}\right)$. The student has a problem correctly perceiving a constant in front of a function if it is less than one. The students struggle with these problems primarily in derivation and integration, where it is necessary to rewrite the power function using a rational exponent. The respondents made the fewest errors when multiplying algebraic expressions and arithmetic operations of the expressions with powers with a natural exponent - that is, with basic knowledge of algebra. The analysis of the mentioned test gives us useful feedback. We have information about which algebraic adjustments require more time to explain the correct procedures. The students themselves perceived such a "repetition" of high school mathematics as appropriate and necessary. Especially after two pandemic years, we are registering deeper deficiencies in basic knowledge.

What have we registered? It is not possible to replace intensive training – standard calculation, acquisition of systematic intensive training of work with algebraic expressions. To master mathematics, it is unthinkable to constantly cut math lessons. There were new mistakes, the origin of which is in the formal understanding of mathematical concepts, a deeper understanding and, above all, perception of the mutual connections of concepts is lacking. The "superficiality" of knowledge is literally readable from the occurring errors, the student does not know why the operations are performed as it is presented. It is not connected with a general formula, or on the contrary, cannot turn an abstract general mathematical formula into practice.

For further data analysis, we also used the results of our experiment carried out in the academic year 2019-2020. A randomly selected group of 99 respondents, students of the 1st

year of The Faculty of Operation and Economics of Transport and Communications at the University of Žilina, was back then presented with the same diagnostic test. These students were not affected by the period of the Covid-19 pandemic during their studies, they thus completed full-time education.
We therefore focused on testing hypothesis: *Students educated in a full-time format (in 2020) achieved better results in solving the diagnostic test than students educated in a distance format (in 2022).*

Table 2 shows the descriptive statistics of the 2020 and 2023 ensembles according to the average number of points obtained by individual students.

Groups	Count	Average	Variance
The test results in 2020	99	10,31	10,99
The test results in 2023	103	7,98	15,54

Table 2. Test results by year

To test the hypothesis, we perform a paired t-test. The observed characters are the characters X, Y, where X indicates the success of the correct solution in percentages for the individual tasks of the diagnostic test in 2020 and Y the success in solving the individual tasks in 2023. Thus, we have data

$x_i = 100; 80{,}6; 52{,}4; 73{,}8; 92; 32; 34; 31; 4{,}9; 4{,}9; 27{,}2; 20{,}4; 10{,}7; 48; 26{,}2; 22{,}3; 46; 32; 9{,}7; 16{,}5; 12{,}6$
$y_i = 100; 90{,}6; 60{,}4; 75{,}8; 93; 25{,}2; 44{,}5; 9{,}1; 11; 35{,}2; 25{,}4; 20{,}3; 52; 61{,}2; 25{,}3; 48{,}1; 30; 11{,}3; 15; 20{,}2$

We tested the hypothesis of equality of mean values μ_1, μ_2 with a paired test against the two-sided alternative hypothesis at the level of significance $\alpha = 0{,}05$. The tested problem has the form

$$H_0\colon \mu_1 = \mu_2 \qquad \text{versus} \qquad H_0\colon \mu_1 \neq \mu_2$$

The value of test statistics is $t = -2,11$ and p = 0,004. H_0 hypothesis was rejected. The averages of both samples on the selected significance level differs. When using the stated teaching methods, different results in diagnostic test were obtained.

Unfortunately, the obtained results confirmed the deterioration of the mathematical knowledge of secondary school students after the pandemic period. It turns out that even the maximum effort of the teacher in the distance education of mathematics cannot replace the personal contact with the teacher in this education. Especially mathematics is a subject where the teacher's face-to-face explanation and clarification of concepts is indispensable.

CONCLUSION

Error analysis is a very important tool for a teacher. It helps determine what mistakes the student makes and why they make them. Using error analysis, a teacher can identify either deficiencies in a student's skills or their misconceptions about how to solve problems. Students' mathematical errors are related either to the student's lack of knowledge or to a misunderstanding of the problem. However, sometimes the student makes mistakes due to inattention or fatigue. In conclusion, we can state that we were pleasantly surprised by the feedback from the students, which was primarily positive. The students stated that the group discussion and the analysis of errors in the tests helped them to correctly understand some mathematical operations that they had previously done mechanically and without a deeper understanding. Students also noted that error analysis has more pros than cons. Analyzing solutions with errors gave students the opportunity to be more involved in the discussion, "explaining" and correcting the errors of the presented task and their own errors, which were activities that increased their interest in the learning process. The error acted as a specific "element of surprise" in teaching; such tasks interested them, motivated them more. An open discussion about solutions revealed the causes of errors and gave us many clues on how to work with students in the future.

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Some Application in the Theory of the Interval Analysis in Algebraic Hyperstructures

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Abstract: *The contribution studies hyperstructions, carrier sets of which are formed by chains of ordered groups of linear ordinary differential operators. Constructions presented in this contribution, are based on classical concepts of interval analysis applied on the ordered additive monoid of all open bounded intervals of real numbers.*

Keywords: linear differential operator, open interval, interval analysis.

INTRODUCTION

Methods of the Interval Analysis (briefly called also "Interval Mathematics") are a large and important part of contemporary mathematics. Many papers on this topic were published in 1970s and 1980s, however some concepts of the theory had appeared earlier. Methods and ideas of Interval Analysis are very constructive and useful, especially in applied mathematics. The Interval Mathematics includes e.g. the interval arithmetic, the interval analysis, the interval algebra, the interval topology, interval solutions of differential equations, solutions of problems of numerical mathematics including the error analysis and the other areas – [11, 13, 15]. In this contribution some ideas of functional analysis and theory of linear differential operators are included and applied. Papers devoted to differential operators using intervals of functions of one real variable include [4]. It is to be noted that the classical Interval Mathematics considers intervals, also called segments, which are bounded as well as unbounded – [13]. Our paper deals with open bounded intervals of the set $\mathbb{R}$ of all real numbers, which is a substantial modification of the classical approach. The foundations of interval arithmetic were laid by R.E. Moore, the topic first appeared in 1959, for more information see for example [15]. Paper [15] contains algebraic structures of intervals, especially lattices.

1 Preliminaries

A binary hypergroupoid is a non-empty set H endowed with a multivalued operation (called also hyperoperation), which is a function mapping pairs (x,y) of elements of H to a non empty subset $x \circ y$ of H. If this hyperoperation "$\circ$" is associative, i.e., $(x \circ y) \circ z = x \circ (y \circ z)$ for any triad of elements $x,y,z \in H$ (where we define $x \circ M = \bigcup_{y \in M} x \circ y$ for any $x \in H$ and any non empty subset $M \subset H$), the hypergroupoid $(H,\circ)$ is called a semihypergroup. If a semihypergroup $(H,\circ)$ satisfies the reproduction axiom, i.e., if there is

$$x \circ H = H = H \circ x$$

for each element $x \in H$, the corresponding semihypergroup is called hypegroup. It is easy to see that the reproduction axiom is equivalent to the following condition: For any pair $a,b \in H$ there exists a pair $x,y \in H$ such that $b \in a \circ x$, $b \in y \circ a$. Notice that first paper on hypergroups appeared in 1934, written by the French mathematician Marty, discussed some properties of hypergroups and applied these to groups and algebraic functions. The algebraic theory of hypergroups has been investigated in many countries including the Czech Republic; see e.g. [3, 5, 6, 9, 10, 17, 18, 19, 20, 21, 22].

In our contribution we also deal with multiautomata considered as actions of hypergroupoids on sets defined using so called Generalized Mixed Associativity Condition. Recall that by a quasi-ordered set we mean a set endowed with a binary relation which is reflexive and transitive. If this relation is moreover antisymmetric, we speak about an ordered set. By an ordered semigroup we mean a triad $(S,\cdot,\leq)$, where $(S,\cdot)$ is a semigroup on the set S and the ordering $\leq$ on the set S has a substitution property on $(S,\cdot)$, i.e., if for any quadruple of elements a,b,c,d, such that $a \leq b$, $c \leq d$, there is $a \cdot c \leq b \cdot d$.

By a group we mean a classical structure, i.e., a non-empty set endowed with an associative binary operation with the neutral element and with uniquely defined inverse element to any element of the group. The interval arithmetic is taken from [13] and also from papers [11, 12, 15].

In what follows we use interval constructions – this theory is mentioned as important part of applied mathematics. Algebraic investigations of posets of interval are well-known from literature – see e.g. [13, 15] and further papers included in references of the mentioned paper.

2 Groups of differential operators ordered in chains

A certain motivation for the study of sequences of hypergroups and their homomorphisms can be traced to ideas of classical homological algebra which comes from the algebraic description of topological spaces. A homological algebra assigns to any topological space a family of abelian groups and to any continuous mapping of topological spaces a family of group homomorphisms. This allows us to express properties of spaces and their mappings (morphisms) by means of properties of groups or modules or their homomorphisms. Notice that a substantial part of homology theory is devoted to the study of exact short and long sequences of the above mentioned structures.

It is crucial that one understands the notation used in this paper. Recall that we study, by means of algebra, linear ordinary differential operators. Therefore, our notation, which follows the original model of Borůvka and Neuman, uses a mix of algebraic and functional notation.

First, we denote intervals by J and regard open intervals (bounded or unbounded). Systems of functions with continuous derivatives of order k on J are denoted by $\mathbb{C}^k(J)$; for $k = 0$ we write $\mathbb{C}(J)$ instead of $\mathbb{C}^0(J)$. We treat $\mathbb{C}^k(J)$ as a ring with respect to the usual addition and multiplication of functions. We denote by δ_{ij} the Kronecker delta, $i, j \in \mathbb{N}$, i.e., $\delta_{ii} = \delta_{jj} = 1$ and $\delta_{ij} = 0$, whenever $i \neq j$; by $\overline{\delta_{ij}}$ we mean $1 - \delta_{ij}$. Since we will be using some notions from the theory of hypercompositional structures, recall that by $\mathscr{P}(X)$ (sometimes denoted as $expX$) one means the power set of X while $(\mathscr{P})^*(X)$ means $\mathscr{P}(X) \setminus \emptyset$.

We regard linear homogeneous differential equations of order $n \geq 2$ with coefficients, which are real and continuous on J, and – for convenience reasons – such that $p_0(x) > 0$ for all $x \in J$, i.e., equations

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = 0. \tag{1}$$

By $\mathbb{A}_n$ we, adopting the notation of Neuman [16], mean the set of all such equations.

Example 1. *The above notation can be explained on an example taken from [16], in which Neuman considers the third-order linear homogeneous differential equation*

$$y'''(x) - \frac{q_1'(x)}{q_1(-x)}\, y''(x) + (q_1(x) - 1)\, y'(x) - \frac{q_1'(x)}{q_1(x)} y(x) = 0.$$

on the open interval $J \in \mathbb{R}$. One obtains this equation from the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + q_1(x)y_3 \\ y_3' &= -q_1(x)y_2 \end{aligned}$$

Here $q_1 \in \mathbb{C}^+(J)$ satisfies the condition $q_1(x) \neq 0$ on J. In the above differential equation we have $n = 3$, $p_0(x) = -\frac{q_1'(x)}{q_1(x)}$, $p_1(x) = (q_1(x) - 1)^2$ and $p_2(x) = -\frac{q_1'(x)}{q_1(-x)}$. It is to be noted that the above three equations form what is known as set of global canonical forms for the third-order differential equation on the interval J.

Denote $L_n(p_{n-1}, \ldots, p_0) : \mathbb{C}^n(J) \to \mathbb{C}^n(J)$ the above linear differential operator defined by

$$L_n(p_{n-1}, \ldots, p_0)y(x) = y^{(n)}(x) + \sum_{k=0}^{n-1} p_k(x)y^{(k)}(x), \tag{2}$$

where $y(x) \in \mathbb{C}^n(J)$ and $p_0(x) > 0$ for all $x \in J$. Further, denote by $\mathbb{LA}_n(J)$ the set of all such operators, i.e.,

$$\mathbb{LA}_n(J) = \{L(p_{n-1}, \ldots, p_0) | p_k(x) \in \mathbb{C}(J), p_0(x) > 0\}. \tag{3}$$

By $\mathbb{LA}_n(J)_m$ we mean subsets of $\mathbb{LA}_n(J)$ such that $p_m \in \mathbb{C}^+(J)$, i.e., there is $p_m(x) > 0$ for all $x \in J$. If we want to explicitly emphasize the variable, we write $y(x), p_k(x)$, etc. However, if there is no specific need to do this, we write y, p_k, etc. Using vector notation $\vec{p}(x) = = (p_{n-1}(x), \ldots, p_0(x))$, we can write

$$L_n(\vec{p})y = y^{(n)} + \sum_{k=0}^{n-1} p_k y^{(k)}. \tag{4}$$

Writing $L(\vec{p}) \in \mathbb{LA}_n(J)$ (or $L(\vec{p}) \in \mathbb{LA}_n(J)_m$) is a shortcut for writing $L_n(\vec{p})y \in \mathbb{LA}_n(J)$ (or, $L_n(\vec{p})y \in \mathbb{LA}_n(J)_m$).

On the sets of linear differential operators, i.e., on sets $\mathbb{LA}_n(J)$, or their subsets $\mathbb{LA}_n(J)_m$, we define some binary operations, hyperoperations or binary relations. This is possible because our considerations happen within a ring (of functions).

For an arbitrary pair of operators $L(\vec{p}), L(\vec{q}) \in \mathbb{LA}_n(J)_m$, where $\vec{p} = (p_{n-1}, \ldots, p_0), \vec{q} = (q_{n-1}, \ldots, q_0)$, we define an operation "$\circ_m$" with respect to the m-th component by $L(\vec{p}) \circ_m L(\vec{q}) = L(\vec{u})$, where $\vec{u} = (u_{n-1}, \ldots, u_0)$ and

$$u_k(x) = p_m(x)q_k(x) + (1 - \delta_{km})p_k(x) \tag{5}$$

for all $k = n-1, \ldots, 0$, $k \neq m$ and all $x \in J$. Obviously, such an operation is not commutative.

Moreover, apart from the above binary operation we can define also a relation "$\leq_m$" comparing the operators by their m-th component, putting $L(\vec{p}) \leq_m L(\vec{q})$ whenever, for all $x \in J$, there is

$$p_m(x) = q_m(x) \text{ and at the same time } p_k(x) \leq q_k(x) \tag{6}$$

for all $k = n-1, \ldots, 0$. Obviously, $(\mathbb{LA}_n(J) \leq_m)$ is a partially ordered set. At this stage, in order to simplify the notation, we write $\mathbb{LA}_n(J)$ instead of $\mathbb{LA}_n(J)_m$ because the lower index m is kept in the operation and relation. The following lemma is proved in [4].

***Lemma* 1.** *Triads $(\mathbb{LA}_n(J), \circ_m, \leq_m)$ are partially ordered (noncommutative) groups.*

Now we can use Lemma 1 to construct a (noncommutative) hypergroup. In order to do this, we will need the following lemma, known as the Ends lemma; for details see e.g., [18]. Notice that a join space is a special case of a hypergroup–in this paper we speak of hypergroups because we want to stress the parallel with groups.

***Lemma* 2.** *Let $(H, \cdot, \leq)$ be a partially ordered semigroup. Then $(H, *)$, where $* : H \times H \to \mathscr{P}^*(H)$ is defined, for all $a, b \in H$ by*

$$a * b = [a \cdot b)_{\leq} = \{x \in H | a \cdot b \leq x\},$$

is a semihypergroup, which is commutative if and only if "$\cdot$" is commutative. Moreover, if $(H, \cdot)$ is a group, then $(H, *)$ is a hypergroup.

Thus, to be more precise, defining

$$\star_m : \mathbb{LA}_n(J) \times \mathbb{LA}_n(J) \to \mathscr{P}(\mathbb{LA}_n(J)), \tag{7}$$

by

$$L(\vec{p}) \star_m L(\vec{q}) = \{L(\vec{u}) | L(\vec{p}) \circ_m L(\vec{q}) \leq_m L(\vec{u})\} \tag{8}$$

for all pairs $L(\vec{p}), L(\vec{q}) \in \mathbb{LA}_n(J)_m$, lets us state the following lemma.

***Lemma* 3.** *Triads $(\mathbb{LA}_n(J), \star_m, \leq_m)$ are (noncommutative) hypergroups.*

Notation 1. *Hypergroups $(\mathbb{LA}_n(J), \star_m)$ will be denoted by $\mathbb{HLA}_n(J)_m$ for an easier distinction.*

Remark 1. *As a parallel to (2) and (3) we define*

$$\overline{L}(q_n,\dots,q_0)y(x)=\sum_{k=0}^{n} q_k(x)y^{(k)}(x), q_0\neq 0, q_k\in\mathbb{C}(J) \tag{9}$$

and

$$\overline{\mathbb{LA}}_n(J)=\{(q_n,\dots,q_0)\mid q_0\neq 0, q_k(x)\in\mathbb{C}(J)\} \tag{10}$$

and, by defining the binary operation "$\circ_m$" and "$\leq_m$" in the same way as for $\mathbb{LA}_n(J)_m$, it is easy to verify that also $\overline{\mathbb{LA}}_n(J)$ are noncommutative partially ordered groups. Moreover, given a hyperoperation defined in a way parallel to (8), we obtain hypergroups $(\overline{\mathbb{LA}}_n(J)_m,\star_m)$, which will be, in line with Notation 1, denoted $\overline{\mathbb{HLA}}_n(J)_m$.

Now, we will construct certain mappings between groups or hypergroups of linear differential operators of various orders. The result will have a form of sequences of groups or hypergroups.

Define mappings $F_n:\mathbb{LA}_n(J)\to\mathbb{LA}_{n-1}(J)$ by

$$F_n(L(p_{n-1},\dots,p_0))=L(p_{n-2},\dots,p_0)$$

and $\phi_n:\mathbb{LA}(J)\to\overline{\mathbb{LA}}_{n-1}(J)$ by

$$\phi_n:(L(p_{n-1},\dots,p_0))=\overline{L}(p_{n-2},\dots,p_0).$$

It can be easily verify that both F_n and ϕ_n are, for an arbitrary $n\geq 2$, group homomorphisms.

Evidently, $\mathbb{LA}_n(J)\subset\overline{\mathbb{LA}}_n(J),\overline{\mathbb{LA}}_{n-1}(J)\subset\overline{\mathbb{LA}}_n(J)$ for all admissible $n\in\mathbb{N}$. Thus we obtain two complete sequences of ordinary linear differential operators with linking homomorphisms F_n and ϕ_n :

$$\begin{tikzcd}
\overline{\mathbb{LA}}_0(J) \arrow[r, "\overline{\mathrm{id}}_{0,1}"] & \overline{\mathbb{LA}}_1(J) \arrow[r, "\overline{\mathrm{id}}_{1,2}"] & \overline{\mathbb{LA}}_2(J) \arrow[r, "\overline{\mathrm{id}}_{2,3}"] & \dots \\
\mathbb{LA}_0(J) \arrow[u, "\mathrm{id}_0"'] & \mathbb{LA}_1(J) \arrow[l, "F_1"'] \arrow[u, "\mathrm{id}_1"'] \arrow[ul, "\phi_1"'] & \mathbb{LA}_2(J) \arrow[l, "F_2"'] \arrow[u, "\mathrm{id}_2"'] \arrow[ul, "\phi_2"'] & \dots \arrow[l, "F_3"'] \arrow[ul, "\phi_3"']
\end{tikzcd} \tag{11}$$

$$\begin{tikzcd}
\dots\overline{\mathbb{LA}}_{n-2}(J) \arrow[r, "\overline{\mathrm{id}}_{n-2,n-1}"] & \overline{\mathbb{LA}}_{n-1}(J) \arrow[r, "\overline{\mathrm{id}}_{n-1,n}"] & \overline{\mathbb{LA}}_n(J) \arrow[r, "\overline{\mathrm{id}}_{n,n+1}"] & \dots \\
\dots\mathbb{LA}_{n-2}(J) \arrow[u, "\mathrm{id}_{n-2}"'] & \mathbb{LA}_{n-1}(J) \arrow[l, "F_{n-1}"'] \arrow[u, "\mathrm{id}_{n-1}"'] \arrow[ul, "\phi_{n-2}"'] & \mathbb{LA}_n(J) \arrow[l, "F_n"'] \arrow[u, "\mathrm{id}_n"'] \arrow[ul, "\phi_n"'] & \dots \arrow[l, "F_{n+1}"'] \arrow[ul, "\phi_{n+1}"']
\end{tikzcd}$$

where $\overline{id}_{k,k+1}$, id_k are corresponding inclusion embeddings.

Notice that this diagram, presented at the level of groups, can be lifted to the level of hypergroups. In order to do this, one can use Lemma 3 and Remark 1. However, this is not enough. Yet, as Lemma 4 suggests, it is possible to show that the below presented assigning is functorial, i.e., not only objects are mapped onto objects but also morphisms (isotone group homomorphisms) are mapped onto morphisms (hypergroup homomorphisms).

Lemma 4. *Let $(G_k,\cdot_k,\leq_k), k=1,2$ be preordered groups and $f:(G_1,\cdot_1,\leq_1)\to(G_2,\cdot_2,\leq_2)$ a group homomorphism, which is isotone, i.e., the mapping $f:(G_1,\leq_1)\to(G_2,\leq_2)$ is order-preserving. Let $(H_k,*_k), k=1,2$ be hypergroups constructed from $(G_k,\cdot_k,\leq_k), k=1,2$ by Lemma 2, respectively. Then $f:(H_1,*_1)\to(H_2,*_2)$ is a homomorphism, i.e., $f(a*_1 b)\subseteq f(a)*_2 f(b)$ for any pair of elements $a,b\in H_1$.*

Proof. There is more reasoning in [6]. □

Consider a sequence of partially ordered groups of linear differential operators

$$\begin{aligned}\mathbb{LA}_0(J) &\xleftarrow{F_1} \mathbb{LA}_1(J) \xleftarrow{F_2} \mathbb{LA}_2(J) \xleftarrow{F_3} \cdots \\ &\cdots \xleftarrow{F_{n-2}} \mathbb{LA}_{n-2}(J) \xleftarrow{F_{n-1}} \mathbb{LA}_{n-1}(J) \xleftarrow{F_n} \mathbb{LA}_n(J) \xleftarrow{F_{n+1}} \mathbb{LA}_{n+1}(J) \leftarrow \cdots\end{aligned}$$

given above with their linking group homomorphisms $F_k : \mathbb{LA}_k(J) \to \mathbb{LA}_{k-1}(J)$ for $k = 1,2,\ldots$ Since mappings $F_n : \mathbb{LA}_n(J) \to \mathbb{LA}_{n-1}(J)$, or rather

$$F_n : (\mathbb{LA}_n(J), \circ_m, \leq_m) \to (\mathbb{LA}_{n-1}(J), \circ_m, \leq_m),$$

for all $n \geq 2$, are group homomorphisms and obviously mappings isotone with respect to corresponding orderings, we immediately get the following theorem.

Theorem 1. *Suppose $J \subseteq \mathbb{R}$ is an open interval, $n \in \mathbb{N}$ is an integer $n \geqq 2, m \in \mathbb{N}$ such that $m \leqq n$. Let $(\mathbb{HLA}_n(J)_m, *_m)$ be the hypergroup obtained from the group $(\mathbb{LA}_n(J)_m, \circ_m)$ by Lemma 2. Suppose that $F_n : (\mathbb{LA}_n(J)_m, \circ_m) \to (\mathbb{LA}_{n-1}(J)_m, \circ_m)$ are the above defined surjective group-homomorphisms, $n \in \mathbb{N}, n \geqq 2$. Then $F_n : (\mathbb{HLA}_n(J)_m, *_m) \to \mathbb{HLA}_{n-1}(J)_m, *_m)$ are surjective homomorphisms of hypergroups.*

Proof. See the reasoning preceding the theorem (also in [8]). □

Chains from the second row of (11) will be briefly denoted by $\langle \mathbb{LA}_n(J), F_n; n \in \mathbb{N}_0 \rangle$.

Remark 2. *It is easy to see that the second sequence from* (11) *can be mapped onto the sequence of hypergroups*

$$\begin{aligned}\mathbb{HLA}_0(J)_m &\xleftarrow{F_1} \mathbb{HLA}_1(J)_m \xleftarrow{F_2} \mathbb{HLA}_2(J)_m \xleftarrow{F_3} \cdots \\ &\cdots \xleftarrow{F_{n-2}} \mathbb{HLA}_{n-1}(J)_m \xleftarrow{F_{n-1}} \mathbb{HLA}_n(J)_m \leftarrow \cdots\end{aligned}$$

This mapping is bijective and the linking mappings are surjective homomorphisms F_n,. Thus this mapping is functorial.

3 Classical mathematical structures using for constructions of hyperstructures

Denote by $\mathfrak{I}_n(\mathbb{R})$ the set of all open bounded intervals $(\underline{a}, \overline{a}) \subseteq \mathbb{R}$ (cf. denoting that is used in the Interval Analysis – [11, 12, 13, 15]) and denote $\mathfrak{I}n_0(\mathbb{R}) = \mathfrak{I}_n(\mathbb{R}) \cup \{0\}$. As has been mentioned above, elements of the interval arithmetic can be taken from titles [11, 12, 13, 15]. So, it is well known from elementary real analysis that if a,b,c,d are real numbers such that $a < b, c < d$, then

$$(a,b) + (c,d) = (a+c, b+d),$$

moreover, if $0 \leq a < b$ and $0 \leq c < d$, then

$$(a,b) \cdot (c,d) = (ac, bd).$$

Evidently, the groupoid $(\mathfrak{I}n_0(\mathbb{R}), +)$ is a commutative monoid with the neutral element $0 \in \mathbb{R}$.

We describe a certain hyperstructural constructure based on centres of intervals. First, let us consider an interval $J_0 \in \mathfrak{I}_n(\mathbb{R}), J_0 = (a,b), a < b,$ where $a,b \in \mathbb{R}$. The center of this interval

J_0 is the number $m(J_0) = m(a,b) = \frac{1}{2}(a+b)$. Consider the following special system $\mathscr{S}(J_0)$ of subintervals of $J_0 = (a,b)$:

$$\mathscr{S}(J_0) = \{(c,d); [c,d] \in (a, m(a,b)) \times (m(a,b), b)\} \cup \{J_0\}.$$

For any bounded subset $M \subseteq \mathbb{R}$ we denote by $Covin^+(M)$ the intersection of all bounded intervals covering the set M, i.e.,

$$Covin^+(M) = \bigcap\{J; J \in \mathfrak{In}(\mathbb{R}), M \subset J\}.$$

We define a mapping

$$\Phi_0 : \mathfrak{In}(\mathbb{R}) \times \mathfrak{In}(\mathbb{R}) \to exp\mathfrak{In}(\mathbb{R})$$

by the rule $\Phi_0(K,L) = \{Covin^+(K \cup L \cup S); S \in \mathscr{S}(J_0)\}$ for any pair of intervals $K, L \in \mathfrak{In}(\mathbb{R})$. Now we define a binary hyperoperation

$$*_0 : \mathbb{CLA} \times \mathbb{CLA} \to exp\mathbb{CLA}$$

by

$$\langle \mathbb{LA}_n(K), F_n; n \in N_0 \rangle *_0 \langle \mathbb{LA}_n(L), F_n; n \in N_0 \rangle = \{\langle \mathbb{LA}_n(I), F_n; n \in N_0 \rangle; I \in \Phi_0(K,L)\}$$

for any pair of chains $\langle \mathbb{LA}_n(K), F_n; n \in N_0 \rangle$, $\langle \mathbb{LA}_n(L), F_n; n \in N_0 \rangle$ belonging to the system $\mathbb{CLA}$. Evidently we have obtained

Proposition 1. *The hyperstructure $(\mathbb{CLA}, *_0)$ is a commutative hypergroupoid.*

In the monograph [3, chap. IV, p. 150, Theorem 2.1], there is presented and proved the following theorem:

Theorem 2.

1° Let (G,R) be a quasi–ordered set. For any pair $a, b \in G$ define

$$a *_R b = R(a) \cup R(b) = R(\{a,b\}).$$

Then $(G, *_R)$ is an extensive commutative hypergroup.

2° Let (G,R), (H,S) be quasi–ordered sets, $f : (G,R) \to (H,S)$ be isotone mapping. Then $f : (G, *_R) \to (H, *_S)$ is a hyperhomomorphism.

Proof.

1° From the reflexivity of the quasi–ordering R there follows immediately $a, b \in a *_R b$ for every pair $a, b \in G$, thus $a *_R b \neq \emptyset$, the hyperoperation $*_R$ is extensive and the commutativity of this hyperoperation is also evident. We verify associativity:

Suppose $a, b, c \in G$. There holds

$$a *_R (b *_R c) = a *_R (R(b) \cup R(c)) = a *_R R(\{b,c\}) = \bigcup_{x \in R(\{b,c\})} a *_R x =$$

$$= R(a) \cup \bigcup_{x \in R(\{b,c\})} R(x) = R(\{b,c\}) \cup R(a) = R(\{a,b,c\}) = c *_R (b *_R a) = (a *_R b) *_R c.$$

Further, for each $a \in G$ there holds

$$a *_R G = \bigcup_{x \in G} a *_R x = R(a) \cup \bigcup_{x \in G} R(x) = G,$$

thus the hyperoperation satisfies the reproduction condition, consequently $(G, *_R)$ is an extensive commutative hypergrupoid.

2° If $f : (G,R) \to (H,S)$ is an isotone mapping then for any pair of elements $a, b \in G$ we have $f(a *_R b) = f(R(a) \cup R(b)) = f(R(a)) \cup f(R(b)) \subseteq \mathscr{S}(f(a)) \cup \mathscr{S}(f(b)) = f(a) *_{\mathscr{S}} f(b)$. □

Now, let us consider the following concrete interval $J = (0,2)$. We assigne $J_k = (k, k+2)$, $k \in \mathbb{Z}$ and moreover we define a binary relation $\leq$ on $\mathbb{CLA}$ by

$$\langle \mathbb{LA}_n(J_m), F_n; n \in \mathbb{N}_0 \rangle \leq \langle \mathbb{LA}_n(J_k), F_n; n \in \mathbb{N}_0 \rangle,$$

where $J_m = (m, m+2)$, $J_k = (k, k+2)$, if there exists $n \in \mathbb{N}_0$ such that $k = m + 2n$. It is easy to show that the relation $\leq$ is a quasi–ordering, i.e., it is reflexive and transitive on the system $\mathbb{CLA}$. Indeed, if $n = 0$ then $k = m$. Now suppose $\langle \mathbb{LA}_n(J_m), F_n; n \in \mathbb{N}_0 \rangle \leq \langle \mathbb{LA}_n(J_k), F_n; n \in \mathbb{N}_0 \rangle$ and $\langle \mathbb{LA}_n(J_k), F_n; n \in \mathbb{N}_0 \rangle \leq \langle \mathbb{LA}_n(J_s), F_n; n \in \mathbb{N}_0 \rangle$, which means $k = m + 2n$, $s = k + 2l$ for suitable numbers $n, l \in \mathbb{N}_0$. Then $s = m + 2(n+l)$, thus $\langle \mathbb{LA}_n(J_m), F_n; n \in \mathbb{N}_0 \rangle \leq \langle \mathbb{LA}_n(J_s), F_n; n \in \mathbb{N}_0 \rangle$.

For $A, B \in \mathfrak{I}n_0(\mathbb{R})$ define $A \circ B = [A+B)_{\leq} = \{K \in \mathfrak{I}n_0(\mathbb{R}); A + B \leq K\}$ and for

$$\langle \mathbb{LA}_n(K), F_n; n \in \mathbb{N}_0 \rangle \boxplus \langle \mathbb{LA}_n(L), F_n; n \in \mathbb{N}_0 \rangle = \{\langle \mathbb{LA}_n(I), F_n; n \in \mathbb{N}_0 \rangle; I \in K \circ L\}.$$

Theorem 3. *Hypergroupoids* $(\mathfrak{I}n_0(\mathbb{R}), \circ)$, $(\mathbb{CLA}, \boxplus)$ *are isomorphic commutative hypergroups.*

The monograph [3] Theorem 1.4, page 147 (chapter IV) yields the following assertion:

Theorem 4. *The following conditions are equivalent for an ordered semigroups* $(S, \cdot, \leq)$:

1° For any pair of elements $a, b \in S$ *there exists a pair of elements* $c, c' \in S$ *such that* $b \cdot c \leq a$, $c' \cdot b \leq a$.

2° The semigroup $(S, *)$ *(defined by the rule* $a * b = [a \cdot b)_{\leq}$*) satisfies the reproduction condition (i.e.,* $t * S = S * t = S$*, for each element* $t \in S$*), thus* $(S, *)$ *is a hypergroup.*

The monoid $(\mathfrak{I}n_0(\mathbb{R}), +)$, which is ordered by the ordering " $\leq$ ", satisfies the above condition 1. Consider intervals $A = (-1,4)$, $B = (2,3) \in \mathfrak{I}n_0(\mathbb{R})$. For the interval $Y = (-4,-1) \in \mathfrak{I}n_0(\mathbb{R})$ we have $B + Y = (2,3) + (-4,-1) = (-2,2) < (-1,4) = A$ and for $X = (-3,-2)$ there holds $A + X = (-1,4) + (-3,-2) = (-4,2) < (2,3) = B$.

Remark 3. *It is to be noted that for each pair of intervals* $A, B \in \mathfrak{I}n(\mathbb{R})$ *there exists infinitely many solutions of the above considered inequalities. On the other hand there exists infinitely many pairs of intervals* $A, B \in \mathfrak{I}n(\mathbb{R})$ *such that the equation* $A + X = B$ *has no solution within* $\mathfrak{I}n(\mathbb{R})$*. Simple counterexample: If* $A = (1,4)$, $B = (2,3)$ *then the equation* $A + X = B$ *has no solution in* $\mathfrak{I}n(\mathbb{R})$*. This fact also implies that the grupoid* $(\mathfrak{I}n_0(\mathbb{R}), +)$ *is monoid (commutative) with the neutral element* $0 \in \mathbb{R}$*, not a group.*

Theorem 5. *For any pair of intervals* $A, B \in \mathfrak{I}n(\mathbb{R})$ *there exists a pair of intervals* $X, Y \in \mathfrak{I}n(\mathbb{R})$ *such that* $A + X \leq B$ *and* $B + Y \leq A$.

Proof. Suppose that $(\underline{a},\overline{a}) = A \in \mathfrak{I}n(\mathbb{R})$, $(\underline{b},\overline{b}) = B \in \mathfrak{I}n(\mathbb{R})$ are intervals of real numbers. Let us denote $X = (x_1,x_2), Y = (y_1,y_2)$. Suppose $a \in \mathbb{R}$ is a positive number and $b \in \mathbb{R}$ such that $\underline{b}-\underline{a}-a < b < \underline{b}-a$. Denote $x_1 = \underline{b}-\underline{a}-a$, $x_2 = b$. Then we have $x_1 < x_2$ and $X = (x_1,x_2) \leq (\underline{b}-\underline{a},\overline{b}-\overline{a}) = B-A$. The inequality $X \leq B-A$ implies $A+X \leq B$. Similarly, if $c \in \mathbb{R}$, $0 < c$ and $d \in \mathbb{R}$ are numbers satisfying the condition $\underline{a}-\underline{b}-c < d < \overline{a}-\overline{b}$ then denoting $y_1 = \underline{a}-\underline{b}-c$ and $y_2 = d$, we have $y_1 < y_2$ and $Y = (y_1,y_2) = (\underline{a}-\underline{b}-c,d) \leq (\underline{a}-\underline{b},\overline{a}-\overline{b}) = A-B$, which implies $B+Y \leq A$. This inequality completes the proof. $\square$

As a specific application of Theorem 5 we obtain the following construction.

Let $\circ : \mathfrak{I}n_0(\mathbb{R}) \times \mathfrak{I}n_0(\mathbb{R}) \to exp\mathfrak{I}n_0(\mathbb{R})$ and $\boxplus : \mathbb{CLA} \times \mathbb{CLA} \to exp\mathbb{CLA}$ be binary hyperoperations defined by

$$A \circ B = \{K \in \mathfrak{I}n_0(\mathbb{R}); A+B \leq K\}$$

for each pair of intervals $A,B \in \mathfrak{I}n_0(\mathbb{R})$ and

$$\langle \mathbb{LA}_n(J),F_n;n \in \mathbb{N}_0\rangle \boxplus \langle \mathbb{LA}_n(L),F_n,n \in \mathbb{N}_0\rangle = \{\langle \mathbb{LA}_n(I),F_n;n \in \mathbb{N}_0\rangle; I \in J \circ L\}.$$

for any pair of chains $\langle \mathbb{LA}_n(J),F_n;n \in \mathbb{N}_0\rangle$, $\langle \mathbb{LA}_n(L),F_n;n \in \mathbb{N}_0\rangle \in \mathbb{CLA}$. Then we obtain:

Theorem 6. *Hypergroupoids* $(\mathfrak{I}n_0(\mathbb{R}),\circ)$, $(\mathbb{CLA},\boxplus)$ *are isomorphic commutative hypergroups.*

Proof. Notice that the presented constructions are based on results related to the Ends lemma; see e.g. [3, 18, 19, 20]. The algebraic structure $(\mathfrak{I}n_0(\mathbb{R}),+,\leq)$ is a commutative (additive) ordered monoid with subtraction, which means that the equality $A+B = A+C$ implies $B = C$ for $A,B,C \in \mathfrak{I}n_0(\mathbb{R})$. By Theorem 5, the monoid $(\mathfrak{I}n_0(\mathbb{R}),+,\leq)$ satisfies the condition 1° of Theorem 4. Since by Theorem 3 the hypergroupoid $(\mathfrak{I}n_0(\mathbb{R}),\circ)$ is a commutative semigroup, by condition 2° of Theorem 4 the semigroup $(\mathfrak{I}n_0(\mathbb{R}),\circ)$ is a commutative hypergroup. Now define a mapping

$$f : \mathfrak{I}n_0(\mathbb{R}) \to \mathbb{CLA} \text{ by } f(L) = \langle \mathbb{LA}_n(L),F_n;n \in \mathbb{N}_0\rangle$$

for each interval $L \in \mathfrak{I}n_0(\mathbb{R})$.

Suppose $J,K \in \mathfrak{I}n_0(\mathbb{R})$ are arbitrary intervals such that $J \neq K$. Then $\langle \mathbb{LA}_n(J),F_n;n \in \mathbb{N}_0\rangle \neq$ $\neq \langle \mathbb{LA}_n(K),F_n;n \in \mathbb{N}_0\rangle$ thus $f(J) \neq f(K)$. Further for any $\langle \mathbb{LA}_n(J),F_n;n \in \mathbb{N}_0\rangle \in \mathbb{CLA}$ then $J \in \mathfrak{I}n_0(\mathbb{R})$, thus the mapping is also surjective, hence f is bijective. Now we show that the bijection $f : (\mathfrak{I}n_0(\mathbb{R}),\circ) \to (\mathbb{CLA},\boxplus)$ is a strong homomorphism. Let $J,K \in \mathfrak{I}n_0(\mathbb{R})$ be arbitrary intervals. Then

$$f(J \circ K) = f(\{L \in \mathfrak{I}n_0(\mathbb{R}); J+K \leq L\}) = \{f(L); L \in \mathfrak{I}n_0(\mathbb{R}); J+K \leq L\} =$$

$$= \{\langle \mathbb{LA}_n(L),F_n;n \in \mathbb{N}_0\rangle; J+K \leq L\} = \langle \mathbb{LA}_n(J),F_n;n \in \mathbb{N}_0\rangle \boxplus \langle \mathbb{LA}_n(K),F_n;n \in \mathbb{N}_0\rangle =$$

$$= f(J) \boxplus f(K).$$

Hence the mapping $f : \mathfrak{I}n_0(\mathbb{R}) \to \mathbb{CLA}$ is an isomorphism of the hypergroupoid $(\mathfrak{I}n_0(\mathbb{R}),\circ)$ onto the hypergoupoid $(\mathbb{CLA},\boxplus)$. Since the hypergroupoid $(\mathfrak{I}n_0(\mathbb{R}),\circ)$ is a commutative hypergroup, the hypergroupoid $(\mathbb{CLA},\boxplus)$ is a commutative hypergroup as well. $\square$

Theorem 7. *Let* $(S,\cdot,\leq)$ *be an ordered semigroup. The binary hyperoperation* $* : S \times S \to expS$ *defined by the rule* $a * b = [a \cdot b)_{\leq}$ *is associative. The semigroup* $(S,*)$ *is commutative if and only if the semigroup* $(S,\cdot)$ *is commutative.*

Proof. Suppose $a,b,c \in S$ are arbitrary elements. First, we show that

$$\bigcup_{t\in[b\cdot c)_\le} [a\cdot t)_\le = \bigcup_{x\in[a\cdot b)_\le} [x\cdot c)_\le. \tag{$\circledast$}$$

Consider $s \in \bigcup_{t\in[b\cdot c)_\le} [a\cdot t)_\le$, thus $a\cdot t_0 \le s$ for a suitable element $t_0 \in S$, $b\cdot c \le t_0$. Then $(a\cdot b)\cdot c = a\cdot(b\cdot c) \le a\cdot t_0$ and putting $x_0 = a\cdot b$, we have $x_0\cdot c \le s$, $x_0 \in [a\cdot b)_\le$, thus $s \in [x_0\cdot c)_\le \subset \bigcup_{x\in[a\cdot b)_\le} [x\cdot c)_\le$. Similarly, we obtain the the opposite inclusion thus the equality ($\circledast$) is valid. Now, we obtain with respect to ($\circledast$)

$$a*(b*c) = \bigcup_{t\in[b*c)_\le} [a*t)_\le = \bigcup_{t\in[b\cdot c)_\le} [a\cdot t)_\le = \bigcup_{x\in[a\cdot b)_\le} [x\cdot c)_\le = \bigcup_{x\in[a*b)_\le} [x*c)_\le = (a*b)*c,$$

hence the hyperoperation $*$ is associative. Further, if the semigroup $(S,\cdot)$ is commutative, then $(S,*)$ is commutative as well. If $(S,*)$ is commutative, then for any pair of elements $a,b \in S$ there holds $[a\cdot b)_\le = a*b = [b\cdot a)_\le$ which implies $b\cdot a \le a\cdot b$ and immediately $a\cdot b \le b\cdot a$, thus $a\cdot b = b\cdot a$. □

4 Multiautomata formed by actions of hypergroupoids

Recall that a multiautomaton is a triad (S,H,δ), where H is a hypergrupoid, S is a nonempty set and $\delta : S\times H \to S$ is a mapping such that

$$\delta(\delta(s,a),b) \in \delta(s,a\cdot b) \tag{GMAC}$$

for any triad $(s,a,b) \in S\times H\times H$, where $\delta(s,a\cdot b) = \{\delta(s,x); x \in a\cdot b\}$. The mapping δ is called the transition function or the next–state function or the action of the input hypergroupoid or the phase hypergroupoid H on the state or phase set S. The acronym GMAC means the Generalised Mixed Associativity Condition [2].

It is to be noted that $(\mathfrak{I}n_0(\mathbb{R}),+)$ is a quasi-linear space over the field $\mathbb{R}$ formed by all open, bounded intervals $(\underline{a},\overline{a}) \subset \mathbb{R}$ (set of which is denoted by $\mathfrak{I}n(\mathbb{R})$).

The actions of that quasi-linear space are the following presented in [13, part 1.2.2, p. 12,13].

(Q1) $\alpha(A+B) = \alpha A + \alpha B$

(Q2) $(\alpha+\beta)A = \alpha A + \beta A$ if $|\alpha+\beta| = |\alpha|+|\beta|$

(Q3) $\alpha(\beta A) = (\alpha\beta)A$. Moreover $1\cdot A = A$, $0\cdot A = 0$

(Q4) $A+B = A+C \implies B = C$ (this implication is valid within $(\mathfrak{I}_n(\mathbb{R}),+)$)

We have $A+B = A+C$, i.e., $(\underline{a},\overline{a}) + (\underline{b},\overline{b}) = (\underline{a},\overline{a}) + (\underline{c},\overline{c})$ i.e., $(\underline{a}+\underline{b},\overline{a}+\overline{b}) = (\underline{a}+\underline{c},\overline{a}+\overline{c}) \iff \underline{a}+\underline{b} = \underline{a}+\underline{c}$, $\overline{a}+\overline{b} = \overline{a}+\overline{c}$ –since $\underline{a},\overline{a},\underline{b},\overline{b},\underline{c},\overline{c} \in \mathbb{R}$ then $\underline{b} = \underline{c}$, $\overline{b} = \overline{c}$, i.e., $B = (\underline{b},\overline{b}) = (\underline{c},\overline{c}) = C$. Hence the axiom $(Q4)$ is valid. So the monoid $(\mathfrak{I}n_0(\mathbb{R}),+)$ is a quasi–linear space satisfying also the condition $(Q4)$, i.e., the monoid $(\mathfrak{I}n_0(\mathbb{R}),+)$ is with the subtraction.

In the classical monograph [13], part 1.2.2 p. 15, which is devoted to quasi–linear spaces (which is in fact the additive commutative semigroup $(\mathfrak{I}n(\mathbb{R}),+)$ – here $K+L = \{x+y; x \in K, y \in L\}$ for any $K,L \in \mathfrak{I}n(\mathbb{R})$) there is simply defined an ordering on the system of intervals

$I(M)$ of an ordered linear space M. As usually $\mathscr{P}(M)$ is a set of all subsets of the space M. So, in [13] there is defined, in systems $I(M), \mathscr{P}(M)$, the relation $<$:

$$A < B \text{ iff } ((\forall a \in A) \vee (\forall b \in B | a < b).$$

There holds inclusions

$$(M,<) \subset (I(M),<) \subset (\mathscr{P}(M),<),$$

however systems $(I(M),<),(\mathscr{P}(M),<)$ are not lattices. Moreover in [13], p.15 there is also defined a relation $\leq$ for intervals $A,B \in I(M)$:

If $A = (a,b), B = (c,d)$ then

$$A \leq B \text{ iff } (\forall x \in A\, \exists y \in B | x \leq y) \wedge (\forall y' \in B\, \exists x' \in A | x' < y').$$

Also, there holds

$$A \leq B \text{ iff } (a \leq b) \wedge (b \leq d).$$

We transfer the above defined relation onto systems $\mathfrak{I}n(\mathbb{R})$ and $\mathscr{P}(\mathfrak{I}n(\mathbb{R}))$. However, we should once again emphasiz that we consider open bounded intervals of $\mathbb{R}$.

Now we put $H = (\mathbb{CLA},*), S = \mathfrak{I}n_0(\mathbb{R}) = \mathfrak{I}n(\mathbb{R}) \cup \{0\}$. The transition function

$$\delta : \mathfrak{I}n_0(\mathbb{R}) \times (\mathbb{CLA},*) \to \mathfrak{I}n_0(\mathbb{R})$$

can be defined in the following way: For $\langle \mathbb{LA}_n, F_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}$ and $K \in \mathfrak{I}n_0(\mathbb{R})$ we put down

$$\delta(K, \langle \mathbb{LA}_n, F_n; n \in \mathbb{N}_0 \rangle) = K + J = [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}],$$

where $[\underline{a}, \bar{a}] = K, [\underline{b}, \bar{b}] = J$. Since $K + J \in \mathfrak{I}n(\mathbb{R})$, we obtain that the triad $(\mathfrak{I}n_0(\mathbb{R}), (\mathbb{CLA},*), \delta)$ is a multiautomaton with the input hypegroupoid $(\mathbb{CLA},*)$ and phase set $\mathfrak{I}n_0(\mathbb{R})$ of all bounded open intervals in the set $\mathbb{R}$ including the zero. Indeed, consider an arbitrary chains $\langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}, \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle \in \mathbb{CLA}$ and an arbitrary interval $K \in \mathfrak{I}n_0(\mathbb{R})$. Then we have

$$\begin{aligned} &\delta(\delta(K, \langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle), \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle) = \\ &\qquad = \delta(K + J_1, \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle) = K + J_1 + J_2. \end{aligned}$$

For the purpose of the construction of a multiautomaton fulfilling the *GMAC* condition, we define a binary hyperproduct which is a certain modification of the above one: Suppose $\langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle$, $\langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle$ is a pair of arbitrary chains from the system $\mathbb{CLA}$. We have defined $\mathfrak{I}n_0(\mathbb{R}) = \mathfrak{I}n(\mathbb{R}) \cup \{0\}$ and

$$\begin{aligned} &\langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle \cdot \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle = \\ &\qquad = \{\langle \mathbb{LA}_n(I), F_n; n \in \mathbb{N}_0 \rangle; I \in \{J_1 + J_2 + J; J \in \mathfrak{I}n(\mathbb{R})\}\}. \end{aligned}$$

Evidently, the pair $(\mathbb{CLA}, \cdot)$ is a commutative hypegroupoid. Further, we define a mapping

$$\delta : \mathfrak{I}n_0(\mathbb{R}) \times \mathbb{CLA} \to \mathfrak{I}n_0(\mathbb{R})$$

$$\text{by } \delta(K, \langle \mathbb{LA}_n(J), F_n; n \in \mathbb{N}_0 \rangle) = K + J \in \mathfrak{I}n_0(\mathbb{R})$$

Then we obtain the following result:

Theorem 8. *The triad $(\mathfrak{I}n_0(\mathbb{R}), (\mathbb{CLA}, \cdot), \delta)$ is a multiautomaton, that satisfies the Generalized Mixed Associativity Condition (GMAC).*

Proof. Let $K \in \mathfrak{I}n_0(\mathbb{R})$ be an arbitrary interval or $K = 0$, let $\langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle$, $\langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle$ be arbitrary chains belonging to $\mathbb{CLA}$. Then we have

$$
\begin{aligned}
&\delta(\delta(K, \langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle, \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle) \\
&= \delta(K + J_1, \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle) \\
&= K + J_1 + J_2 = K + J_1 + J_2 + 0 \in \{K + S; S \in \{J_1 + J_2 + J; J \in \mathfrak{I}n_0(\mathbb{R})\} \\
&= \delta(K, \langle \mathbb{LA}_n(I), F_n; n \in \mathbb{N}_0 \rangle; I \in \{J_1 + J_2 + J; J \in \mathfrak{I}n_0(\mathbb{R})\}) \\
&= \delta(K, \langle \mathbb{LA}_n(J_1), F_n; n \in \mathbb{N}_0 \rangle \cdot \langle \mathbb{LA}_n(J_2), F_n; n \in \mathbb{N}_0 \rangle).
\end{aligned}
$$

Hence the Generalized Mixed Associativity Condition (GMAC) is satisfied. □

CONCLUSION

As has been mentioned, some papers discussing topics included in our contribution present ideas which originated in the scientific school of Otakar Borůvka, František Neuman and their collaborates who studied ordinary differential equations using the algebraic approach based on the group theory. Our study thus represents a certain generalization of this direction. Generalization of presented constructions can be obtained in several directions. Firstly, we can consider the direct product of ordered groups which with constructions of chains of groups of differential operators. This approach can be generalized up to direct products of finite but sufficiently many factors. Alternatively, one can investigate differentiable functions of n variables in an n–dimensional space having partial derivatives of sufficiently high order, and then restrict such functions onto one–dimensional intervals. Nevertheless these directions will be explored in some forthcoming papers. The authors' contributions to the creation of the article are equal - 33%.

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On the teaching of the infinite series in high school

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Abstract: *In the article, we will describe the development of the treatment of the topic of infinite series in selected textbooks and books for high school students in our territory. We will show some of Zeno's paradoxes, an approach to terminology and different approaches to processing this issue.*

Keywords: infinite series, teaching, high school, Zeno's paradoxes.

INTRODUCTION

Infinite series are not listed in the State Educational Program in the educational standard for the course of mathematics – grammar school with a four-year and five-year educational program, nor in the Target requirements for the knowledge and skills of high school graduates in mathematics. They are missing even in our current grammar school textbooks. In this paper we shall look at how is this topic developed in selected textbooks and books for high school students in the last decades in the territory of Czechoslovakia, the Czech Republic and the Slovak Republic in 16 textbooks, publications and related problem books. In my research I focused on the following indicators:

- Motivational tasks from history (Zeno's paradoxes)
- Interpretation of an infinite geometric series
- Introduction of the concepts of series, infinite series, terms of the series
- Introduction of the designation of an infinite series and its double meaning (series and the sum of the series)
- Construction of a sequence of partial sums
- Investigation of the limits of the sequence of partial sums
- Divergence of infinite series
- Methods of denoting sequences
- Theorems about infinite series (necessary conditions of convergence, etc.)
- Definition of the term infinite geometric series
- The formula for the sum of an infinite geometric series
- Proof of the formula for the sum of an infinite geometric series
- Conditions for the divergence of an infinite geometric series

- Examples of evaluating sums and using infinite geometric series
- Determining the sum of an infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
- Conversion of a periodic decimal number to a fraction
- Determining the convergence of an infinite series of the Grandi series type
- Adding and rearranging adjacent members of infinite series
- Infinite series with geometric interpretation
- Problems on infinite geometric series with a parametric quotient
- Harmonic series and its divergence
- Infinite series related to fractals
- The quantity of solved and unsolved problems

In this paper, we shall focus on two of these points – motivation and terminology.

1 Motivation

As taught by didactic theory, the topic and its interpretation begin with motivational tasks in most publications. Usually, it is a task related to one of the infinite geometric series. In the overwhelming majority, it is the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. However, there are different approaches to this series. The first (predominant) approach is straightforward – a suitable sequence is presented (often with a geometric interpretation) and the individual terms of the series are calculated. The second approach is historical and based on the so-called Zeno's paradoxes. These have almost disappeared from current textbooks – in the present they can be found only in one Czech textbook, and therefore we will focus on them.

Zeno of Elea (ca. 490 – 430 BC) was a Greek mathematician and philosopher. He aimed to disprove the idea of motion by presenting a series of examples in which, with the help of logical arguments, he went from the assumption of motion to logical contradictions. According to historical sources, there were approximately 20 such examples, but only four were preserved. Their wording differs in various literary sources, and in this paper we present the spelling according to Kline (1990) based on Aristotle's quotations:

Dichotomy: *The first asserts the nonexistence of motion on the ground that that which is in motion must arrive at the half-way stage before it arrives at the goal.*

Achilles and the Tortoise: *It says that the slowest moving object cannot be overtaken by the fastest since the pursuer must first arrive at the point from which the pursued started so that necessarily the slower one is always ahead. The argument is similar to that of the Dichotomy, but the difference is that we are not dividing in halves the distances which have to be passed over. Aristotle then says that if the slowly moving object covers a finite distance, it can be overtaken for the same reason he gives in answering the first paradox.*

Arrow: *The third paradox he [Zeno] spoke about, is that a moving arrow is at a standstill. This he concludes from the assumption that time is made up of instants. If it would not be for this supposition, there would be no such conclusion.*

Stadium: *The fourth is the argument about a set of bodies moving on a race-course and passing another set of bodies equal in number and moving in the opposite direction, the one starting from the end, the other from the middle and both moving at equal speed; he [Zeno] concluded that it follows that half the time is equal to double the time. The mistake is to assume that two bodies moving at equal speeds take equal times in passing, the one a body which is in motion, and the other a body of equal size which is at rest, an assumption which is false.*

We will show how the infinite geometric series is hidden in the first two paradoxes. In the Dichotomy paradox, let us imagine that we had to travel a unit distance from point A to point B. To travel this distance, we must first travel half of the total (unit) distance – we must reach the middle of the path – the midpoint of segment AB, which we denote S_1. Apparently $|S_1B| = \frac{1}{2}$. Now we apply the same principle to line segment S_1B, which is one-half in length – we need to get to the middle of segment S_1B. We will mark this point as S_2. Its distance from point B is $|S_2B| = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, so $|S_2B| = (\frac{1}{2})^2$. We can continue this way and get that in the n-th step we will be at point S_n, while its distance from point B will be $|S_nB| = (\frac{1}{2})^n$. Thus, in each step, we will be a certain positive distance away from point B, and thus we will never arrive at point B. To get there, we would have to take an infinite number of steps in finite time, which is impossible.

The paradox of Achilles and the tortoise works on a similar principle – if Achilles gives the tortoise a certain head start, it takes a certain amount of time to reach the point from which got started (regardless of how many times faster Achilles is, this time is non-zero). During this time, however, the turtle moves a certain non-zero distance. Achilles must run over this distance again in non-zero time, and so on. But this means that the tortoise will always be a certain distance ahead of Achilles, so Achilles will never overtake it.

The solution to these two paradoxes lies in the way how we understand infinity. Zeno's argumentation is based on the so-called potential infinity. However, from the point of view of the so-called actual infinity, we can count the individual distances and even if there are infinitely many of them, we can cross them in a finite time (in both paradoxes, the length of the section also shortens the time required to cover it, which is why it is possible to cover an infinite number of sections in a finite time).

These problems with grasping the concept of infinity probably caused paradoxes to disappear from the textbooks, and in the only one where they remain, they occur at the very end of the exposition. This classification has the advantage that the students already have their first experience with evaluating sums of infinite series, and thus they can calculate when the ”key moment” will occur – when ”what moves will reach the end”, or when Achilles overtakes the tortoise. At the same time, we also see that both analyzed paradoxes, Dichotomy and Achilles and the Tortoise, also lead to the sum of an infinite geometric series.

2 Terminology

Most of the authors first define the terms of the infinite series, sequence of partial sums, and then define the sum of the infinite series as the limit of its sequence of partial sums. The definition of Smítal and Šalát (1986, p. 46) is very interesting and unique:

Let $a_n \in \mathbb{R}$ $(n = 1, 2, \dots)$. We call the sequence $(s_n)_{n=1}^{\infty}$, $s_n = a_1 + a_2 + \cdots + a_n$ $(n = 1, 2, \dots)$ an infinite series (more briefly, a series)

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

In this definition, the infinite series is not introduced as a formal infinite sum written using the symbol $\sum$, but as a specially constructed sequence (sequence of partial sums). This approach can cause a significant problem with the philosophical grasp of this concept for high school (as well as university) students. This is the reason why this definition cannot be recommended for use in high school. The intention of the authors is described in detail by Šalát (1975) – the goal of this definition was to *give a real definition of a series without the indefinite term "symbol", a definition that ultimately converts the concept of a series into a concept of a set*. This unusual definition then facilitated the interpretation of other concepts (among them, for example, the concepts of convergence/divergence, which are usually previously introduced for a general sequence, the limit of which in our case is called the sum of a series, or the term *a series with a bounded sequence of partial sums*, which can be shortened to *the bounded series*).

Similarly, it is necessary to emphasize to the students that by the symbol $\sum_{n=1}^{\infty} a_n$ we must understand not only the number as the sum of this series but also the series as such. Jarník (1979), Odvárko (2019), Odvárko, O., & Řepová, J. (1986), Smida (1988), Tlustý (2020) and Zemek & Zemková (2017) directly draw attention to this fact. Other authors do not address this issue.

The issue of comparing terms of infinite sums and placing parentheses in them is only mentioned in current textbooks by Odvárko (2019) and Zemek and Zemková (2017). In the last-mentioned textbook, the authors show that a suitable rearrangement, or placing parentheses between members of this series, can lead to three different sums (0, 1, and -1) of the Grandi series $\sum_{n=1}^{\infty} (-1)^n$.

The concept of divergent infinite series is introduced by all authors except Smida (1988). Liška (2019a, p. 88 – 95) limits himself to infinite geometric series in his entire publication, and we will not draw attention to this fact further. In their publications, the authors mentioned most often as examples of divergent series an infinite geometric series with a quotient greater than 1, a Grandi series, or an infinite series formed from members of an arithmetic sequence with a non-zero difference. Apart from books intended for gifted students (where its divergence is also proven), the harmonic series is mentioned only by Zemek and Zemková (2017).

From a terminological point of view, it is interesting to observe the denotation of sequences – in publications published before 2000, curly brackets are the preferred marking (the exception is Smítal and Šalát (1986), where, however, the preference for round brackets can be attributed to the fact that this publication was published as a script written on a typewriter machine, and therefore curly brackets would have to be written by hand; this theory is also supported by other publications of these authors from the same period, where they use curly brackets), in publications published after 2000 (the exception is Králiková (2006)), the authors use round brackets.

Theorems on the convergence of infinite series can be found in only three publications – Hecht (2000), Králiková (2006) and Smítal & Šalát (1986). The first one is more than 20 years old, last two books are focused on talented students. In the textbooks that are currently used we cannot find any theorems on the convergence.

CONCLUSION

Infinite series are an interesting and stimulating topic that should not be missing in the secondary school curriculum (even more so in the grammar school curriculum). With a little effort, it is possible to carry out an interpretation of the topic (of course, not in its entire breadth, but to the extent that will be sufficient for understanding the issue) even during one lesson, which is also proven by Kudláček, Válka & Burian (1963) for the first and second-year secondary industrial schools for workers. This book shows that the foundations of infinite series can be taught during only one lesson.

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TWO CALCULATION METHODS OF THE SUM OF RECIPROCALS OF ALL PRODUCTS GENERATED BY PRIME FACTORS OF THE NUMBER 2022

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Abstract: *This paper is inspired by one task on the sum of a special infinite series on the Berkeley Math Circle. The task was to determine the sum of all the unit fractions that have denominators with only factors consisting of all prime divisors of the number 2002, i.e. the primes 2, 7, 11, 13. We updated this task to determine the sum of reciprocals of all products generated by all prime divisors of the number 2022. This series can be consider as the reduced harmonic series generated by three primes 2, 3, 337. The sum of this series was calculated analytically by two methods and also numerically by using the programming language of the computer algebra system Maple 2022. By these three attempts we obtained the same result. Finally, we generalized the obtained result and presented a formula for the sum of these series with n generating positive integers different from* 1.

Keywords: infinite series, geometric series, harmonic series, reduced harmonic series, CAS Maple 2022.

INTRODUCTION

The Berkeley Math Circle is one of the most famous math circles in North America. Math circles can be characterized as a learning space where participants engage in the depths and intricacies of mathematical thinking, propagate the culture of doing mathematics, and create knowledge. To reach these goals, participants partake in problem-solving. Some circles involve competition, such as the Berkeley Math Circle, others do not.

The Berkeley Math Circle study material from 2002 (see [1]) includes the following unsolved task 5.4 from the paragraph 5 Harmonic Series:

> $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Find the sum of all the unit fractions that have denominators with only factors from the set $\{2, 7, 11, 13\}$. That is, find the following sum:
> $$\frac{1}{2}+\frac{1}{4}+\frac{1}{7}+\frac{1}{8}+\frac{1}{11}+\frac{1}{13}+\frac{1}{14}+\frac{1}{16}+\frac{1}{22}+\frac{1}{26}+\frac{1}{28}+\cdots.$$

Infinite series are still a hot topic in mathematical analysis. In addition to harmonic series – see e.g. papers [2], [3], [4], [5], [6], attention has recently been paid to time series – see e.g. papers [7], [8], [9], [10] and Fourier series – see e.g. paper [11].

In this article, that was written last year, we solve an analogous task related to the year 2022. Because this task concerns the harmonic series and the so called reduced harmonic series or modified harmonic series, let us recall the necessary notions from infinite series theory.

1 BASIC NOTIONS

For any sequence $\{a_k\}$ of numbers the associated *infinite series* or more briefly *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n .$$

The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$.

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the square numbers (the *Basel problem*) is $\pi^2/6$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \doteq 1.644934 ,$$

Generally,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is the *Riemann zeta function*, which is a function of a complex variable s. This series converges when the real part of s is greater than 1. Thus $\zeta(2) = \pi^2/6$.

The sum of the reciprocals of the cube numbers is the *Apéry's constant* $\zeta(3)$ which is given by the formula

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \doteq 1.202057 .$$

The *reduced harmonic series* is defined as the subseries of the harmonic series, which arises by omitting some its terms. As an example of the reduced harmonic series we can take the series formed by reciprocals of primes and number one

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots .$$

This reduced harmonic series is divergent. The first proof of its divergence was made by Leonhard Euler (see e.g. book [12]).

A very interesting example of reduced harmonic series are *Kempner's series* K_a (see [13]). The Kempner series is a modification of the harmonic series, formed by omitting all terms whose denominator expressed in base 10 contains a digit a. That is, it is the sum of fractions $1/n$ where n takes only values whose decimal expansion has no digit a. The series K_9 with omitted the 9 digit, i.e. the series

$$K_9 = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{18} + \frac{1}{20} + \frac{1}{21} + \cdots + \frac{1}{88} + \frac{1}{100} + \frac{1}{101} + \cdots,$$

was first studied by A. J. Kempner in 1914 in the paper [14].

This series is interesting because of the counter-intuitive result that unlike the harmonic series it converges. Kempner showed this value was less than 80. The upper bound of 80 is very crude, and F. Irwin showed in 1916 in the paper [15] by a slightly finer analysis of the bounds that the value of this Kempner series is between 22.4 and 23.3. T. Schmelzer and R. Baillie in their paper [16] showed that up to 20 decimals. The actual sum is 22.920676619264150348 16.

2 THE FIRST METHOD OF ANALYTICAL SOLUTION

Let us consider the following series T formed of all the unit fractions that have denominators with only prime factors of the number 2022, i.e. with factors from the set $\{2, 3, 337\}$, as

$$2022 = 2 \cdot 3 \cdot 337.$$

We gradually calculate the sum S of the series T by rearranging it, appropriately dividing it into subseries and using the well-known formula for the sum of an infinite geometric series. The series T is thus the infinite series of the form

$$\begin{aligned} T = {} & \frac{1}{2} + \frac{1}{3} + \frac{1}{337} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{337^2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 337} + \frac{1}{3\cdot 337}\right) + \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{337^3} + \right. \\ & \left. + \frac{1}{2^2\cdot 3} + \frac{1}{2^2\cdot 337} + \frac{1}{3^2\cdot 2} + \frac{1}{3^2\cdot 337} + \frac{1}{337^2\cdot 2} + \frac{1}{337^2\cdot 3} + \frac{1}{2\cdot 3\cdot 337}\right) + \\ & + \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{337^4} + \frac{1}{2^3\cdot 3} + \frac{1}{2^3\cdot 337} + \frac{1}{3^3\cdot 2} + \frac{1}{3^3\cdot 337} + \frac{1}{337^3\cdot 2} + \frac{1}{337^3\cdot 3} + \right. \\ & \left. + \frac{1}{2^2\cdot 3\cdot 337} + \frac{1}{3^2\cdot 2\cdot 337} + \frac{1}{337^2\cdot 2\cdot 3} + \frac{1}{2^2\cdot 3^2} + \frac{1}{2^2\cdot 337^2} + \frac{1}{3^2\cdot 337^2}\right) + \cdots. \end{aligned} \tag{1}$$

Assume that its sum S is finite and that the series (1) converges. Because all its terms are positive, then the series (1) converges absolutely and so we can rearrange it. For easier determination the sum S it is necessary to rearrange it and divide it into six subseries T_2, T_3, T_{337}, $T_{2\cdot 3}$, $T_{2\cdot 337}$ and $T_{3\cdot 337}$, where

$$T_2 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right), \tag{2}$$

$$T_3 = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots = \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right), \tag{3}$$

$$T_{337} = \frac{1}{337} + \frac{1}{337^2} + \frac{1}{337^3} + \cdots = \frac{1}{337}\left(1 + \frac{1}{337} + \frac{1}{337^2} + \cdots\right), \tag{4}$$

$$T_{2\cdot 3} = \frac{1}{2\cdot 3} + \frac{1}{2^2\cdot 3} + \frac{1}{3^2\cdot 2} + \frac{1}{2\cdot 3\cdot 337} + \frac{1}{2^3\cdot 3} + \frac{1}{3^3\cdot 2} + \\ + \frac{1}{2^2\cdot 3\cdot 337} + \frac{1}{3^2\cdot 2\cdot 337} + \frac{1}{337^2\cdot 2\cdot 3} + \frac{1}{2^2\cdot 3^2} + \cdots,$$

i.e.

$$T_{2\cdot 3} = \frac{1}{2\cdot 3}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{337} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{337^2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 337} + \frac{1}{3\cdot 337} + \cdots\right), \quad (5)$$

$$T_{2\cdot 337} = \frac{1}{2\cdot 337} + \frac{1}{2^2\cdot 337} + \frac{1}{337^2\cdot 2} + \frac{1}{2^3\cdot 337} + \frac{1}{337^3\cdot 2} + \frac{1}{2^2\cdot 337^2} + \cdots,$$

i.e.

$$T_{2\cdot 337} = \frac{1}{2\cdot 337}\left(1 + \frac{1}{2} + \frac{1}{337} + \frac{1}{2^2} + \frac{1}{337^2} + \frac{1}{2\cdot 337} + \cdots\right), \quad (6)$$

$$T_{3\cdot 337} = \frac{1}{3\cdot 337} + \frac{1}{3^2\cdot 337} + \frac{1}{337^2\cdot 3} + \frac{1}{3^3\cdot 337} + \frac{1}{337^3\cdot 3} + \frac{1}{3^2\cdot 337^2} + \cdots,$$

i.e.

$$T_{3\cdot 337} = \frac{1}{3\cdot 337}\left(1 + \frac{1}{3} + \frac{1}{337} + \frac{1}{3^2} + \frac{1}{337^2} + \frac{1}{3\cdot 337} + \cdots\right). \quad (7)$$

Now, we determine unknown sum S by means of the sums of the series (2) – (7). By the formula

$$s = \frac{a}{1-q}, \quad (8)$$

for the sum s of the convergent infinite geometric series with the first term a and with the ratio q, $|q| < 1$, we get the sums S_2, S_3 and S_{337} of the series (2) – (4):

$$S_2 = \frac{1}{2}\cdot\frac{1}{1-1/2} = \frac{1}{2}\cdot\frac{2}{2-1} = 1, \quad (9)$$

$$S_3 = \frac{1}{3}\cdot\frac{1}{1-1/3} = \frac{1}{3}\cdot\frac{3}{3-1} = \frac{1}{2}, \quad (10)$$

$$S_{337} = \frac{1}{337}\cdot\frac{1}{1-1/337} = \frac{1}{337}\cdot\frac{337}{337-1} = \frac{1}{336}. \quad (11)$$

It is clear that the sum $S_{2\cdot 3}$ of the series (5) we can write in the form

$$S_{2\cdot 3} = \frac{1}{2\cdot 3}\left(1+S\right) = \frac{1}{6}\left(1+S\right). \quad (12)$$

The sums $S_{2\cdot 337}$ and $S_{3\cdot 337}$ of the series (6) and (7) are the sums of all the unit fractions that have in denominators all powers of the product of two primes (2, 337 and 3, 337).

The expression in the brackets of the sum $S_{2\cdot 337}$ of the series (6) we rearrange and write as the sum of number 1 and three subseries T_2, T_{337} and $T_{2\cdot 337}$ with the sums $S_2 = 1$, $S_{337} = 1/336$ and $S_{2\cdot 337}$. Therefore we have the equation

$$S_{2\cdot 337} = \frac{1}{2\cdot 337}\left(1 + 1 + \frac{1}{336} + S_{2\cdot 337}\right).$$

From this equation we obtain $673\,S_{2\cdot 337} = 673/336$, thus

$$S_{2\cdot 337} = \frac{1}{336}\,. \tag{13}$$

In the same way as above, for the sum $S_{3\cdot 337}$ we obtain the equation

$$S_{3\cdot 337} = \frac{1}{3\cdot 337}\left(1 + \frac{1}{2} + \frac{1}{336} + S_{3\cdot 337}\right).$$

From this equation it follows that $1010\,S_{3\cdot 337} = 505/336$, thus

$$S_{3\cdot 337} = \frac{1}{672}\,. \tag{14}$$

By the assumption of the absolute convergence of the series (1) we can write its sum S in the form

$$S = S_2 + S_3 + S_{337} + S_{2\cdot 3} + S_{2\cdot 337} + S_{3\cdot 337}\,.$$

According to (9) – (14) we get the equation

$$S = 1 + \frac{1}{2} + \frac{1}{336} + \frac{1}{6}(1+S) + \frac{1}{336} + \frac{1}{672}\,.$$

Multiplying both sides of this equation by 672, we obtain the equation

$$672\,S = 672 + 336 + 2 + 112 + 112\,S + 2 + 1\,,$$

i.e. $560\,S = 1125$, thus we get

$$S = \frac{225}{112} = 2.008\overline{928\,597\,14}\,. \tag{15}$$

Interestingly, the numerator of the fraction $225/112$ is 1 greater than twice the denominator and the sum of the numerator and denominator is 337.

3 THE SECOND METHOD OF ANALYTICAL SOLUTION

Let us consider three convergent geometric series $T_{1,2}$, $T_{1,3}$ and $T_{1,337}$ with the ratios $2^{-1} = \frac{1}{2}$, $3^{-1} = \frac{1}{3}$ and $337^{-1} = \frac{1}{337}$ and unlike the previous paragraph with the first term 1, so the geometric series of the form

$$T_{1,2} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\,, \tag{16}$$

$$T_{1,3} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots\,, \tag{17}$$

$$T_{1,337} = 1 + \frac{1}{337} + \frac{1}{337^2} + \frac{1}{337^3} + \frac{1}{337^4} + \cdots\,, \tag{18}$$

with the sums $S_{1,2}$, $S_{1,3}$ and $S_{1,337}$, which have according to formula (8) the values

$$S_{1,2} = \frac{1}{1-1/2} = \frac{2}{2-1} = 2\,, \tag{19}$$

$$S_{1,3} = \frac{1}{1 - 1/3} = \frac{3}{3-1} = \frac{3}{2}\,, \tag{20}$$

$$S_{1,337} = \frac{1}{1 - 1/337} = \frac{337}{337-1} = \frac{337}{336}\,. \tag{21}$$

Because the series (16) – (18) are absolutely convergent, their product

$$\begin{aligned} T_{1,2} \cdot T_{1,3} \cdot T_{1,337} = {} & \Big(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\Big) \cdot \\ & \cdot \Big(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots\Big) \cdot \\ & \cdot \Big(1 + \frac{1}{337} + \frac{1}{337^2} + \frac{1}{337^3} + \frac{1}{337^4} + \cdots\Big) \end{aligned}$$

is again an absolutely convergent series. It is obvious that by multiplying the terms of these three series we obtain a series of the form

$$\begin{aligned} T_{1,2} \cdot T_{1,3} \cdot T_{1,337} = 1 & + \frac{1}{2} + \frac{1}{3} + \frac{1}{337} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{337^2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 337} + \frac{1}{3 \cdot 337} + \\ & + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{337^3} + \frac{1}{2^2 \cdot 5} + \frac{1}{2^2 \cdot 337} + \frac{1}{3^2 \cdot 2} + \frac{1}{3^2 \cdot 337} + \\ & + \frac{1}{337^2 \cdot 2} + \frac{1}{337^2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 337} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{337^4} + \cdots\,, \end{aligned}$$

that is, the series $1 + T$ with the sum $1 + S$. From the equation

$$1 + S = S_{1,2} \cdot S_{1,3} \cdot S_{1,337}$$

we get

$$S = S_{1,2} \cdot S_{1,3} \cdot S_{1,337} - 1\,.$$

According to (19) – (21) we have

$$S = 2 \cdot \frac{3}{2} \cdot \frac{337}{336} - 1\,,$$

so

$$S = \frac{337}{112} - 1 = \frac{225}{112} = 2.008\overline{92859714}\,. \tag{22}$$

We thus obtained the same result with both methods of calculation.

It is obvious that the second method of calculation with notation of the given series in the form of a product of partially convergent geometric series is significantly simpler and for more than two prime factors generating these series also significantly clearer. Therefore, we can now generalize the result obtained by the second calculation method for any three and for arbitrary n generating positive integers different from one.

Theorem 1 *The sum $S(a_1, a_2, \ldots, a_n)$ of the series of reciprocals of all products generated by n different positive integers $a_1, a_2, \ldots, a_n$, greater than* 1*, is given by the formula*

$$S(a_1, a_2, \ldots, a_n) = \prod_{k=1}^{n} \frac{1}{1 - 1/a_k} - 1\,,$$

i.e. by the formula

$$S(a_1, a_2, \ldots, a_n) = \prod_{k=1}^{n} \frac{a_k}{a_k - 1} - 1\,. \tag{23}$$

4 NUMERICAL SOLUTION

Now, we will solve the task to determine the sum S of reciprocals of all products generated by all prime divisors $2, 3, 337$ of the number 2022 numerically by using the basic programming language in the computer algebra system Maple 2022. We use the following simple procedure `part2022`:

```
part2022:=proc(p)
    local d,e,i,j,k,n,s,sa;
    s:=0;
    sa:=2.00892857142857142857;
    for n from 1 to p do
        for i from 0 to n do
            for j from 0 to n-i do
                for k from 0 to n-i-j do
                    if i+j+k=n-1 then
                       d:=2^i*3^j*337^k;
                       s:=s+1/d;
                    end if;
                end do;
            end do;
        end do;
    end do;
    e:=abs(s-sa)/s;
    print("sum for the most power",p,"is",evalf[21](s),
        "relative error is",evalf[21](e));
end proc:
```

This procedure generates and sums first p numbers of 3-combinations with repetitions, i.e.

$$c = \sum_{m=1}^{p} \binom{3+m-1}{m} = \sum_{m=1}^{p} \binom{m+2}{m} = \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \cdots + \binom{p+2}{p}$$

unit fractions that have denominators with only factors from the set $\{2, 3, 337\}$, i.e. with denominators of the form $2^i \cdot 3^j \cdot 337^k$, up to the sum $p = i + j + k$ of their exponents.

Next, we will use the following basic formulas from combinatorics:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \qquad \text{and} \qquad \binom{n}{k} = \binom{n}{n-k}.$$

By adding the binomial coefficient $\binom{3}{0} (= 1)$ and subsequently subtracting 1 we can easily show that

$$c = \binom{p+3}{p} - 1 = \binom{p+3}{3} - 1 = \frac{(p+3)(p+2)(p+1)}{6} - 1.$$

The result of the procedure `part2022` is the sum S, which for $p \geq 70$ (the total exponent $p = 70$ corresponds to $c = 62\,195$ unit fractions) gives the value

$$S = 2.008\overline{928\,597\,14}, \tag{24}$$

likewise by using both analytical solutions above.

The approximate values of the sums $S = S(p)$ for $p = 5, 10, \dots, 70, 75$, rounded to 20 decimals and obtained by one for statement
`for p from 5 by 5 to 75 do part2022(p); end do;`
are written into the following table:

p	$S(p)$	$\eta(p)$
5	1.918771059384298081 00	$4.699 \cdot 10^{-2}$
10	2.005998480456305209 40	$1.461 \cdot 10^{-3}$
15	2.008836542428400526 19	$4.581 \cdot 10^{-5}$
20	2.008925693614261406 46	$1.432 \cdot 10^{-6}$
25	2.008928481489022163 50	$4.480 \cdot 10^{-8}$
30	2.008928568617928200 96	$9.956 \cdot 10^{-10}$
35	2.008928571340738694 73	0
40	2.008928571425826655 09	0
45	2.008928571428485654 40	0
50	2.008928571428568748 13	0
55	2.008928571428571344 81	0
60	2.008928571428571425 95	0
65	2.008928571428571428 49	0
70	2.008928571428571428 57	0
75	2.008928571428571428 57	0

Tab. 1. The approximate values of the sums $S(p)$ and their relative errors $\eta(p)$
Source: own modelling in Maple 2022

Note that the calculation of these 15 values of the sums $S(p)$ took less than 4 seconds.

CONCLUSION

In this paper the sum S of all the unit fractions that have denominators with only factors from the set $\{2, 3, 337\}$ was determined by two analytical methods and subsequently numerically verified. By these three calculation methods we obtain the same result

$$S = \frac{225}{112} = 2.008\,9\overline{28\,597\,14}\,.$$

It can be said that the infinite series we have dealt with, i.e. the series the convergent reduced harmonic series generated by three primes $2, 3, 337$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{337} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{337^2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 337} + \frac{1}{3 \cdot 337} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{337^3} + \cdots,$$

belongs to special types of convergent infinite series, such as geometric and telescoping series, which sum can be found analytically by means of a finite formula.

At the end of the paper, we generalized the obtained result and determined the simple formula for the sum of these series with n generating positive integers different from 1.

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MULTI-CRITERIA EVALUATION OF VARIANTS – PROBLEMS OF THE GRAPHICAL METHOD AND THEIR SOLUTIONS

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Abstract: *In multi-criteria variant evaluation tasks, the order of the variants is determined from best to worst, considering all criteria. The article describes two common methods of solving such problems, namely the graphical method and the weighted sum method. The authors notice the negatives of the graphical method and propose its modifications that eliminate these negatives. It involves the introduction of a weighted graphical method and also the elimination of the problem of non-invariance of the order of variants for changing the order of criteria.*

Keywords: multi-criteria evaluation of variants, graphical method, weighted graphical method, non-invariance of the graphical method, changing the order of criteria, weighted sum method.

INTRODUCTION

In the article, the authors describe an area of operational research called multi-criteria evaluation of variants (MCEV). The subject Operational research is a part of the study program at the Faculty of Military Leadership of the University of Defense (FVL UO) in Brno. Its teaching is provided by the Department of Quantitative Methods. As part of the course, students learn to solve problems of an economic nature, which are also closely related to military issues. These are mainly linear programming problems, transport problems, matrix games and precisely problems of multi-criteria evaluation of variants; see [8].

In its first chapter, the article presents the basic methods of solving MCEV tasks as they are commonly taught at universities. The second chapter summarizes the advantages and disadvantages of the graphical method and the weighted sum method. It turns out that the negatives of the graphical method are so serious that they can even be called disqualifying for the correct use of this method. Furthermore, in this chapter, the authors present proposals for modifications of the graphical method, which remove these negatives.

The issue of multi-criteria evaluation of variants is a commonly used method, there are a large number of articles describing the use of MCEV. For example the contribution [4] provides an overview of MCEV methods as well as methods for determining criteria weights. The [9] contribution also deals with the sensitivity analysis of the evaluation of the variants according to the criteria weights. The article [7] examines the methods of evaluation of tenders in public contracts, applies MCEV methods in the selection of the most suitable tender. The [6] paper shows how decision theory combined with operational research techniques could be applied in practice to solve complex water management and planning problems to reduce water loss. The application of the MCEV method in 3D printing to support the protection of personnel during the Covid pandemic is part of the article [3]. Many articles dealing with the MCEV issue contain the use of Saaty's method for determining the weights of the criteria as well as the application of fuzzy issues. An example of such an article is [2].

There are more ways to apply the graphical method when solving the problem of multi-criteria evaluation of variants. A purely numerical form of the graphical method can be used, as well as a graphical form of this method. It displays graphically the individual variants and their evaluation according to individual criteria. There are more ways of such visualizations, they are clearly summarized in the article [5].

We did not find a publication that mentioned the negatives of the graphical method. Similarly, we did not find the procedures mentioned in this article, i.e. taking into account the weights of the criteria in the graphical method (weighted graphical method) or eliminating the problem of non-invariance of the graphical method. In the book [1] on p. 29–30, the authors list seven properties that a method leading to finding compromise solutions should fulfill. The third property "Invariance with respect to permutations of the criteria" is a key property that – as will be shown later in the article – the classical graphical method does not fulfill.

1 MULTI-CRITERIA EVALUATION OF VARIANTS

This chapter serves to introduce the reader to the issue of multi-criteria evaluation of variants (MCEV). An introduction to the MCEV will be presented here, i.e. the role of multi-criteria evaluation of variants is introduced and the procedure common to both methods is described. Both commonly used methods, i.e. the graphical method and the weighted sum method, are then described in the following subsections. The key part is chapter 2. In it, the authors present the advantages and disadvantages of both methods of solving MCEV problems, and propose such modifications of the graphical method, which then eliminate the negatives of the graphical method.

1.1 Introduction to multi-criteria evaluation of variants

We consider the task specified as follows:
We have p variants $X_1, X_2, ..., X_p$ that are evaluated according to k criteria $A_1, A_2, ..., A_k$. These evaluations form the so-called criterion matrix

$$Y = (y_{ij}), \tag{1}$$

where the element y_{ij} indicates the evaluation of the variant X_i according to the criterion A_j, $i = 1, 2, ..., p,\ j = 1, 2, ..., k$. The goal is to find the order of all variants (from best to worst) taking into account all criteria.

There are two methods commonly used in solving the MCEV problem – the graphical method and the weighted sum method. Both of them are based on the so-called normalized criterion matrix $R = (r_{ij})$. First, we describe how we obtain the matrix R.

The criteria are of maximization or minimization type. The minimization criteria are converted to maximization by expressing the savings against the worst variant, i.e.

$$y'_{ij} = \max_i (y_{ij}) - y_{ij},\ i = 1, 2, ..., p,\ j = 1, 2, ..., k. \tag{2}$$

In the criterion matrix $Y' = (y'_{ij})$ there are already all criteria of the maximization type – the higher the value in the given column, the better the variant according to the given criterion. However, the values in different columns are still in different units (price in CZK, display resolution in MPx, weight in grams, etc.), and therefore cannot be compared with each other. Therefore, the so-called normalized criterion matrix R is introduced, the elements of which are

$$r_{ij} = \frac{y'_{ij} - d_j}{h_j - d_j},\ i = 1,2,...,p,\ j = 1,2,...,k,\ \text{where} \tag{3}$$

$$d_j = \min_i(y'_{ij}) \text{ and } h_j = \max_i(y'_{ij}).$$

The matrix $R = (r_{ij})$ has all elements unitless numbers from the interval $\langle 0,1\rangle$. Now the values from different columns (evaluation according to different criteria) are comparable to each other.

In the next section, we will describe two methods to determine the order of the variants. It is necessary, taking into account all the criteria, to obtain some single quantitative characteristic (for each variant), on the basis of which we can easily determine this ranking.

1.2 Graphical method

A star coordinate system with semi-axes $A_1, A_2, ..., A_k$ is introduced. Two adjacent semi-axes are at an angle $\alpha = \frac{2\pi}{k} = \frac{360^\circ}{k}$. For each variant X_i, we will plot the corresponding values r_{ij}, $i = 1,2,...,p,\ j = 1,2,...,k$ on individual semi-axes A_j. We connect every two adjacent values (on adjacent semi-axes) with a line segment. In the graph, each variant X_i is displayed as a corresponding polygon – generally a k-angle. The more content a given polygon has, the better the respective variant is.

The given polygon X_i, i.e. the respective k-angle, can be divided into k triangles, where we always know one angle (angle α) and the lengths of the two sides that enclose this angle. If we denote these sides b, c, then the content of the triangle is given by the relation

$$S_\triangle = \frac{1}{2} bc \sin\alpha.$$

Then the content S of the polygon X_i is

$$S(X_i) = \frac{1}{2}\sin\alpha(r_{i1}r_{i2} + r_{i2}r_{i3} + ... + r_{i,k-1}r_{ik} + r_{ik}r_{i1}).$$

Since the value of $\sin\alpha$ is the same for all variants, the order of the variants can also be obtained by simply comparing the expressions in brackets, i.e. the sums

$$S'(X_i) = r_{i1}r_{i2} + r_{i2}r_{i3} + ... + r_{i,k-1}r_{ik} + r_{ik}r_{i1}. \tag{4}$$

The larger the value of the function $S'(X_i)$, the better the variant X_i is.

1.3 Weighted sum method

As the name of the method suggests, this method is based on determining the weighted sum of r_{ij} values from the criterion matrix, i.e. standardized evaluations of the variant according to the criteria. The weights express the importance of the criteria. The relative importance of individual criteria can be expressed by means of a vector of criteria weights

$$\vec{v} = (v_1, v_2, ..., v_k), \text{ where } v_j \geq 0, \sum_{j=1}^{k} v_j = 1. \tag{5}$$

Thus, the weighted sum method uses knowledge of criteria weights. For each variant X_i, the sum of the products of the evaluation r_{ij} and the respective weight v_j is calculated, i.e. the value of the function is

$$VS(X_i) = \sum_{j=1}^{k} v_j r_{ij} \longrightarrow \max. \tag{6}$$

The larger this value is, the better the variant X_i is. The order of variants (from best to worst) is then given by this function.

Note: Due to the fact that the sum of the weights is equal to 1, it is clear that the weighted sum is also a weighted average here.

2 DISADVANTAGES OF THE GRAPHICAL METHOD AND THEIR ELIMINATION

2.1 Methods – advantages and disadvantages

In this chapter, we present a list of the advantages and disadvantages of the graphical method and the weighted sum method in the area of multi-criteria evaluation of variants, see table 1. In doing so, we assess the methods as described in the previous section, i.e. as they are commonly defined.

Method	Advantages	Disadvantages
Graphical method	- color graph, impressive in presentations and annual reports, ...	- all criteria are equally important, respectively the importance of criteria is not considered; - the results are not invariant, see further.
Weighted sum method	- the importance of the criteria is taken into account using the criteria's weights; - the results are invariant, see further.	- unattractive look, just "a stark table with many numbers" (contrast with a color graph)

Tab. 1. Advantages and disadvantages of both methods

The authors are aware that this may not be a complete list of the positives and negatives of these methods. Above all, however, the table clearly shows the areas on which the article focuses. These are the disadvantages (negatives) of the graphical method, as well as suggestions for modifications to the graphical method that eliminate these negatives.

2.2 Weighted graphical method

This is a modification of the graphical method with consideration of criteria weights. This modification therefore eliminates the first negative of the graphical method.

We will consider the weights of the criteria as they are introduced in subsection 1.3, see the formula (5). Instead of the normalized criterion matrix $R = (r_{ij})$ we will use the matrix $R' = (r'_{ij})$, where

$$r'_{ij} = v_j r_{ij},\ i = 1,2,...,p,\ j = 1,2,...,k. \tag{7}$$

We get the order of the variants using the function

$$S'_v(X_i) = r'_{i1}r'_{i2} + r'_{i2}r'_{i3} + ... + r'_{i,k-1}r'_{ik} + r'_{ik}r'_{i1}. \tag{8}$$

The higher the value of $S'_v(X_i)$, the better the variant X_i is. It is obvious that the different importance of the criteria is already taken into account here. However, as we will see in the next subsection, there is a much more serious problem with the graphical method.

2.3 The problem of invariance of the graphical method

The problem we are going to describe now is a very fundamental problem. On the following example and its purely formal change (not factual), we will show this problem.

2.3.1 Example 1a

Let us consider the task of multi-criteria evaluation of variants given by the criterion matrix

$$Y = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0,7 & 0,7 & 0,2 & 0,2 \\ 0 & 0,5 & 0,5 & 1 \\ 0,7 & 0,2 & 0,7 & 0,2 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{9}$$

We will solve this task using a graphical method. It is obvious that the matrix Y is already a normalized criterion matrix, i.e. $R = Y$. There are four criteria in the task, therefore there will be four semi-axes, each of the two adjacent ones making an angle $\alpha = \frac{\pi}{2} = 90°$. Since $\sin\alpha = 1$, then $S(X_i) = \frac{1}{2}S'(X_i)$.

Table 2 summarizes the calculated values of the function $S'(X_i)$ for individual variants X_i, $i = 1,2,...,p$. The table also shows the order of individual variants, determined according to the value of $S'(X_i)$, see the formula (4).

Variant X_i	$S'(X_i) = r_{i1}r_{i2} + r_{i2}r_{i3} + r_{i,3}r_{i4} + r_{i4}r_{i1}$		**Order**
X_1	$1\cdot 1 + 1\cdot 0 + 0\cdot 0 + 0\cdot 1$	$= 1$	1.
X_2	$0{,}7\cdot 0{,}7 + 0{,}7\cdot 0{,}2 + 0{,}2\cdot 0{,}2 + 0{,}2\cdot 0{,}7$	$= 0{,}81$	2.
X_3	$0\cdot 0{,}5 + 0{,}5\cdot 0{,}5 + 0{,}5\cdot 1 + 1\cdot 0$	$= 0{,}75$	3.
X_4	$0{,}7\cdot 0{,}2 + 0{,}2\cdot 0{,}7 + 0{,}7\cdot 0{,}2 + 0{,}2\cdot 0{,}7$	$= 0{,}56$	4.
X_5	$1\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 1$	$= 0$	5.

Tab. 2. Calculation of the values of $S'(X_i)$ for the variants $X_i,\ i = 1,2,...,p$, and the order of the variants, Ex. 1a

Graphically, the corresponding polygons are shown in color (red, blue, violet, green and yellow) in the figure 1.

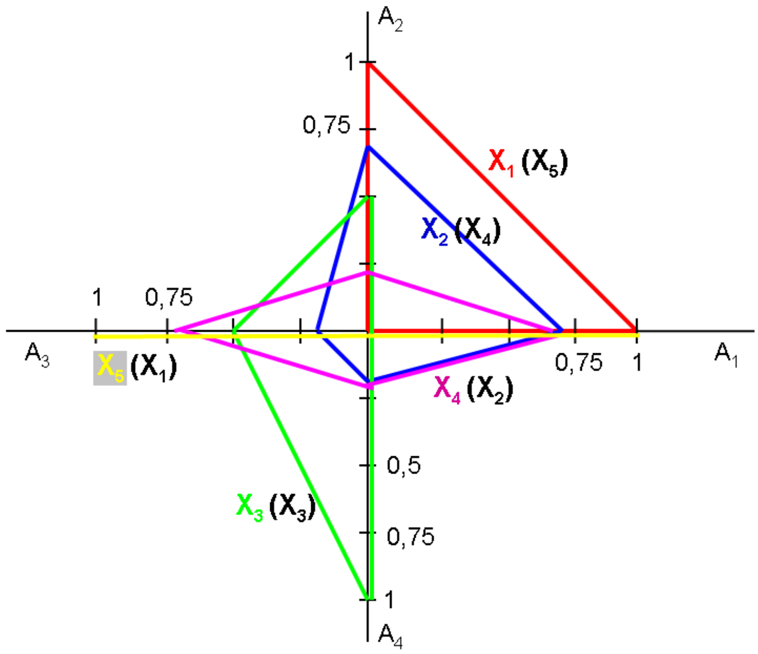

Fig. 1. Graphs of polygons from Ex. 1a (in color) and 1b (in black)
Source: own

2.3.2 Example 1b

Let us consider exactly the same example as Ex. 1a, we just switch the order of the second and third criteria. We will again determine the order of the variants using the graphical method.

It is clear that in the normalized criterion matrix the second and third columns are swapped with respect to the matrix (9). Thus, the normalized criterion matrix now has the form

$$R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0{,}7 & 0{,}2 & 0{,}7 & 0{,}2 \\ 0 & 0{,}5 & 0{,}5 & 1 \\ 0{,}7 & 0{,}7 & 0{,}2 & 0{,}2 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

Table 3 recalculates the values of the function $S'(X_i)$ for the individual variants $X_i,\ i = 1,2,...,p$, and lists the order of the individual variants – according to the value of $S'(X_i)$.

Variant X_i	$S'(X_i) = r_{i1}r_{i2} + r_{i2}r_{i3} + r_{i,3}r_{i4} + r_{i4}r_{i1}$	**Order**
X_1	$1\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 1 = 0$	5.
X_2	$0{,}7\cdot 0{,}2 + 0{,}2\cdot 0{,}7 + 0{,}7\cdot 0{,}2 + 0{,}2\cdot 0{,}7 = 0{,}56$	4.
X_3	$0\cdot 0{,}5 + 0{,}5\cdot 0{,}5 + 0{,}5\cdot 1 + 1\cdot 0 = 0{,}75$	3.
X_4	$0{,}7\cdot 0{,}7 + 0{,}7\cdot 0{,}2 + 0{,}2\cdot 0{,}2 + 0{,}2\cdot 0{,}7 = 0{,}81$	2.
X_5	$1\cdot 1 + 1\cdot 0 + 0\cdot 0 + 0\cdot 1 = 1$	1.

Tab. 3. Calculation of the values of $S'(X_i)$ for the variants $X_i,\ i = 1,2,...,p$, and the order of the variants, Ex. 1b

If we compare table 2 with table 3, we see that the order of variants is now completely reversed from the order in Ex. 1a.

The respective polygons are graphically represented by black labels in the figure 1.

2.3.3 Conclusion from examples – non-invariance

The conclusion that we can make, which is a direct consequence of the results of Examples 1a and 1b, is a serious one. It turns out that the order of variants (from best to worst) obtained when solving the MCEV problem by the graphical method depends on the order of the criteria. Therefore, the findings can be formulated as follows:

The order of the variants (in the task of multi-criteria evaluation of variants) obtained by the graphical method *is not invariant to a change in the order of criteria!*

At the same time, this is purely a formal (administrative) change in the order of the criteria, it does not change the task itself in any way. It is clear that such a finding means a big problem for the method as such and makes the use of the graphical method very problematic.

The exception, when the graphical method works correctly (unequivocally, it is invariant), is the problem of multi-criteria evaluation of variants with only two or three criteria – in this case, the order of the variants is invariant to the order of the criteria.
The proof is obvious. The contents of $S(X_i)$ and also the value of $S'(X_i)$ of the polygon do not change. E.g. for three criteria, when the order of the criteria is changed, the polygons (triangles) corresponding to the variants only change in the mirror image.

At this point it is good to point out that the weighted sum method does not have such a problem, in all cases the order of variants determined by the weighted sum method is invariant to changing the order of the criteria.

2.3.4 Invariant modification of the graphical method

Let us consider all permutations of the criteria $A_1, A_2, ..., A_k$. Denote the set of all these permutations by $P(k)$. Then for each permutation P we can calculate the content of $S_P(X_i)$, or the value of the function $S'_P(X_i)$ (acording to the formula (4)). We determine the order of variants using the average $M(X_i)$ of these values $S'_P(X_i)$ over all permutations $P(k)$,

$$M(X_i) = \frac{\sum_{P \in P(k)} S'_P(X_i)}{k!} = \frac{2}{k-1} \sum_{j=1}^{k-1} \sum_{\substack{m=2 \\ j<m}}^{k} r_{ij} r_{im}.$$

It is obvious that the constant $\frac{2}{k-1}$ has no effect on determining the order of variants from best to worst, and the corresponding order of variants can only be determined by comparing the function values

$$M'(X_i) = \sum_{j=1}^{k-1} \sum_{\substack{m=2 \\ j<m}}^{k} r_{ij} r_{im}. \tag{11}$$

2.3.5 Modification of the graphical method that removes both disadvantages

It is possible to eliminate both disadvantages of the classically defined graphical method, see table 1, simultaneously. So it is enough to combine the procedures described in the sections 2.2 and 2.3.4.

The function

$$M''(X_i) = \sum_{j=1}^{k-1} \sum_{\substack{m=2 \\ j<m}}^{k} r'_{ij} r'_{im}, \tag{12}$$

can be used to determine the order of the variants from best to worst considering all criteria. The values of r'_{ij} are defined in the formula (7). After calculating the values of $M''(X_i)$ for all considered variants and arranging them from the highest value to the smallest, we obtain the order of the variants from best to worst. In doing so, the importance of the criteria (given by the weights of the criteria) is taken into account, and the invariance of the order of the variants with respect to the order of the criteria is also guaranteed.

2.3.6 Example 1a solved by the invariant graphical method

It is proposed to solve the test example 1a from the section 2.3.1 by the modified graphical method just presented. We solve the MCEV problem with a criterion matrix (9). As criteria weights are not considered in the example, we use the formula (11), see table 4. The obtained order of variants is different from the order obtained both in example 1a (see table 2) and from the order obtained in example 1b (see table 3). This could be expected, as well as the location of variants X_1 and X_5, respectively variants X_2 and X_4 at the same position in the order of variants.

Variant X_i	$M'(X_i) = r_{i1}r_{i2} + r_{i1}r_{i3} + r_{i1}r_{i4} + r_{i2}r_{i3} + r_{i2}r_{i4} + r_{i3}r_{i4}$		**Order**
X_1	$1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 0$	$= 1$	4.–5.
X_2	$0{,}7 \cdot 0{,}7 + 0{,}7 \cdot 0{,}2 + 0{,}7 \cdot 0{,}2 + 0{,}7 \cdot 0{,}2 + 0{,}7 \cdot 0{,}2 + 0{,}2 \cdot 0{,}2$	$= 1{,}09$	2.–3.
X_3	$0 \cdot 0{,}5 + 0 \cdot 0{,}5 + 0 \cdot 1 + 0{,}5 \cdot 0{,}5 + 0{,}5 \cdot 1 + 0{,}5 \cdot 1$	$= 1{,}25$	1.
X_4	$0{,}7 \cdot 0{,}2 + 0{,}7 \cdot 0{,}7 + 0{,}7 \cdot 0{,}2 + 0{,}2 \cdot 0{,}7 + 0{,}2 \cdot 0{,}2 + 0{,}7 \cdot 0{,}2$	$= 1{,}09$	2.–3.
X_5	$1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0$	$= 1$	4.–5.

Tab. 4. Calculation of the values of $M'(X_i)$ for variants $X_i,\ i = 1, 2, \ldots, p$, and the order of the variants determined by the invariant graphical method

If we were to solve the MCEV task considering the weights of the criteria, we would determine the order of the variants based on the values of the function $M''(X_i)$, see the formula (12).

CONCLUSION

The article deals with the issue of multi-criteria evaluation of variants (MCEV) with special regard to the graphical method. The authors described the negatives of the commonly used procedure of this method and presented modifications of the graphical method that eliminate both of these disadvantages. Normally, the graphical method does not work with criteria scales. Usually, the graphical method does not take into account the different importance of individual criteria, it works with the criteria as if they all have the same importance. In subsection 2.2, the authors proposed a modification of the graphical method that takes the weights of the criteria into account.

Using an example, the authors showed that the common graphical method for solving the MCEV problem provides a solution (order of variants from best to worst) that is not invariant to changing the order of criteria. In section 2.3.4, the authors present a procedure, a modification of the graphical method that solves this problem. In section 2.3.5, the combination of both of these new modifications into a single procedure, which solves both of the current shortcomings of the common graphical method, is then presented. In the 2.3.6 section, the proposed new method was applied to a concrete example.

The authors believe that the mentioned modifications of the graphical method make this work unique in comparison to other works dealing with the issue of multi-criteria evaluation of variants. To find out the relevance of such a statement, the authors conducted a relatively detailed research, the most interesting findings of which are commented in the introduction of the article.

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DESIGNING AND TEACHING AN ADVANCED COURSE COVERING OPERATING SYSTEM KERNEL DEVELOPMENT (FOR EMBEDDED AND REAL-TIME SYSTEMS)

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Abstract: *In this article, we discuss the motivation for including an operating system kernel development course in the master curriculum of the Faculty of Informatics at Czech Technical University (CTU) at Prague. We provide arguments for having this course included in the master's curriculum. We present selected options for the content of the course lectures and labs and options for technical infrastructure. Experience from delivering the mentioned course is shared.*

Keywords: operating system, operating system kernel, embedded systems, real-time systems, device drivers, LINUX.

INTRODUCTION – WHY/WHEN SHOULD OPERATING SYSTEM KERNEL DEVELOPMENT BE PART OF THE CURRICULUM?

Understanding the functionality and implementation of operating systems is essential knowledge for any IT engineer. Although recent trends in demand for engineering education show a significant focus on the development of WEB and mobile applications, without understanding the functionality of operating systems it is not possible to complete advanced IT tasks (deployment, troubleshooting, performance, and tuning, improving the security of IT solutions).

The current portfolio of operating system courses at *the Faculty of Informatics of Czech Technical University (CTU)* is focused on the functionality and administration of operating systems. UNIX/LINUX is used for practical labs, mainly because of free-of-charge availability, portability, good documentation, and mainly because of widespread usage.

The industrial demand (especially around embedded and real-time systems), interest from students, and the meaning of experts resulted in the requirement for the development of a course covering the development of operating system kernels. We have naturally chosen LINUX as a system used for demonstration of implementation techniques and practical labs. The main reasons are the source code's availability, the kernel's clear structure, and the fact that LINUX is an open system.

1 OPERATING SYSTEMS CURRICULUM AT CTU

1.1 Traditional Operating Systems Courses at CTU

The university curriculum at the faculty of informatics at CTU traditionally consists of three courses covering the area of operating systems:

- *Introduction to the UNIX Operating System* - This course is obligatory for all approximately 1000 students in the first year of their studies at the Faculty of Informatics. The main objective of this course is to make students familiar (as soon as possible) with the technical infrastructure of IT solutions. LINUX is used for labs – students are made familiar with processes, files, text processing of configuration files, and SHELL scripting.
- *Operating Systems* – classical course covering
    - architecture of operating systems (OS) and architecture of OS kernel
    - implementation of processes and threads
    - classical synchronization problems and their solution
    - virtual memory and implementation of memory management
    - technologies of storing data on fixed media (disks),
    - virtualization of disk space
    - monitoring, performance and tuning, and security of operating systems

  Laboratory exercises for this course are being delivered on LINUX, SOLARIS, and Windows.

- *UNIX and LINUX System Administration* – The objective of this course is to make students familiar with the main areas of operating systems administration

1.2 Operating Systems and System Programming course

Growing demand from academia and industry resulted in the introduction of a new course – *Operating Systems and System Programming.* The objective of this course is to make students familiar with the knowledge and methods needed for the modification and development of operating system kernels. We have naturally chosen the LINUX kernel. Students will learn about the implementation of the LINUX kernel, and they complete lab exercises consisting of the creation of new kernel modules or modification of existing modules (subsystems).

The main topics of this course are:

- tools for debugging LINUX kernel and dynamic kernel modules
- implementation of systems calls and interrupt handlers
- kernel synchronization mechanisms
- kernel support for symmetric multiprocessing
- development of device drivers
- address space of processes and memory management
- implementation of file systems, Virtual File System (VFS)
- kernel tuning and kernel portability
- implementation of network services (implementation of sockets)
- architecture (hardware) dependent kernel services, system booting, and startup

- implementation of virtualization support (e.g. `KVM`, **`cgroups`**)
- specifics of LINUX kernel development for embedded systems and real-time systems

The aim of this course is not to teach students how to develop a complete OS kernel. Such work is outside the standard knowledge of university students. The number of lines of LINUX kernel source code currently exceeds 35 million lines (See **Fig. 1.**).

The second drawback is that such work needs to be distributed among several (groups of) students and putting together the work of these (groups of) students represents an enormous amount of work. We rather focus on giving students smaller (but realistic) tasks, like developing (part of) a kernel subsystem or modifying an existing kernel subsystem.

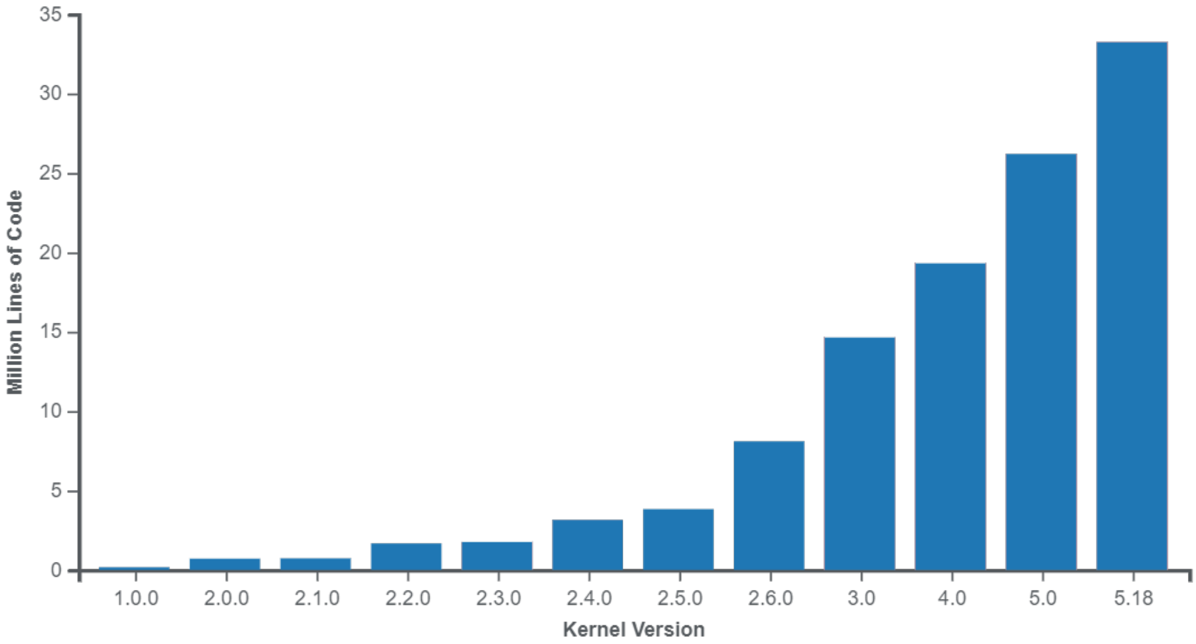

Fig. 1. Number of lines of the LINUX kernel source code
Source: https://en.wikipedia.org/wiki/Linux_kernel

Besides standard OS kernel topics (processes, files, synchronization primitives, device drivers, network interfaces), we have put extra focus on kernel subsystems supporting ***virtualization*** (i.e., fundamentals for ***cloud computing***) and on the development of kernels for ***embedded systems.*** The main reason is the growing demand for engineers to be familiar with the development of software for embedded systems. Significant demand is coming from the automotive and precise machinery industries.

Recommended books for this course are [1], [2] and [3]. The main source of labs is the LINUX Kernel Teaching Project [7]. This project contains lab exercises of reasonable complexity (creating dynamically loadable kernel modules providing modifications of the functionality of kernel subsystems).

2 LABORATORY INFRASTRUCTURE FOR LINUX KERNEL DEVELOPMENT

2.1 Virtualized Lab Development Environment

The software development environment for LINUX Kernel development is quite complex. The main drawback is that when making changes in the developed kernel, we may introduce situations when the system will not be able to boot. This means that we cannot use a kernel of a *“native host”* system in the development process. We need to do the development in an emulated HW environment. Once the development of a kernel (module) is successful, we can replace the *“native host”* kernel with the developed kernel and boot the system.

To mitigate this situation, we need to use a stable HW emulator – e.g. QEMU [4]. QEMU is an emulator that runs on top of the host system (See **Fig. 2**.). Performance of QEMU can be improved using LINUX KVM (Kernel Virtual Machine, see also **Fig. 2**.). More details about KVM are available at [5]. We have not considered using XEN (or other hypervisors supporting paravirtualization).

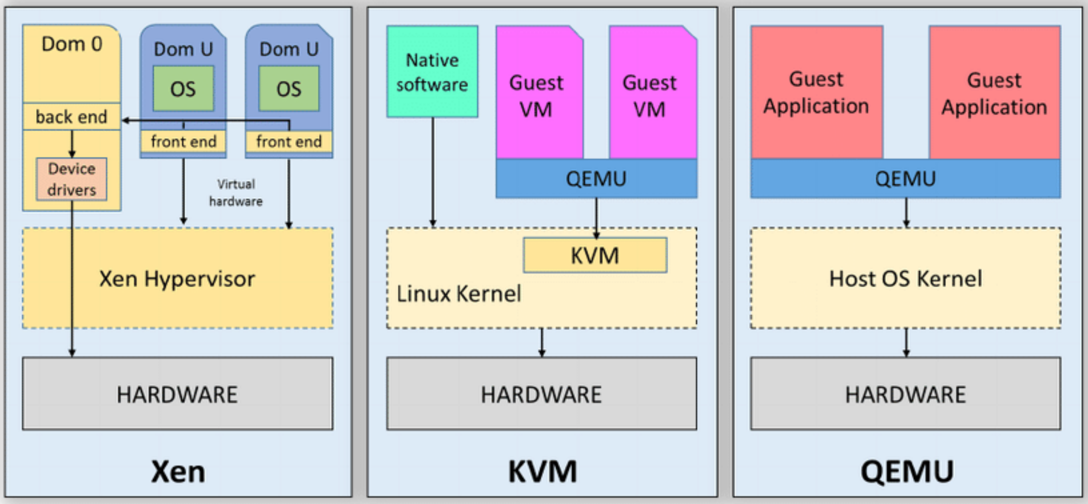

Fig. 2. Comparison of XEN, KVM and QEMU
Source: https://www.researchgate.net/figure/Comparison-of-Xen-KVM-and-QEMU_fig1_281177318

During the kernel development process, we usually need to work with several instances of the development environment. The reason is that we usually need to develop multiple different instances of the LINUX kernel. We recommend virtualizing the host operating system using *Oracle Virtual Box* [6] of a similar hypervisor. This is a solution we have recommended to our students.

Creating a described development environment is quite challenging. We have therefore prepared ready-made images that students can use instantly.

2.2 LINUX Kernel Teaching Project

After details research, we have chosen lab exercises publicly available in the *LINUX Kernel Teaching (LKT)* project. The main arguments for this decision are:

- the whole lab environment is available free of charge

- labs are to be used with the latest version of the LINUX kernel
- the complexity of lab exercises is adequate – there are sets of labs with different complexity
- ARM (embedded) hardware emulator is part of the LKT package
- labs are available as a **`GitHub`** project (students can contribute to this project as well)
- labs demonstrate all the most important kernel subsystems:
  - process and memory management
  - file systems (including VFS – Virtual File System)
  - device drivers, handling of interrupts
  - kernel synchronization primitives (also for symmetric multiprocessing)
  - networking
  - security
- most exercises consist of modification of functionality of kernel subsystems (no need for developing a kernel module from scratch)

Our experience with using *the LINUX Kernel Teaching* project for lab exercises is very positive. We have given students a basic set of exercises to be completed to receive credit for completing the course. Additional points (leading to better final evaluation) can be gained when completing extra exercises.

Special SW development tools need to be used for debugging kernel modules. We lack *a "high-level development environment"* when working with the LINUX kernel. We mostly use debugging prints, **`kdb`**, **`gdb`**, and **`kgdb`** debuggers, and various tracing utilities (**`ftrace, strace, ltrace, dtrace`**).

A good survey of LINUX kernel debugging tools is available at [8].

2.2 LINUX Kernel Teaching Project – Lab exercise example

Let's demonstrate briefly how kernel development works. Below is a very simple example of a kernel module. When loading into the kernel, the message "`Hi`" will be generated. When unloading the kernel module, the "`Bye`" message will be generated:

```
#include <linux/kernel.h>
#include <linux/init.h>
#include <linux/module.h>

MODULE_DESCRIPTION("My kernel module");
MODULE_AUTHOR("Me");
MODULE_LICENSE("GPL");

static int dummy_init(void)
{
        pr_debug("Hi\n");
        return 0;
}

static void dummy_exit(void)
{
        pr_debug("Bye\n");
}

module_init(dummy_init);
module_exit(dummy_exit);
```

A complete example of compiling and loading/unloading a kernel module is presented below:

```
faust:~/lab-01/modul-lin# ls
Kbuild  Makefile  modul.c
faust:~/lab-01/modul-lin# make
make -C /lib/modules/`uname -r`/build M=`pwd`
make[1]: Entering directory `/usr/src/Linux-2.6.28.4'
  LD      /root/lab-01/modul-lin/built-in.o
  CC [M]  /root/lab-01/modul-lin/modul.o
  Building modules, stage 2.
  MODPOST 1 modules
  CC      /root/lab-01/modul-lin/modul.mod.o
  LD [M]  /root/lab-01/modul-lin/modul.ko
make[1]: Leaving directory `/usr/src/linux-2.6.28.4'
faust:~/lab-01/modul-lin# ls
built-in.o  Kbuild  Makefile  modul.c  Module.markers
modules.order  Module.symvers  modul.ko  modul.mod.c
modul.mod.o  modul.o
faust:~/lab-01/modul-lin# insmod modul.ko
faust:~/lab-01/modul-lin# dmesg | tail -1
Hi
faust:~/lab-01/modul-lin# rmmod modul
faust:~/lab-01/modul-lin# dmesg | tail -2
Hi
Bye
faust:~/lab-01/modul-lin#
```

We have used many complex examples in real lab work. Examples will be given during the presentation of this article – presentation of more complex examples is limited by the available space in this text.

CONCLUSION

We have delivered the course described in this article two times, each time for 30+ students. The feedback from students was very positive. Students very much appreciate realistic lab exercises with adequate complexity.

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