# Mathematics, Information Technologies and Applied Sciences 2019

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**Editors:**

**Jaromír Baštinec and Miroslav Hrubý**

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## Aims and target group of the conference:

The conference **MITAV 2019** is the sixth annual MITAV conference. It should attract in particular teachers of all types of schools and is devoted to the most recent discoveries in mathematics, informatics, and other sciences as well as to the teaching of these branches at all kinds of schools for any age groups, including e-learning and other applications of information technologies in education. The organizers wish to pay attention especially to the education in the areas that are indispensable and highly demanded in contemporary society. The goal of the conference is to create space for the presentation of results achieved in various branches of science and at the same time provide the possibility for meeting and mutual discussions of teachers from different kinds of schools and orientation. We also welcome presentations by (diploma and doctoral) students and teachers who are just beginning their careers, as their novel views and approaches are often interesting and stimulating for other participants.

## Organizers:

Union of Czech Mathematicians and Physicists, Brno branch (JČMF),
in co-operation with
Faculty of Military Technology, University of Defence in Brno,
Faculty of Science, Faculty of Education and Faculty of Economics and Administration,
Masaryk University in Brno,
Faculty of Electrical Engineering and Communication, Brno University of Technology.

## Venue:

Club of the University of Defence in Brno, Šumavská 4, Brno, Czech Republic
June 20 and 21, 2019.

## Conference languages:

Czech, Slovak, English, Russian

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## Programme of the conference:

*Thursday, June 20, 2019*

| | |
|---|---|
| 12:00-13:15 | Registration of the participants |
| 13:15-13:30 | Opening of the conference |
| 13:30-14:20 | Keynote lecture No. 1 (Hana Nečasová and Iveta Wilnerová, Czech Republic) |
| 14:20-14:30 | Break |
| 14:30-15:20 | Keynote lecture No. 2 (Aleš Nekvinda, Czech Republic) |
| 15:20-16:00 | Break |
| 16:00-16:50 | Keynote lecture No. 3 (Pavel Šišma, Czech Republic) |
| 16:50-17:00 | Break |
| 17:00-17:50 | Presentations of papers in session I |
| 18:00-18:50 | Conference dinner |
| 19:00 | Bus departure to the Brno lake |
| 19:30-22:00 | Social event (evening on the boat) |

*Friday, June 21, 2019*

| | |
|---|---|
| 09:00-10:45 | Presentations of papers in two parallel sessions II and III |
| 10:45-11:15 | Break |
| 11:15-12:45 | Presentations of papers in session IV |
| 12:45 | Closing |

Each MITAV 2019 participant received printed collection of abstracts **MITAV 2019** with ISBN 978-80-7582-097-6. CD supplement of this printed volume contains all the accepted contributions of the conference.

Now, in autumn 2019, this **post-conference CD** was published, containing extended versions of selected MITAV 2019 contributions. The proceedings are published in English and contain extended versions of 12 selected conference papers. Published articles have been chosen from 32 conference papers and every article was once more reviewed.

## Webpage of the MITAV conference:

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# REMARKS ON SEPARATE CONTINUITY VERSUS JOINT CONTINUITY OF FUNCTIONS $f\colon \mathbb{R}^m \to \mathbb{R}$


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**Abstract:** *In this paper the notions of dual separately continuous functions and dominance of one variable for joint continuity of functions $f\colon \mathbb{R}^m \to \mathbb{R}$ are introduced and properties of such functions are investigated. Moreover this notion is compared with the notion of separately continuous functions and strongly separately continuous functions defined on $\mathbb{R}^m$ respectively.*
*Every dual separately continuous function $f\colon \mathbb{R}^m \to \mathbb{R}$ is separately continuous function. It means that the dual separate continuity is the stronger property for functions defined on $\mathbb{R}^m$ than the separate continuity but not so much stronger to guarantee the joint continuity as it is in the case of the strong separate continuity.*



**Keywords:** separate continuity, continuity, strong separate continuity, dual separate continuity, dominance of a variable for continuity.


## INTRODUCTION

The consideration of separate continuity vis-à-vis joint continuity goes back, at least, to R. Baire (1899) [1]. Since then separate continuity penetrated a variety of mathematical problems, namely into functional analysis ( see e.g. [16],[19]). Separately continuous functions $f\colon \mathbb{R}^m \to \mathbb{R}$ were investigated in several papers (see e.g. [4],[6],[15],[20], [21]). Separately continuous functions from the product of certain type of Hausdorff topological spaces $X$ and $Y$ into a metrizable space $Z$, $f\colon X \times Y \to Z$ were also investigated by many authors for instance [3],[7],[8],[10],[16],[17], [18], [23]. In a lot of papers classes of Hausdorff spaces $X$ and $Y$ are defined by a game theoretic conditions ( see e.g. [2],[10],[13]). Recently there exist papers which investigated separately continuous functions $f\colon \ell^2 \to \mathbb{R}$ (see e.g. [5],[11],[12],[22]).

In this paper we turn back our attention to the functions $f\colon \mathbb{R}^m \to \mathbb{R}$ $(m \geq 2)$.

## 1 SEPARATE CONTINUITY AND STRONG SEPARATE CONTINUITY

First of all we recall some basic notions.

**Definition 1.** A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is said to be *jointly continuous* (usually it is called *continuous*) *at a point* $x^0 = (x_1^0, x_2^0, \ldots, x_m^0) \in \mathbb{R}^m$ if $\lim_{x\to x^0} f(x) = f(x^0)$. Moreover if a function $f$ is jointly continuous at every point $x^0 \in \mathbb{R}^m$, then $f$ is said to be *jointly continuous* on $\mathbb{R}^m$.

**Definition 2.** A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is said to be *separately continuous at a point* $x^0 = (x_1^0, x_2^0, \dots, x_{k-1}^0, x_k^0, x_{k+1}^0, \dots, x_m^0) \in \mathbb{R}^m$ *with respect to the variable* $x_k$ if the function $f_k\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(x) = f(x_1^0, x_2^0, \dots, x_{k-1}^0, x, x_{k+1}^0, \dots, x_m^0)$$

is continuous at the point $x_k^0$. If a function $f$ is separately continuous at the point $x^0 \in \mathbb{R}^m$ with respect to each variable $x_k$, $k = 1, 2, \dots, m$, then $f$ is called *separately continuous at the point* $x^0$. Moreover, if a function $f$ is separately continuous at every point $x^0 \in \mathbb{R}^m$ then $f$ is said to be *separately continuous* on $\mathbb{R}^m$.

It is well known that a function $f$ can be separately continuous at a point $x^0$ without being jointly continuous at $x^0$. The standard example illustrating this phenomenon is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = 0$ if the product of $x_1$ and $x_2$ is different from zero and $f(x_1, x_2) = 1$ elsewhere. This function is separately continuous at $(0,0)$ without being jointly continuous at $(0,0)$. Such example, where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is separately continuous everywhere on $\mathbb{R}^2$ without being jointly continuous at a point can be found [9]. Moreover, there exists a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ discontinuous at the point $(0,0)$ which is continuous along every straight line passing through $(0,0)$ (e.g. [9]). H. Lebesque [14] found a discontinuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,0)$ even if it is continuous along every analytic curve passing through $(0,0)$. Finally A. Rosenthal [20] proved the following proposition: If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at the point $(x_1^0, x_2^0)$ along every convex curve passing through this point, which is (at least) once differentiable, then $f$ is also jointly continuous at the point $(x_1^0, x_2^0)$.

On the other hand, if a function $f\colon \mathbb{R}^m \to \mathbb{R}$ is jointly continuous at $x^0 \in \mathbb{R}^m$ then it is separately continuous at $x^0$ as well.

O. P. Dzagnidze [6] introduced the notion of strongly separately continuous $f\colon \mathbb{R}^m \to \mathbb{R}$ as it follows.

**Definition 3.** A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is said to be *strongly separately continuous at a point* $x^0 = (x_1^0, x_2^0, \dots, x_{k-1}^0, x_k^0, x_{k+1}^0, \dots, x_m^0) \in \mathbb{R}^m$ *with respect to the variable* $x_k$ if the following limit

$$\lim_{x \to x^0} \left( f(x_1, x_2, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_m) - f(x_1, x_2, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_m) \right)$$

is equal to zero. If a function $f$ is strongly separately continuous at the point $x^0 \in \mathbb{R}^m$ with respect to each variable $x_k$, $k = 1, 2, \dots, m$, then $f$ is called *strongly separately continuous at the point* $x^0$. Moreover if a function $f$ is strongly separately continuous at every point $x^0 \in \mathbb{R}^m$ then $f$ is said to be *strongly separately continuous* on $\mathbb{R}^m$.

It is clear that the joint continuity of a function $f\colon \mathbb{R}^m \to \mathbb{R}$ at a point $x^0$ implies the strong separate continuity of $f$ at $x^0$ and also reversely the strong separate continuity of $f\colon \mathbb{R}^m \to \mathbb{R}$ at a point $x^0 \in \mathbb{R}^m$ guarantees the joint continuity of $f$ at $x^0$ (see [6, Theorem 2.1]). Moreover there is proved the following result.

**Theorem 1** ([6, Theorem 2.2])**.** *A function* $f\colon \mathbb{R}^m \to \mathbb{R}$ *is jointly continuous at a point* $x^0 = (x_1^0, x_2^0, \dots, x_m^0) \in \mathbb{R}^m$ *if and only if it is strongly separately continuous at the point* $x^0$ *with respect to only one of the variables and separately continuous at* $x^0$ *with respect to all other variable collectively.*

## 2 DUAL SEPARATE CONTINUITY AND DOMINANCE OF A VARIABLE FOR JOINT CONTINUITY

**Definition 4.** A function $f\colon \mathbb{R}^m \to \mathbb{R}$ is said to be *dual separately continuous at a point* $x^0$, $x^0 = (x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_m^0) \in \mathbb{R}^m$ *with respect to the variable* $x_k$ if

$$\lim_{x \to x^0} f(x_1, x_2, \ldots, x_{k-1}, x_k^0, x_{k+1}, \ldots, x_m) = f(x^0).$$

If a function $f$ is dual separately continuous at $x^0$ with respect to each variable $x_k$, $k = 1, 2, \ldots, m$, then $f$ is called *dual separately continuous at the point* $x^0$. If $f$ is dual separately continuous at every point $x^0 \in \mathbb{R}^m$ then $f$ is said to be *dual separately continuous* on $\mathbb{R}^m$.

The dual separate continuity of a function $f\colon \mathbb{R}^m \to \mathbb{R}$ at a point $x^0$ with respect to the variable $x_k$ means that $f$ is jointly continuous at the point $x^0$ on the hyperplane $x_k = x_k^0$. From this we immediately obtain that $f$ is separately continuous at the point $x^0$ with respect to all other variables except of the variable $x_k$ and we gain the following easy proposition.

**Proposition 1.** *A dual separate continuous function $f\colon \mathbb{R}^m \to \mathbb{R}$ at a point $x^0 \in \mathbb{R}^m$ with respect to the one variable and separately continuous at $x^0$ with respect to the same variable is separately continuous function at $x^0$.*

**Remark 1.** For $m = 2$ the dual separate continuity of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at a point $x^0$ $x^0 = (x_1^0, x_2^0)$ coincides with the separate continuity at $x^0$. Since a dual separately continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at a point $x^0$ with respect to the variable $x_1$ is separately continuous at $x^0$ with respect to the variable $x_2$ and vice versa for other variable.

**Proposition 2.** *For a function $f\colon \mathbb{R}^m \to \mathbb{R}$ to be jointly continuous at a point $x^0 \in \mathbb{R}^m$ it is necessary and sufficient that it be strong separately continuous at a point $x^0$ with respect to the one variable and dual separately continuous at $x^0$ with respect to the same variable.*

*Proof.* Since the $f$ is jointly continuous at a point $x^0$ it is clear that it is strong separately continuous at a point $x^0$ with respect to the variable $x_k$ and also dual separately continuous at $x^0$ with respect to the same variable for every $k$, $1 \le k \le m$. The necessity is proved.
To prove the sufficiency we take any $k$, $1 \le k \le m$, and considere the obvious inequality.

$$| \, f(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m) - f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_m^0) \, | \le$$

$$| \, f(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m) - f(x_1, x_2, \ldots, x_{k-1}, x_k^0, x_{k+1}, \ldots, x_m) \, | \, +$$
$$| \, f(x_1, x_2, \ldots, x_{k-1}, x_k^0, x_{k+1}, \ldots, x_m) - f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_m^0) \, | \, .$$

Since the $f$ is strong separately continuous at a point $x^0$ with respect to the variable $x_k$ and also dual separately continuous at $x^0$ with respect to the same variable, expressions in absolute values on the right-hand side of the previous inequality are arbitrary small. Thefore the expression in absolute value on the left-hand side of that inequality is also arbitrary small. This is equivalent to the fact that $f$ is jointly continuous at a point $x^0$. □

**Definition 5.** For a function $f\colon \mathbb{R}^m \to \mathbb{R}$ a variable $x_k$ is said to be *dominant for the joint continuity at a point* $x^0 = (x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_m^0) \in \mathbb{R}^m$ if

$$\lim_{x \to x^0} f(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m) - f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k, x_{k+1}^0, \ldots, x_m^0)$$

is equel zero. If a variable $x_k$ is dominant for the joint continuity for $f$ at every point $x^0 \in \mathbb{R}^m$ then the variable $x_k$ is said to be *dominant for the joint continuity of* $f$ on $\mathbb{R}^m$.

Directly from the previous definitions we obtain the following propositions.

**Proposition 3.** *Let $k$ be arbitrary, $1 \le k \le m$ and a variable $x_k$ be dominant for the joint continuity at a point $x^0$ for the function $f\colon \mathbb{R}^m \to \mathbb{R}$ and $f$ be separate continuous function at a point $x^0$ with respect to the variable $x_k$ then $f$ is separately continuous function at $x^0$.*

**Proposition 4.** *For a function $f\colon \mathbb{R}^m \to \mathbb{R}$ to be jointly continuous at a point $x^0 \in \mathbb{R}^m$ it is necessary and sufficient that it be separately continuous at a point $x^0$ with respect to the one variable and the same variable is dominant for the joint continuity at a point $x^0$ for the function $f$.*

*Proof.* Since the $f$ is jointly continuous at a point $x^0$ it is clear that it is separately continuous at a point $x^0$ with respect to the variable $x_k$ and also variable $x_k$ is dominant for the joint continuity at a point $x^0$ for the function $f$ for every $k$, $1 \le k \le m$. The necessity is proved.
To prove the sufficiency we take any $k$, $1 \le k \le m$, and considere the following inequality.

$$\mid f(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m) - f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_m^0) \mid \le$$

$$\mid f(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m) - f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k, x_{k+1}^0, \ldots, x_m^0) \mid +$$
$$\mid f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k, x_{k+1}^0, \ldots, x_m^0)) - f(x_1^0, x_2^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_m^0) \mid .$$

Since the variable $x_k$ is dominant for the joint continuity at a point $x^0$ for the function $f$ and also separately continuous at $x^0$ with respect to the variable $x_k$, expressions in absolute values on the right-hand side of the previous inequality are arbitrary small. Thefore the expression in absolute value on the left-hand side of that inequality is also arbitrary small. This again is equivalent to the fact that $f$ is jointly continuous at a point $x^0$. □

Again directly from the previous definitions we obtain the following proposition.

**Proposition 5.** *The strong separate continuity at $x^0 \in \mathbb{R}$ implies the dual separate continuity at $x^0$ which implies the separate continuity at $x^0$ for a function $f\colon \mathbb{R}^m \to \mathbb{R}$.*

No implication in the Proposition 5 can be reversed as it shown by means of the following examples. The first example shows that the separate continuity $f$ at $x^0 \in \mathbb{R}$ does not imply the dual separate continuity of $f$ at $x^0$. On the basis of the Remark 1 we take into consideration the case when $m > 2$.

**Example 1.** Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a function given by $f(x, y, z) = 0$ if $(x, y, z) = (0, 0, 0)$ and $f(x, y, z) = \frac{xy+xz+yz}{x^2+y^2+z^2}$ otherwice.

If $(x, y, z) \ne (0, 0, 0)$ then $f$ is jointly continuous at $(x, y, z)$. At the point $x^0 = (0, 0, 0)$ the function $f$ is separately continuous with respect to each variable, without being dual separately continuous at $x^0$ with respect to none of variables. We show it only for the first variable. Since $f_1(x) = f(x, 0, 0) = 0$ everywhere, we have that $f$ is separately continuous at $x^0$ with respect to the variable $x$. $f$ is not dual separately continuous at $x^0$ with respect to none of variables. We only show it for the variable $x$. We need to proove that $f(0, y, z)$ is not jointly continuous at $x^0$ as a function of two variables. To show this, it is enough to realize that every neighborhood of the point $x^0$ in the hyperplane $x = 0$ contains points of the form $(0, a, a)$ and $f(0, a, a) = \frac{1}{2}$.

The next example shows that the dual separate continuity $f$ at $x^0$ does not imply the strong separate continuity $f$ at $x^0$.

**Example 2.** Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a function given by $f(x, y, z) = 0$ if $(x, y, z) = (0, 0, 0)$ and $f(x, y, z) = \frac{x^2yz}{x^4+y^4+z^4}$ otherwice.

If $(x, y, z) \neq (0, 0, 0)$ then $f$ is jointly continuous at $(x, y, z)$. At the point $x^0 = (0, 0, 0)$ the function $f$ is dual separately continuous with respect to each variable, without being strong separately continuous at $x^0$ with respect to none of variables. We show it only for the first variable. Since $f(0, y, z) = 0$ everywhere, we have that $f$ is dual separately continuous at $x^0$ with respect to variable $x$. $f$ is not strong separately continuous at $x^0$ with respect to none of variables. We again to prove it only for the first variable. Consider the sequence $\{X_n\}_{n=1}^{\infty}$, $X_n = (1/n, 1/n, 1/n)$ which is going to $x^0$ but

$$\lim_{n\to\infty} (f(1/n, 1/n, 1/n) - f(0, 1/n, 1/n)) = 1/3.$$


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# INVESTIGATING THE QUALITY OF THE SILICON AND BLACK SILICON INTERFACE FOR PHOTOVOLTAIC APPLICATIONS

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**Abstract:** *Effective utilisation of solar panels is a crucial factor of the future technical advancement of humankind. Great attention is focused on solar cells, hence an effective photovoltaic conversion of solar radiation into electricity. Most widely used material in commercial applications is crystalline silicon, although an intensive effort is devoted to replacing this material with a more cost-friendly and effective compound. One possible solution could be black silicon. The contribution discusses this promising material in terms of photovoltaic applications, influences of technological processes, especially chemical and electrochemical etching, on the quality of silicon-black silicon interface.*

**Keywords:** solar panels, black silicon (b-Si), nanostructured porous silicon (PS), electrically active defects

## INTRODUCTION

Mankind has been already comforted by the fact that for more advanced and convenient life, we need energy in a pure and environmental-friendly form. Energy, whether it is heat, light or fuel need to be rethought and re-evaluated. Nevertheless, only a few of us is willing to admit that classical energy sources such as coal, oil, gas or uranium, and how these are utilised, is limited by time. Human activities affect nature more than before. Reducing negative environmental impacts in this field is a hot topic of the European Union (EU). Countries of the EU have a common intention to increase the share of renewable energies on a wider scale and to replace a great portion of the total energy consumption by these solutions. The main goal is targeted to be 21% for 2020 of the overall electricity production to be purely generated by renewable energy sources such as solar energy, hence the photovoltaic conversion of solar radiation into electricity [1].

Solar energy is a perspective renewable energy source [2]. It is becoming more and more important, and it has been forecast that the following decade 2025-2035 will be significant in this field due to our climate change. Many referring these years to the decade of solar energy.

Commercial, silicon-based solar panels is a widespread technology of the solar panel market. However, it should be noted that while their efficiency is still the highest (around 25% [3]), the manufacturing process is demanding a more advanced technology (less complicated fabrication process, more cost-friendly solution at room temperature). To find suitable low cost, high-efficiency replacement compounds is a real challenge for scientists [4, 5, 6]. Even complementary technologies, e.g. utilisation of an antireflective coating to reduce energy losses has proved to be expensive for the commercial market. A possible solution that came to light recently could be the involvement of 'black silicon' (b-Si) or nanostructured 'porous silicon' (PS) [7, 8].

## 1 WHAT IS BLACK SILICON?

Silicon is the second most frequent material in the Earth's core (26% - 28%). This dark grey semiconductor is the most widely used material in electronics. Black silicon, nanostructured porous silicon, respectively, as its name implies, is silicon with an altered surface structure, which is capable to reduce the reflection of sunlight; therefore it appears like a really "black" surface (Fig. 1). Silicon possessed as a nanostructured surface layer effectively reduces the reflection of broadband of light, and as a direct consequence, these silicon wafers appear black, instead of the typical silver-grey colour of planar silicon wafers (Fig. 1). This unique property makes black silicon a promising solution as the anti-reflection coating of silicon solar cells.

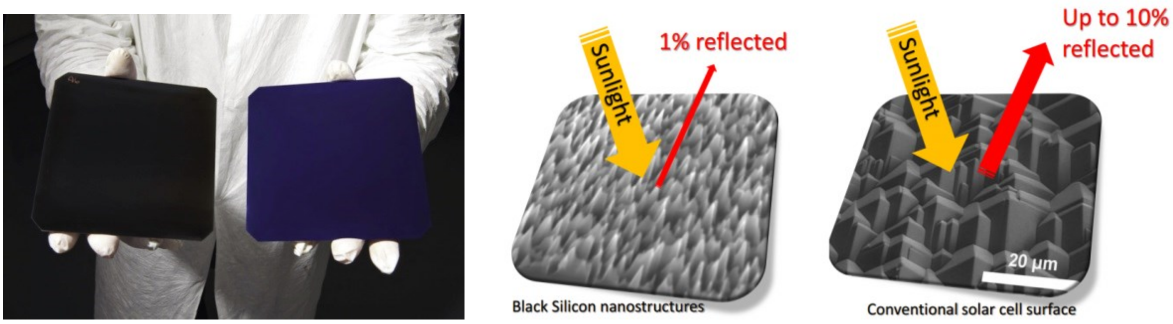


**Fig. 1.** Black silicon and silicon solar cells. Black silicon nanostructures and conventional solar cell surface reflections.
Source: [9] and [10]

In the most basic sense, black silicon, also well-known by the name porous silicon, is a network of air holes within an interconnected silicon matrix. The size of these air holes, called pores (Fig. 2 and Fig. 3), can vary from a few nanometers to a few microns depending on the conditions of formation and the characteristics of the silicon wafer [11].

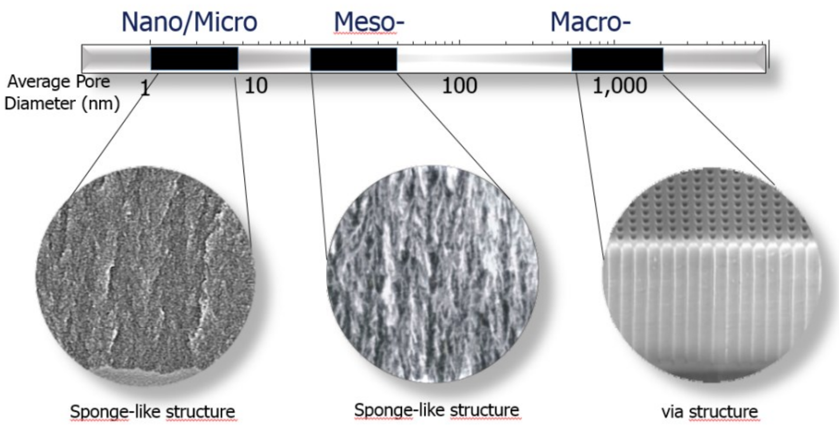


**Fig. 2.** Nano/Micro, Meso and Macro pores.
Source: [11]

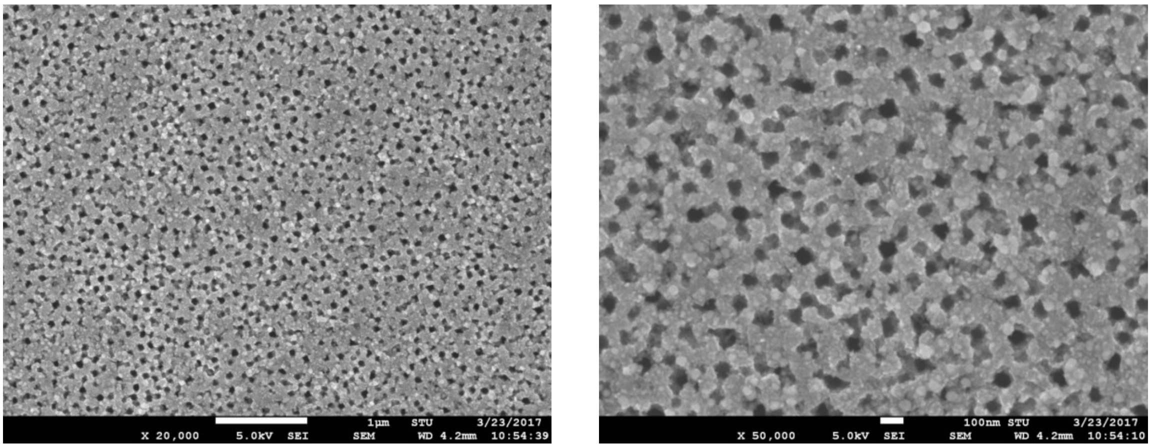


**Fig. 3.** Scanning electron microscope images of a porous silicon structure with about 100 nm wide pores.
Source: own

This structural constellation allows black silicon to have a lower reflectance compared to the classical silicon structure (Fig. 4), moreover lower costs by surface area, higher absorption in wavelengths from 300 to 1200 nm, hence higher effectivity.

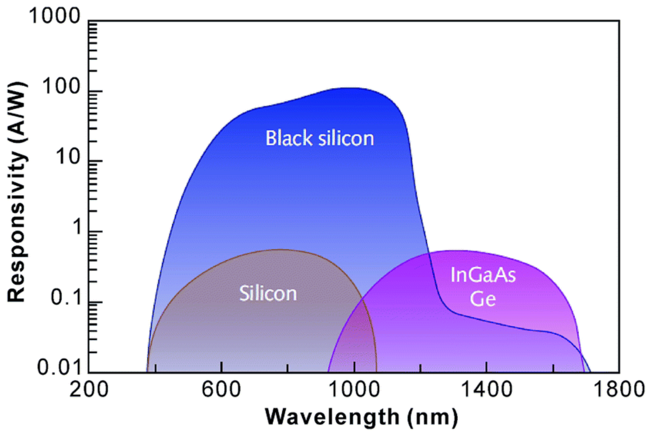


**Fig. 4.** The responsivity of black silicon, silicon and InGaAs.
Source: [12]

The utilization of black silicon has a wide spectrum from solar cells (more effective absorption of sunlights), to micro-mechanical systems (MEMS), optoelectronic and photonic devices (high sensitivity image sensors), TV monitors (ultra-thin monitors), sensors (biological to chemical), satellites (lack silicon field emitters to steer small satellites), telecommunication along fibre optic cables, lithium-ion batteries and $H_2$ production via electrochemically splitting water [13, 14].

The production of black silicon can be achieved by various methods [15]. Schematic diagram of the four basic methods is shown in Fig. 5, where (a) shows the inductive coupled plasma reactive ion etching (ICPRIE, or short ICP) process in an atmosphere of $SF_6$ and $O_2$, (b) depicts the metal-assisted wet chemical etching (MACE) process based on Ag or Au catalyst particles in aqueous solutions of HF and $H_2O_2$ (c) shows the electrochemical etch cell used for macro-porous silicon (mac P-Si) fabrication and (d) schematically, shows the experimental setup for the fs-laser-treated Si surfaces (L-Si) [12].

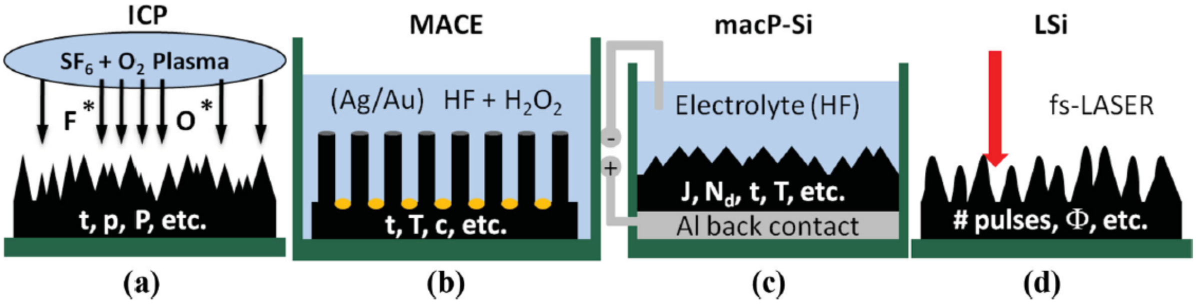


**Fig. 5.** Processes for the production of black silicon
Source: [15]

## 2 BLACK SILICON AND SOLAR CELLS

Solar cells are too expensive compared to other energy sources. The main reason for this a complicated development and many times a complex fabrication process. There is huge effort to reduce manufacturing costs. It is anticipated that black silicon can solve this requirement and will be able to reduce the total manufacturing cost by 10% and shorten the fabrication time of a solar panel by 12% (Tab. 1). Solar cells with black silicon can reach effectivity up to 22.1% [16]. On the market there are some examples of quality solar cell panels, e.g. Suntech Power – producing commercial black silicon solar cells.

Tab. 1 Comparison of solar cell manufacturing complexity [17]

| | Fabrication time / percentual proportion of the process | The percentual proportion of the overall costs |
|---|---|---|
| Commercial Si solar cell – texturizing | 40 min / 14% | 13% |
| Black silicon | 4 min / 2% | 3% |
| Reduction during manufacture | 12% | 10% |

*Disadvantages*
• Influence of etching parameters on the kinetics of photovoltaic processes.
• Contamination with electrically active impurities in the etched part of the silicon substrate.

*Advantages:*
• Achieving higher efficiency (low reflectivity, better sunlight absorption).
• Reduced costs of production as much as possible.

High impact on the above-described advantages is induced electrically active defect states in black silicon, especially at the interface between black and the conventional silicon. To improve the fabrication and the quality of these structures, it is important to understand the mechanism of the generation of these defect states and with this knowledge to optimize the fabrication process.

Widely used method of defect investigation is DLTS (Deep-Level Transient Spectroscopy). This capacitance-based method is sensitive to charge accumulation, trapping, and/or release in barrier structures as metal-insulator-semiconductor (MIS) structures are needed [18]. This method is based on sensing the capacity changes after applying a voltage pulse on the investigated structure. The method follows the relaxation of the system to its original state, which is strongly influenced by the trapping of charges. The obtained capacitive responses are used to construct Arrhenius dependencies, from which we can determine the basic parameters of traps: activation energy and trapping cross-section. These parameters represent a unique picture of the defect state and can serve as a fingerprint to identify the origin of the defect state.

Although black silicon has been envisioned as a promising candidate for future photovoltaics, it still needs optimisation in terms of the fabrication technology, to reduce electrically active defects that can have further downgrading affect on the overall performance. This paper introduces results of a defect study carried out by DLTS on black silicon structures. Here, the PS/$SiO_2$ MIS structures were investigated. The porous p-type Si substrates have been fabricated by the chemical etching with a platinum nanoparticle catalyser with various etch times. This study was realised with aim to contribute to optimization of the chemical PS formation parameters.

## 3 DEFECT DISTRIBUTION IN BLACK SILICON STRUCTURES

Results of DLTS experiments are illustrated on five samples of PS/$SiO_2$ (nanostructured porous Silicon) structures (Tab. 2) based on a p-type (100)-oriented boron-doped silicon substrate (725 µm thick) with resistivity of 9-12 Ωcm. The $SiO_2$ films were prepared by thermal oxidation or by chemical oxidation in $HNO_3$ acid as summarized in Tab. 2. Subsequently, the PS surface was formed by etching in $H_2O_2$ + HF solution for 240, 20, or 7.5 sec using Pt nanoparticles catalyser. Top and bottom contacts were prepared by evaporation of Al through a metal mask. The largest area was 0.25 mm$^2$.

Tab. 2 Samples with various formation conditions of the porous surface of the silicon

<table>
<tr><th rowspan="2">Sample</th><th rowspan="2">Type/ dopant concentration</th><th rowspan="2">Oxidation</th><th rowspan="2">Annealing</th><th colspan="2">Etching</th></tr>
<tr><th>chemical</th><th>time</th></tr>
<tr><td>PS1</td><td rowspan="5">Boron, $10^{16}$ cm$^{-3}$</td><td colspan="2" rowspan="2">Thermal 950°C for 5 minutes in $O_2$</td><td rowspan="5">$H_2O_2$+HF (in Japan)</td><td>240 sec</td></tr>
<tr><td>PS2</td><td>20 sec</td></tr>
<tr><td>PS3</td><td rowspan="3">Chemical $HNO_3$ @ 70 °C/10 min</td><td rowspan="3">in $N_2$ @ 900°C /10 min</td><td rowspan="3">7.5 sec</td></tr>
<tr><td>PS4</td></tr>
<tr><td>PS5</td></tr>
</table>

Electrically active defect states were experimentally evaluated by Deep Level Transient Spectroscopy method. DLTS experiments were carried out in the temperature range from 85K to 420K in vacuum conditions using the BIORAD DL8000 measurement system. Used pulse voltages ranged from -0.5 V up to 0.8 V, while the reverse voltage was set from -0.8 V to -1 V. The pulse width was 20 ms and 100 ms to effectively fill all the electrical active trap levels and the period time was set in a wider range from 0.32 ms to 2 s.

Fig. 6 shows typical DLTFS spectra of the investigated structures PS1, PS2, PS3, PS4 and PS5 at different experimental conditions. We detected four defect states, the parameters compared to literature of Silicon (Fe, Pt, Au or Zn [19]) are showed in the figures for each structure. As illustrated in Fig. 6, defect states of iron (T1, activation energy $\Delta E_T = 0.41$ eV and capture cross-section $\sigma_T = 3.1\times10^{-16}$ cm$^2$), gold (T3 activation energy $\Delta E_T = 0.54$ eV and capture cross-section $\sigma_T = 3.5\times10^{-14}$ cm$^2$) and zinc (T4 activation energy $\Delta E_T = 0.55$ eV and capture cross-section $\sigma_T = 2.7\times10^{-16}$ cm$^2$) were identified in all examined structures.

Traces of Pt [20] (T2 activation energy $\Delta E_T = 0.31$ eV and capture cross-section $\sigma_T = 1.5\times10^{-14}$ cm$^2$) were also observed as a residue of the catalyser, and thus only in PS1, PS2, and PS5. Pt $1.4\times10^{15}$ cm$^{-3}$ defect concentrations were calculated in both samples PS1, PS2 (Fig. 6). In PS5 the concentration of Pt was significantly lower $1.3\times10^{13}$ cm$^{-3}$.

Surface defect states of activation energies 0.06 eV and capture cross-section $\sigma_T = 1.3\times10^{-22}$ cm$^2$ were identified only in PS4 and PS5. Non-homogeneity of the trap distribution was confirmed by repeated DLTS measurements (Fig. 6).

The outcome of the DLTS study showed a significant impact of the experiment alone on the concentration of defects in the measured structures. Each measurement affects localized charges in the structure, however at the same time section of the active defect states are annealed, resulting in changing defect concentrations. Defect concentration is only qualitative, depending on the relaxation time between experiments.

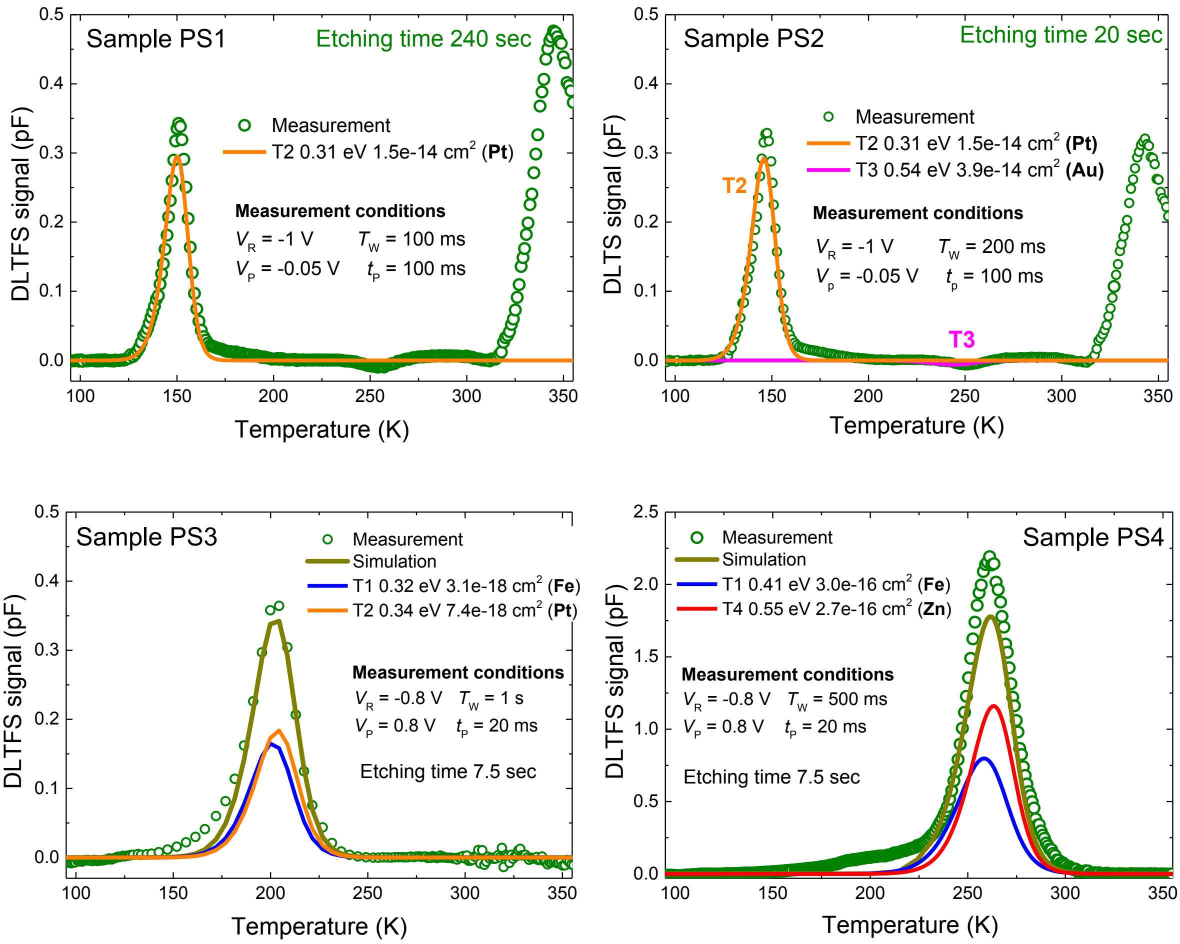


**Fig. 6** Measured DLTS spectra, identified deep levels: values of activation energy and cross capture with simulated curves and measurement conditions parameters).

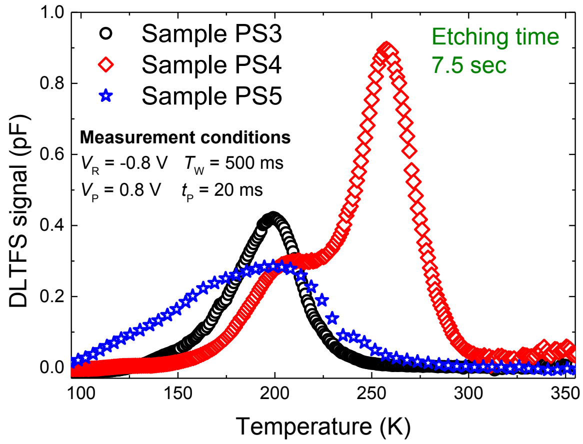


Fig. 6 The non-homogeneity of the defect distribution in investigated samples. DLTS signals of selected samples were measured at the same measuring conditions.

### 4.1 Summary of results

We have investigated five structures with black silicon (metal-porous silicon-silicon) for solar applications by DLTS. The porous p-type Si substrates have been fabricated by the chemical etching with a platinum nanoparticle catalyser with various etch times. Parameters of the deep energy levels T1 (0.41 eV), T2 (0.31 eV), T3 (0.54 eV) and T4 (0.55 eV) were identified with a high level of validity. It is suggested that these levels are related to the impurities in Silicon as Fe, Pt, Au or Zn. Presence of Fe, Au a Zn was confirmed in all samples. Our evaluation indicates that Fe and Zn were induced by the metal parts of the fabrication apparatus. With high probability, gold particles were induced by previous manufacture processes. The change in the distribution of defects and their electrical activity as a function of the etching time of the Si surface was not unambiguously confirmed.

## CONCLUSION

The utilisation of renewable energy sources is one of the ways how to solve the future survival of humankind. Solar energy is the most affordable and cleanest form of renewable energies. Efficient solar cells, hence the photovoltaic conversion of solar radiation into electricity is an important topic. Black silicon, with its non-reflective surface and greater solar absorption angle, is a promising approach to increase efficiency and reduce production costs of solar cells.

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# CONSTRUCTIONS OF GROUPS AND SEMIHYPERGROUPS OF TRANSFORMATION OPERATORS

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**Abstract:** *There is constructed a noncommutative group and its certain normal subgroup from the vector space, vectors of which are formed by continuous real functions of one variable. Then we describe constructions of a noncommutative semihypergroup based on so called Ends-lemma and a commutative hypergroup using upper ends of its ordered support. The presented construction can serve for investigation of linear differential operators of the n-th order and solution spaces of n-th order homogeneous linear differential equations.*

**Keywords:** Vector space; vectors of continuous functions; semigroup; hypergroup.

## INTRODUCTION

It is a well-known fact that linear ordinary differential equations are classifical tools for modelling of processes in many field of science. Groups and hypergroups of linear differential operators which form left-hand sides of the mentioned equations are treated in a series of papers motivated by the algebraic approach of Borůvka`s school of differential equations and their transformations. The present contribution is devoted to certain multistructures based on generalizations of operators treated in [2]. This approach can be also considered as a skeleton of formerly constructed structures and multistructures where binary operations on the vector spaces of continuous functions are motivated by the composition of affine transformations of vector spaces. Motivating papers are e. g. [1, 3, 5, 6]. Notice, that the basic used terms of the hypergroup theory can be found in [11, 13], and, of course, in many other papers from References.

## 1 GROUPS OF VECTORS AND OPERATORS

In some papers, e.g. [10, 14], there are treated operators motivated by the classical Laplace transformation. As a result of discussing the Laplace transform we consider the half-plane of complex numbers $\Omega = \{z, Re z > 0\}$ in [14]. For the purpose of this contribution let us consider $\boldsymbol{C}$ the field of all complex numbers, $\emptyset \neq \Omega \subseteq \boldsymbol{C}$ a compact (i. e. closed and bounded) subset, $\boldsymbol{C}^{\Omega} = \{f\colon \Omega \to \boldsymbol{C}\}$ a normed algebra of all complex functions of one variable. We define an operator $T(\lambda, F, \varphi)\colon \boldsymbol{C}^{\Omega} \to \boldsymbol{C}^{\Omega}$, where $\lambda \in \boldsymbol{C}$ and $F, \varphi \in \boldsymbol{C}^{\Omega}$ by

$$T(\lambda, F, \varphi)\,(f(z)) = \lambda\, F(z)\, f(z) + \varphi(z)$$

for the sake of brevity

$$T(\lambda, F, \varphi)(f) = \lambda F f + \varphi$$

for any $f \in \mathbf{C}^{\Omega}$ and any $z \in \mathbf{C}$. We denote $\mathcal{T}(\Omega)$ the set of all operators $T(\lambda, F, \varphi)$, i. e.

$$\mathcal{T}(\Omega) = \{T(\lambda, F, \varphi);\ \lambda \in \mathbf{C},\ F, \varphi \in \mathbf{C}^{\Omega}\}.$$

For the purpose of our consideration we restrict ourselves to operators $T(1, F, \varphi)$ restricted on real functions defined on interval $I$ of the real line. As usually, $\mathbf{C}^k(I)$ stands for the commutative ring of all real functions defined on an open interval $I$ of reals, and having there continuous derivatives up to order $k \geq 0$; $\mathbf{C}_+(I)$ is the subsemiring of $\mathbf{C}(I) = \mathbf{C}^0(I)$ of all positive continuous functions defined on the interval $I$. Thus

$$T(1, p_1, q_1) \circ T(1, p_2, q_2) = T(1, p_1 p_2, p_1 q_2 + q_1) \text{ and } T(1, p, q)(f) = pf + q$$

for any $f \in \mathbf{C}^k(I)$.

Evidently, if all functions $p: I \to \mathbf{R}$ determining $T(1, p, q)$ are positive, we obtain that $\mathbf{S}(I)$ is a (noncommutative) group. Cf. papers [1, 5, 6].

Now we are concerning to a certain generalization of considerations presented in [2]: As usual $\mathbf{R}$ stands for the set of all reals, $J \subseteq \mathbf{R}$ is an open interval (bounded or unbounded) of real numbers, $\mathbf{C}(J)$ is the ring of continuous functions. Denote $\vec{p}(x) = (p_0(x), \dots, p_{n-1}(x))$, $\vec{q}(x) = (q_0(x), \dots, q_{n-1}(x))$, $x \in J$. Denote $\mathbf{N}_0(n-1) = \{0, 1, \dots, n-1\}$ and $\delta_{ij}$ stands for the Kronecker $\delta$, $\overline{\delta} = 1 - \delta_{ij}$. For any $m \in \mathbf{N}_0(n-1)$ we denote $\mathbf{VA}_n(J)_m$ the set of all vectors of continuous functions $\{(p_0, \dots, p_{n-1}\};\ p_k \in \mathbf{C}(J),\ p_m(x) > 0,\ x \in J\}$. Define a binary operation "$o_m$" and a binary relation "$\leq_m$" on the set $\mathbf{VA}_n(J)_m$ in this way:

For arbitrary pair $\vec{p}, \vec{q} \in \mathbf{VA}_n(J)_m$, where $\vec{p} = (p_0, \dots, p_{n-1})$, $\vec{q} = (q_0, \dots, q_{n-1})$ we put

$$\vec{p}\ o_m\ \vec{q} = \vec{u},\ \vec{u} = (u_0, \dots, u_{n-1}), \text{ where } u_k(x) = p_m(x)\, q_k(x) + (1 - \delta_{km})\, p_k(x),\ x \in J$$

and

$$\vec{p} \leq_m \vec{q}, \text{ whenever } p_k(x) \leq q_k(x),\ k \in \mathbf{N}_0(n-1),\ p_m(x) = q_m(x),\ x \in J.$$

Evidently $(\mathbf{VA}_n(J)_m, \leq_m)$ is an ordered set.

Recall that in paper [8] there are constructed groups and hypergroups of linear partial differential operators of the first order. A certain similar approach as in the mentioned paper is used also in this contribution.

The following lemma can be obtained using the proof of similar lemma from paper [7], Lemma 2, page 281, where considered linear differential operators are in fact determined by vectors of functions forming coefficients of operators in question.

***Lemma 1.*** *The triad* $(\mathbf{VA}_n(J)_m, o_m, \leq_m)$ *is an ordered noncommutative group.*

Let us recall a certain construction of an action of the additive group of all integer on the phase space $\mathbf{VA}_n(J)_m$ (or in terminology of O. Borůvka – an algebraic space with vectors) with the use of the group of vectors $(\mathbf{VA}_n(J)_m, o_m)$.
Let $(\mathbf{Z}, +)$ be the an additive group of all integers. Let $L(\vec{q}) \in \mathbf{VA}_n(J)_m$ be arbitrary but fixed chosen vectors. Denote by $\wedge_q : \mathbf{VA}_n(J)_m \to \mathbf{VA}_n(J)_m$ the left translation determined by $L(\vec{q})$, i. e.

$\wedge_q\ (L(\vec{p})) = L(\vec{q})\ o_m\ L(\vec{p})$ for any vector $L(\vec{p}) \in \boldsymbol{VA}_n(J)_m$. Further, denote by $\wedge_q^r$ the $r$-th iteration of $\wedge_q$ for $r \in \boldsymbol{Z}$. Now define

$$\pi_q : \boldsymbol{VA}_n(J)_m \times \boldsymbol{Z} \to \boldsymbol{VA}_n(J)_m \text{ by } \pi_q(L(\vec{p}), r) = \wedge_q^r\ (L(\vec{p})).$$

It is easy to see that we get a usual (discrete) transformation group, i. e. the action of $(\boldsymbol{Z}, +)$ (as the phase group) onto $\boldsymbol{VA}_n(J)_m$, thus the following two requirements are satisfied:

1. $\pi_q(L(\vec{p}), 0) = L(\vec{p})$ for any $L(\vec{p}) \in \boldsymbol{VA}_n(J)_m$,

2. $\pi_q(L(\vec{p}), r + s) = \pi_q(\pi_q(L(\vec{p}), r), s)$

for any vector $L(\vec{p}) \in \boldsymbol{VA}_n(J)_m$ and any pair of integers $r, s \in \boldsymbol{Z}$ .

Denoting $\boldsymbol{V_1A}_n(J)_m = \{ L(\vec{p});\ \vec{p} = (p_0, \dots, p_{n-1}),\ p_k \in \boldsymbol{C}(J),\ k \in \boldsymbol{N}_0(n),\ p_m(x) \equiv 1\}$ we get the following technical assertion which is also proved in the above mentioned paper [7].

***Proposition***. *Let J be an open interval of the set* $\boldsymbol{R}$. *Then for any* $n \in \boldsymbol{N}$, $n \geq 2$, $0 \leq m \leq n-1$ *we have that the group* $(\boldsymbol{V_1A}_n(J)_m, o_m)$ *is an invariant subgroup of the group* $(\boldsymbol{VA}_n(J)_m, o_m)$.

Now we define the corresponding transformation operator motivated by $T(\lambda, F, \varphi)(f)$:

$$T(1, p_0, \dots, p_{n-1})(f_1, \dots, f_{n-1}) = \sum_{k=1}^{n-1} p_k f_k + p_0 .$$

Definition of transformation operators

$$\boldsymbol{S}(J)_m = \{T(1, p_0, \dots, p_{n-1});\ p_k \in \boldsymbol{C}(J),\ p_m(x) > 0,\ k \in \boldsymbol{N}_0(n-1),\ 0 \leq m \leq n-1\}.$$

Binary operation $T(1, p_0, \dots, p_{n-1}) \bullet T(1, q_0, \dots, q_{n-1}) = T(1, u_0, \dots, u_{n-1})$, where

$$u_k(x) = p_m(x)\, q_k(x) + (1 - \delta_{km})\, p_k(x),\ x \in J,$$

$$T(1, p_0, \dots, p_{n-1}) \leq_m T(1, q_0, \dots, q_{n-1}), \text{ whenever}$$

$$p_k(x) \leq q_k(x),\ k \in \boldsymbol{N}_0(n-1) \text{ and } p_m(x) = q_m(x),\ x \in J.$$

Then $(\boldsymbol{S}(J), \leq_m)$ is an ordered set.

***Theorem 1***: *Let* $J \subseteq \boldsymbol{R}$ *be an open interval,*
$\boldsymbol{S}(J)_m = \{T(1, p_0, \dots, p_{n-1});\ p_k \in \boldsymbol{C}(J),\ p_m(x) > 0,\ k \in \boldsymbol{N}_0(n-1)\},\ 0 \leq m \leq n-1$.
*Suppose* $\bullet : \boldsymbol{S}(J)_m \times \boldsymbol{S}(J)_m \to \boldsymbol{S}(J)_m$ *is the above defined binary operation. Then the groupoid* $(\boldsymbol{S}(J)_m, \bullet)$ *is a (non-commutative) group.*

*Proof*. Evidently the binary operation “$\bullet$” on $\boldsymbol{S}(J)_m$ is not commutative but associative:
Indeed, denoting $\vec{p} = (p_0, \dots, p_{n-1})$, $\vec{q} = (q_0, \dots, q_{n-1})$ and $\vec{u} = (u_0, \dots, u_{n-1})$, where $T(1,\vec{p})$, $T(1,\vec{q})$, $T(1,\vec{u}) \in \boldsymbol{S}(J)_m$ we obtain $(T(1,\vec{p}) \bullet T(1,\vec{q})) \bullet T(1,\vec{u}) = T(1,\vec{v}) \bullet T(1,\vec{u})$, where $\vec{v} = (v_0, \dots, v_{n-1})$ and $\vec{u} = (u_0, \dots, u_{n-1})$. We have $T(1,\vec{p}) \bullet T(1,\vec{q}) = T(1,\vec{v})$, with $v_k(x) = p_m(x)\, q_k(x) + p_k(x)$, for $k \neq m$ and $v_m(x) = p_m(x)\, q_m(x),\ x \in J$.

Further $T(1,\vec{v}) \bullet T(1,\vec{u}) = T(1,\vec{w})$, where $\vec{w} = (w_0, \dots, w_{n-1})$ and $w_k(x) = v_m(x)\, u_k(x) + v_k(x)$, if $k \neq m$ and $w_m(x) = v_m(x)\, u_m(x),\ x \in J$.

On the other hand $T(1,\vec{q}) \bullet T(1,\vec{u}) = T(1,\vec{s})$, $\vec{s} = (s_0, \dots, s_{n-1})$, $s_k(x) = q_m(x)\, u_k(x) + q_k(x)$, if $k \neq m$, $s_m(x) = q_m(x)\, u_m(x),\ x \in J$. Put $T(1,\vec{p}) \bullet T(1,\vec{s}) = T(1,\vec{z})$, $\vec{z} = (z_0, \dots, z_{n-1})$ with

$z_k(x) = p_m(x)\, s_k(x) + p_k(x)$, if $k \neq m$, $z_m(x) = p_m(x)\, s_m(x)$, $x \in J$. Then

$$(T(1,\vec{p}) \bullet T(1,\vec{q})) \bullet T(1,\vec{u}) = T(1,\vec{v}) \bullet T(1,\vec{u})) = T(1,\vec{w}),$$

with

$$w_k(x) = v_m(x)\, u_k(x) + v_k(x) = p_m(x)\, q_m(x)\, u_k(x) + p_m(x)\, q_k(x) + p_k(x)$$

whereas

$$T(1,\vec{p}) \bullet (T(1,\vec{q}) \bullet T(1,\vec{u})) = T(1,\vec{p}) \bullet T(1,\vec{s}) = T(1,\vec{z}),$$

with

$$z_k(x) = p_m(x)\, s_k(x) + p_k(x) = p_m(x)(q_m(x)\, u_k(x) + q_k(x)) + p_k(x) = p_m(x)\, q_m(x)\, u_k(x) + p_m(x)\, q_k(x) + p_k(x),\ x \in J,$$

thus $T(1,\vec{w}) = T(1,\vec{z})$, since

$$v_m(x) = p_m(x)\, q_m(x),\ w_m(x) = v_m(x)\, u_m(x) = p_m(x)\, q_m(x)\, u_m(x),$$

and

$$z_m(x) = p_m(x)\, s_m(x) = p_m(x)\, q_m(x)\, u_m(x),$$

for any $x \in J$. Consequently we have

$$(T(1,\vec{p}) \bullet T(1,\vec{q})) \bullet T(1,\vec{u}) = T(1,\vec{p}) \bullet (T(1,\vec{q})) \bullet T(1,\vec{u})).$$

Denote $T(1,\vec{u}) \in \boldsymbol{S}(J)_m$ such an operator that $u_k = 0$ for $k \neq m$ and $u_m = 1$. Then for arbitrary $T(1,\vec{p}) \in \boldsymbol{S}(J)_m$ we have $T(1,\vec{p}) \bullet T(1,\vec{u}) = T(1,\vec{q})$, where $q_k(x) = p_m(x)\, u_k(x) + p_k(x)$ for $k \neq m$ and $q_m(x) = p_m(x)\, u_m(x)$, i. e. $q_k(x) = 0 + p_k(x) = p_k(x)$, $q_m(x) = p_m(x) \,.\, 1 = p_m(x)$, hence $T(1,\vec{p}) \bullet T(1,\vec{u}) = T(1,\vec{p})$ and similarly $T(1,\vec{u}) \bullet T(1,\vec{p}) = T(1,\vec{p})$ thus the operator $T(1,\vec{u})$ is the identity (the neutral element).

Moreover, for any $T(1,\vec{p}) \in \boldsymbol{S}(J)_m$ denoting $T^{-1}(1,\vec{p}) = T\Big(1,(-\frac{p_0}{p_m}, -\frac{p_1}{p_m}, \dots, \frac{1}{p_m}, \dots, -\frac{p_{n-1}}{p_m})\Big)$ we obtain that $T(1,\vec{p}) \bullet T^{-1}(1,\vec{p}) = T(1,\vec{q})$, where $q_k(x) = -p_m(x) \,.\, \frac{p_k(x)}{p_m(x)} + p_k(x) = -p_k(x) + p_k(x) = 0$, if $k \neq m$ and $q_m(x) = p_m(x) \,.\, \frac{1}{p_m(x)} = 1$.

Hence $T(1,\vec{q}) = T(1,\vec{u})$ and similarly $T^{-1}(1,\vec{p}) \bullet T(1,\vec{p}) = T(1,\vec{u})$.

Consequently the groupoid $\boldsymbol{S}(J)_m$ is a non-commutative group. □

***Theorem 2***. *The triad $(\boldsymbol{S}(J), \bullet_m, \leq_m)$ is an ordered noncommutative group.*

*Proof.* According to Theorem 1 we have that $(\boldsymbol{S}(J), \bullet_m)$ is a noncommutative group. Now suppose $T(1,\vec{p}), T(1,\vec{q}) \in \boldsymbol{S}(J)_m$ are operators such that $T(1,\vec{p}) \leq_m T(1,\vec{q})$, i. e. $p_k(x) \leq q_k(x)$, $p_m(x) = q_m(x)$ for $k \in \boldsymbol{N}_0(n-1)$, where $\vec{p} = (p_0(x), \dots, p_{n-1}(x))$, $\vec{q} = (q_0(x), \dots, q_{n-1}(x))$, $x \in J$. Consider an arbitrary operator $T(1,\vec{u}) = T(1, u_0, \dots, u_{n-1}) \in \boldsymbol{S}(J)_m$ and denote

$$T(1,\vec{p}) \bullet T(1,\vec{u}) = T(1, p_0, \dots, p_{n-1}) \bullet T(1, u_0, \dots, u_{n-1}) = T(1, v_0, \dots, v_{n-1}),$$

$$T(1,\vec{q}) \bullet T(1,\vec{u}) = T(1, q_0, \dots, q_{n-1}) \bullet T(1, u_0, \dots, u_{n-1}) = T(1, w_0, \dots, w_{n-1}),$$

where $v_k(x) = p_m(x)\, u_k(x) + p_k(x)$, $w_k(x) = q_m(x)\, u_k(x) + q_k(x)$, if $k \neq m$ and $v_m(x) = p_m(x)\, u_m(x) = q_m(x)\, u_m(x) = w_m(x)$, $x \in J$.

Since $p_k(x) \leq q_k(x)$, $x \in J$, $k \in \boldsymbol{N}_0(n-1)$, we have $v_k(x) \leq w_k(x)$, $x \in J$, $k \in \boldsymbol{N}_0(n-1)$. Hence $T(1,\vec{p}) \bullet T(1,\vec{u}) = T(1, v_0, \dots, v_{n-1}) \leq_m T(1, w_0, \dots, w_{n-1}) = T(1,\vec{q}) \bullet T(1,\vec{u})$. Similarly we obtain that also $T(1,\vec{u}) \bullet T(1,\vec{p}) \leq_m T(1,\vec{u}) \bullet T(1,\vec{q})$,

consequently the triad $(\boldsymbol{S}(J)_m, \bullet, \leq)$ is an ordered group. □

## 2 ISOMORPHISMS OF CONSTRUCTED BINARY HYPERSTRUCTURES

Now we will consider the submonoid

$\boldsymbol{S}^{+}(J)_m = \{T(1, p_0, \dots, p_{n-1});\ p_k \in \boldsymbol{C}(J),\ p_k \geq 0,\ p_m > 0,\ k \in \boldsymbol{N}_0(n-1)\}$.

If $T(1,\vec{p}),\ T(1,\vec{q}) \in \boldsymbol{S}(J)_m$, we define $T(1,\vec{p}) \leq T(1,\vec{q})$ if $p_k(x) \leq q_k(x)$ for any $k \in \boldsymbol{N}_0(n-1)$ and $x \in J$, where $\vec{p} = (p_0(x), \dots, p_{n-1}(x))$, $\vec{q} = (q_0(x), \dots, q_{n-1}(x))$. Using the Ends-lemma [13, 14, 15, 16] we obtain that $(\boldsymbol{S}^{+}(J)_m, *)$, where
$T(1,\vec{p}) * T(1,\vec{q}) = T(1,\vec{r});\ T(1,\vec{p}) \bullet T(1,\vec{q}) \leq T(1,\vec{r}) = [T(1,\vec{p}) \bullet T(1,\vec{q}))_{\leq}$ is a non-commutative semigroup.

Using the just defined ordering "$\leq$" we can obtain a commutative hypergroup $(\boldsymbol{S}(J)_m, *)$, which is given by this binary commutative hyperoperation

$$T(1,\vec{p}) * T(1,\vec{q}) = [T(1,\vec{p}))_{\leq} \cup [T(1,\vec{q}))_{\leq}.$$

Similarly, we construct a non-commutative semihypergroups and a commutative hypergroups using vector spaces of continuous functions, cf. [13].

So, denote by $\boldsymbol{V}_n\boldsymbol{A}(J)^{+}$ the set of all vectors from the vector spaces $\boldsymbol{V}_n\boldsymbol{A}(J)$ formed by all vectors with positive components, i. e. $\vec{p} = (p_0, \dots, p_{n-1})$ belongs into $\boldsymbol{V}_n\boldsymbol{A}(J)^{+}$ iff $p_k(x) > 0$ for all $k \in \boldsymbol{N}_0(n-1)\}$ and $x \in J$. Then the above defined ordering $\leq$ on $\boldsymbol{V}_n\boldsymbol{A}(J)$ enables to create a non-commutative semihypergroup and a commutative hypergroup:

For $\vec{p}, \vec{q} \in \boldsymbol{V}_n\boldsymbol{A}(J)^{+}$ we define $\vec{p} \,.\, \vec{q} = \{\vec{r};\ \vec{r} \in \boldsymbol{V_n}\boldsymbol{A}(J)^{+},\ \vec{p}\ o\ \vec{q} \leq \vec{r}\}$ and
$\vec{p} * \vec{q} = \{\vec{r};\ \vec{r} \in \boldsymbol{V_n}\boldsymbol{A}(J), \vec{p} \leq \vec{r}\} \cup \{\vec{s};\ \vec{s} \in \boldsymbol{V_n}\boldsymbol{A}(J),\ \vec{q} \leq \vec{s}\}$. Then according to considerations presented in papers [2, 3, 5, 6, 7] or in the monography [13] we have that the hypergroupoid $(\boldsymbol{V}_n\boldsymbol{A}(J)^{+}, o)$ is a non-commutative semihypergroup and the hypergroupoid $(\boldsymbol{V}_n\boldsymbol{A}(J)^{+}, *)$ is a commutative hypergroup. Recall that if $(S, .), (T, .)$ are hyperstructures then $(S, .) \cong (T, .)$ means that corresponding hyperstructures are isomorphic.

***Theorem 3***: *Let $\boldsymbol{V}_n\boldsymbol{A}(J)$ be a vector space of n-tupples of real continuous functions defined on the interval (open or closed) $J \subseteq \boldsymbol{R}$. Let $(\boldsymbol{V}_n\boldsymbol{A}(J)^{+}, o)$, $(\boldsymbol{S}(J)^{+}, o)$ be semihypergroups and $(\boldsymbol{V}_n\boldsymbol{A}(J), *)$, $(\boldsymbol{S}(J), *)$ be commutative hypergroups defined above. Then we have:*
$(\boldsymbol{V}_n\boldsymbol{A}(J)^{+}, o) \cong (\boldsymbol{S}(J)^{+}, o)$, $(\boldsymbol{V}_n\boldsymbol{A}(J), *) \cong (\boldsymbol{S}(J), *)$.

*Proof:* It is sufficient to use the identity transformation *Id* of the vector space $\boldsymbol{V}_n\boldsymbol{A}(J)$ or its restriction onto the subset $\boldsymbol{V}_n\boldsymbol{A}(J)^{+}$. In this way we obtain the isomorphism of semihypergroups $(\boldsymbol{V}_n\boldsymbol{A}(J)^{+}, o)$ and $(\boldsymbol{S}(J)^{+}, o)$ and also the isomorphism between hypergroups $(\boldsymbol{V}_n\boldsymbol{A}(J), *)$ and $(\boldsymbol{S}(J), *)$; for corresponding terms cf. [11]. □

In papers [15], [16] there are constructed hyperstructures, in particular semihypergroups induced by the Ends lemma, called also *EL–semihypergroups* playing the important role in the hyperstructure theory.

## CONCLUSION

The semihypergroup $(\boldsymbol{S}(J)^{+}, o)$ and the hypergroup $(\boldsymbol{S}(J), *)$ are generalizations of the hypergroup $(\boldsymbol{S}(I), \bullet)$ constructed in [2] which is used in that mentioned paper for representation of a certain hypergroup of the second-order linear differential operators. Investigation of differential operators and special differential equations are parts of large theory treated by various authors as J. Diblík, J. Baštinec, Z. Šmarda, Z. Piskořová, G. Vážanová contributions of which are presented in former proceedings of the conference MITAV, for example

Diblík, J., Khusainov, D.Ya., Šmarda, Z. *Construction of the general solution of planar linear discrete systems with constant coefficients and weak delay*. Adv. Difference Equ. 2009, Art. ID 784935, 18 pp;

Baštinec, J., Diblík, J. Two Classes of Positive Solutions of a Discrete Equation. *Mathematics, Information Technologies and Applied Sciences 2017*, post-conference proceedings of extended versions of selected papers. Brno: University of Defence, 2017, p. 21-32;

Diblík, J., Vážanová, G. Global solutions to mixed-type nonlinear functional differential equations. *Mathematics, Information Technologies and Applied Sciences 2018*, post-conference proceedings of extended versions of selected papers. Brno: University of Defence, 2018, p. 44-54.

Moreover, in the mentioned paper there are also constructed actions of commutative transposition hypergroups, i. e. join spaces created from rings of continuous and smooth functions of a given class on semigroups or hypergroups of second order linear differential operators. The other papers devoted to this topic are [3, 4, 7, 8, 9, 10, 12].

Multistructures obtained in this contribution can be also used for generalization of actions treated in [2] and for constructions of various series of generalised multistructures and their actions. These directions of investigations can be realised in a next paper.

Author of papers [15], [16] has applied the so called Ends lemma and some its generalisation in many other papers where he obtained interesting results. Using theorems 1.3 and 1.4 chapter IV of the monography [13] we can reformulated the Ends lemma in this way:

*Let $(S, \cdot, \leq)$ be an ordered semigroup. A binary hyperoperation $*: S \times S \to \mathcal{P}'(S)$ defined by $a * b = [a \cdot b)_{\leq}$ is associated. The semihypergroup $(S, *)$ is commutative, if and only if the semigroup $(S, \cdot)$ is commutative. Moreover, for an ordered semigroup $(S, \cdot, \leq)$ the following conditions are equivalent:*

*$1^{o}$ For any pair of elements $a, b \in S$ there exists a pair of elements $c, c' \in S$ such that $b \cdot c \leq a$, $c' \cdot b \leq a$.*

*$2^{o}$ The semihypergroup $(S, *)$ satisfied the condition of reproduction (i.e. $t * S = S * t$ for any $t \in S$), thus it is a hypergroup.*

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# APPLICATIONS OF GEOGEBRA FOR CALCULATIONS FIGURE AREAS BOUNDED BY ROSES

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**Abstract:** *In this paper, an application of unconventional method for calculations of areas of figures, especially roses, is presented. The contribution shows how to use the dynamic geometric software GeoGebra to solve dynamic problems. An alternative method for calculations of areas of figures can be applied, if theirs boundaries are the smooth (resp. piecewise smooth) closed curves described by suitable parametric equations.*

**Keywords:** Roses, figure areas, parametric equations, GeoGebra.

## INTRODUCTION

Often, we encounter classical two and three-dimensional Euclidean surfaces and volumes of solids. Calculations of figure areas belong to very frequent integral calculus applications. Usually, the real double integrals are used for these calculations. In the papers [1], [3], [4], and [5], that problem is investigated as a problem in the topological sense and the formula for the calculation of the area (resp. volume) of the $n$-dimensional solid in the space $\mathbf{E_n}$ is proved there. The application of that theory is dependent on suitable parametric descriptions of the surface areas of solids. These surface areas must be smooth (respective piecewise smooth) areas in Euclidean space of the corresponding dimensions. The advantage of this method is using integrals of dimension $n-1$ for the $n$-dimensional solids.

In the paper [2], the correspondence between the alternative theory in $\mathbf{E_2}$ and the known result of the curvilinear integral theory is presented. Everything is demonstrated by some examples. Now we will present some applications in $\mathbf{E_2}$, in the continuity with a dynamic geometric software GeoGebra. All presented figures are original. They were created by an author in the GeoGebra program. These applications are suitable especially for students.

## 1 CALCULATION OF AREAS AND VOLUMES IN $\mathbf{E_n}$

Let $\mathbf{x}$ be points of Euclidean $n$–dimensional space $\mathbf{E_n}$, $n \geq 2$, $x^\alpha$ for $\alpha = 1, \ldots, n$ be the Cartesian coordinates of the point $\mathbf{x} \in \mathbf{E_n}$, $u^a$, where $a = 1, \ldots, n-1$, be the Cartesian coordinates of the point $\mathbf{u} \in \mathbf{E_{n-1}}$. Let $\Omega$ be the bounded closed domain in $\mathbf{E_{n-1}}$, $x^\alpha(u^1, \cdots, u^{n-1})$ given functions defined on some domain $O \subset \mathbf{E_{n-1}}$, $\Omega \subset O$. Let us also suppose that the vector function $\mathbf{x(u)} = \{x^\alpha(u^a)\}$ has almost everywhere in $\Omega$ the continuous partial derivatives

$$B_a^\alpha := \frac{\partial x^\alpha}{\partial u^a},\ \alpha = 1, \ldots, n,\ a = 1, \ldots, n-1, \qquad (1)$$

And the rank of the matrix $(B_a^\alpha)_{n\times(n-1)}$ is equal to $n$ – 1 almost everywhere in Ω,

the subset

$$P^O := \{\mathbf{x} \in \mathbf{E}_n;\ \mathbf{X} = \mathbf{x}(\mathbf{u}), \mathbf{u} \in \operatorname{int}\Omega\} \tag{2}$$

of the set

$$P := \{\mathbf{x} \in \mathbf{E}_n;\ \mathbf{X} = \mathbf{x}(\mathbf{u}), \mathbf{u} \in \Omega\} \tag{3}$$

is a homeomorphic range of the set int Ω (of all interior points of the set Ω) in $\mathbf{E_n}$. It follows from the assumptions, that the set $P$ is bounded and it is a piecewise smooth hypersurface by parts in the space $\mathbf{E_n}$. This manifold does not intersect by itself and that divides the space $\mathbf{E_n}$ to two disjoint regions in $\mathbf{E_n}$, in which one is bounded and the second one is unbounded.

The closure of relevant bounded region is called **the *n*-dimensional solid** in space $\mathbf{E_n}$. Let us denote it by $W$. If int $W$ is the set of all internal points of the set $W$ and by $\partial W$ the boundary of it, it is obvious, that $W = P \cup \operatorname{int} W$, where $\partial W = P$.

The **volume *V*** = $\mu_W$ of the $n$-dimensional solid $W$ can be calculated by the formula,

$$\mu_W := \frac{1}{n}\iint\cdots\int_\Omega \Delta(\mathbf{u})\mathrm{d}u^1\mathrm{d}u^2\cdots\mathrm{d}u^{n-1} \tag{4}$$

where the Jacobian of the mapping is

$$\Delta(\mathbf{u}) := \begin{vmatrix} x^1(\mathbf{u}) & x^2(\mathbf{u}) & \dots & x^n(\mathbf{u}) \\ B_1^1 & B_1^2 & \dots & B_1^n \\ \dots & \dots & \dots & \dots \\ B_{n-1}^1 & B_{n-1}^2 & \dots & B_{n-1}^n \end{vmatrix}, \tag{5}$$

which has been proved in [2].

## 2 AREAS OF FIGURES IN $\mathbf{E_2}$

If we want to calculate area of a closed figure $W$ bounded by a closed curve $P$ in $\mathbf{E_2}$ by the method above, we must find the suitable parameterization of the curve $P$ and calculate the determinant $\Delta(\mathbf{u})$. The area $W$ is of the form of a closed sector with the boundary described by the parametric equations $x = \varphi(t)$, $y = \psi(t)$ for each $t \in \langle \alpha;\ \beta \rangle$.
Then

$$\Delta(t) = \begin{vmatrix} \varphi(t) & \psi(t) \\ \varphi'(t) & \psi'(t) \end{vmatrix} = \varphi(t)\psi'(t) - \psi(t)\varphi'(t), \tag{6}$$

$$S = \mu_W = \frac{1}{2}\int_{\alpha}^{\beta} |\Delta(t)| \, \mathrm{d}t = \frac{1}{2}\int_{\alpha}^{\beta} \left| \, \varphi(t)\psi'(t) - \psi(t)\varphi'(t) \, \right| \mathrm{d}t \,. \qquad (7)$$

## 3 FIGURES OF ROSES

The **rose** (the rhodonea, resp. the rosette) is a case of a hypocycloid. It is a very nice and interesting curve with a various number of petals, suitable for many applications. The number of petals is equal to $p = \frac{1}{d}$ , where $d = \frac{1}{2} - \frac{1}{2c}$. The algebraic curve exists for a rational $c$ different from 0 and 1 only. For $c = 1$, we have got a circle. If $p = k$, where $k$ is a natural number, then the rose shape consists of $2k$ petals for $k$ is even, and for an odd number $k$, we have got a rose with $k$ petals.

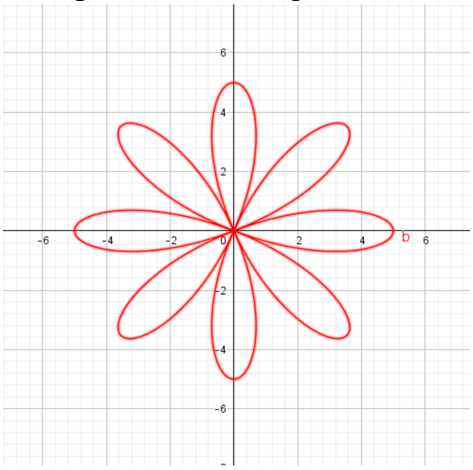

**Fig. 1.** Rose $a = 5$, $k = 4$

Let us consider $k$ natural now. The general equation is not so much practical for calculations, and so we usually calculate with the relation in the polar coordinates

$$\rho = a \cos k\varphi \qquad (8)$$

or we can express a rose in parametric equations:

$$x = a \cos kt \cos t, \qquad (9)$$
$$y = a \cos kt \sin t.$$

Then $t$ is a parameter, $k$ is a natural number indicating a number of petals. Limits of the parameter $t$ also depends on $k$. For an even $k$, $t$ is changing from 0 to $2\pi$, for an odd $k$, $t$ is changing from 0 to $\pi$ (or we can use the limits from $-\pi/2$ to $\pi/2$).

Using the alternative method for an area above, we have got:

$$\Delta(t) = \begin{vmatrix} a\cos kt \; \cos t & a\cos kt \; \sin t \\ -ka\sin kt \; \cos t - a\cos kt \; \sin t & -ka\sin kt \sin t + a\cos kt \; \cos t \end{vmatrix} = \qquad (10)$$

$$= a^2 \cos kt\left(-k \sin kt \sin t \cos t + \cos kt \; \cos^2 t + k \sin kt \sin t \cos t + \cos kt \; \sin^2 t\right) =$$

$$= a^2 \cos kt \left(\cos kt \left( \cos^2 t + \sin^2 t \right)\right) = a^2 \cos^2 kt$$

The area for **odd** number of petals is equal to $k\,S_1$, where $S_1$ is area of one petal, thus

$$S = \frac{k}{2}a^2 \int_{-\pi/2}^{\pi/2} \cos^2 kt \,\mathrm{d}t = \frac{k}{2}a^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2kt}{2} \,\mathrm{d}t = \tag{11}$$

$$= \frac{k}{2}a^2 \left[\frac{t}{2} + \frac{\sin 2kt}{4k}\right]_{-\pi/2}^{\pi/2} = \frac{ka^2}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi a^2}{4}$$

If the number of petals is **even**, the integral must be multiplied by $2k$, i. e. the area is equal to $a^2\,\pi / 2$.

## 4 USING GEOGEBRA

The following figures Fig. 2 – Fig. 7 are a view of some applications of roses. All is drawn in the mathematical software GeoGebra. It is the free dynamic mathematical software, suitable especially for students, which however needs the last version of Java. In the GeoGebra, we have got two different views on one curve (resp. mapped object generally) at the same time. The first, we see the expression (in our case parametric equations of the given curve) in the algebra window, and the second, we see also the corresponding figure in the geometry window. An object drawn in the geometry window has the same colour as the expression, which belongs to it, has in the algebra window. GeoGebra is freeware and it is very easy to download and install. It is not negligible that it can be opened in 45 different languages and that the dynamic worksheets can be exported as web pages.
On the first applet (Fig. 8), two roses are drawn, on the second one (Fig. 9), the area of the roses is calculated. Changing parameters, we can created many figures and calculates areas. It is clear that the different parameters were used for different curves.

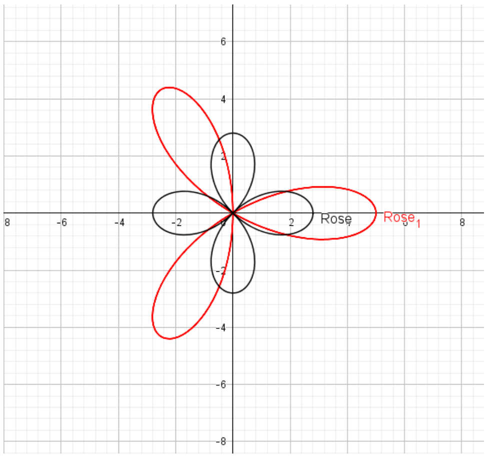


**Fig. 2.** Roses $a$ = 5, $b$ = 2.8, $k$ = 3, $l$ = 2

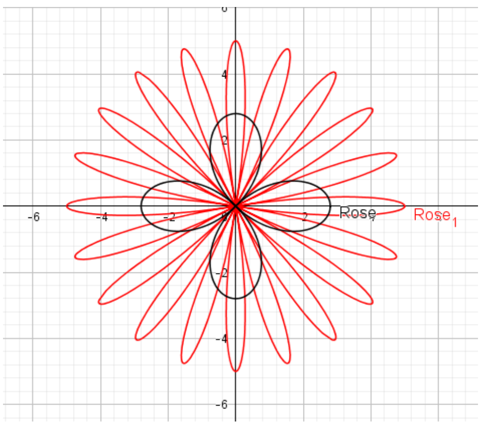


**Fig. 3.** Roses $a$ = 5, $b$ = 2,8, $k$ = 10, $l$ = 2

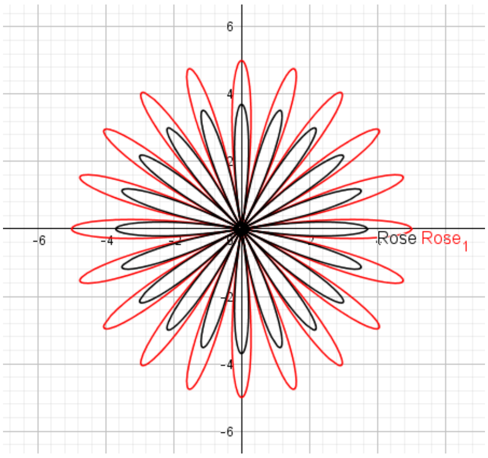


**Fig. 4.** Roses $a = 5$, $b = 3.7$, $k = 10$, $l = 10$

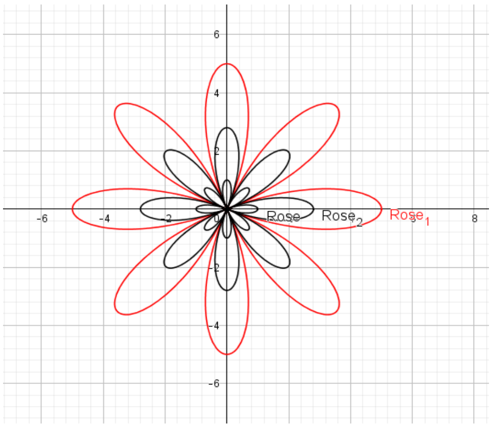


**Fig. 5.** Roses $a = 5$, $b = 2.8$, $c = 1$, $k = l = m = 4$

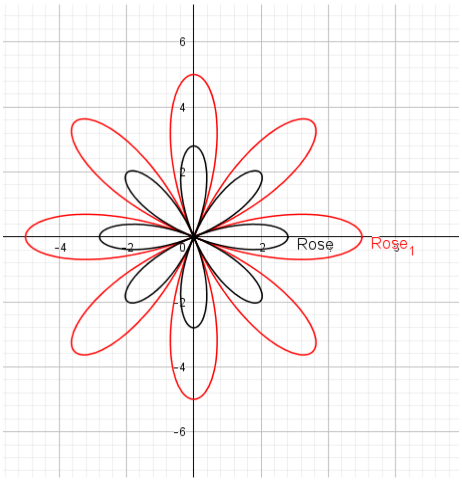


**Fig. 6.** Roses $a = 5$, $b = 2.8$, $k = 4$, $l = 4$

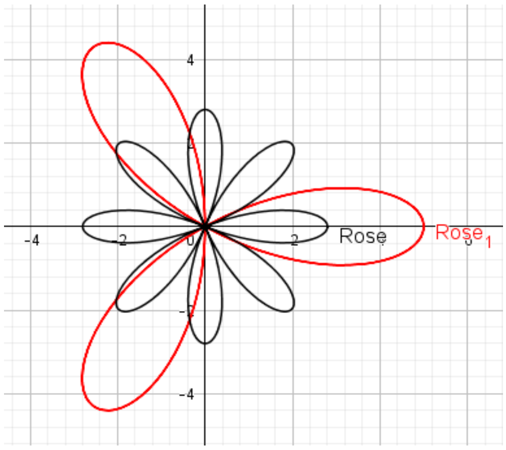


**Fig. 7** Roses $a = 5$, $b = 2,8$, $k = 3$, $l = 4$

In the Fig. 5, we drew three curves. Students can create many pictures and calculate areas. To program formulas it is not difficult. The appropriate manual in English can be found on the internet. Similarly, we can map also figures in $\mathbf{E}_3$ and calculate volumes of solids. There exists lots of different software to draw geometric figures but GeoGegra has a lot of advantages for students. It is free and very intuitive.

The numbers of petals in the applets (and the figures) are denoted by $k$, $l$, and $m$, resp. $2k$, $2l$, and $2m$.

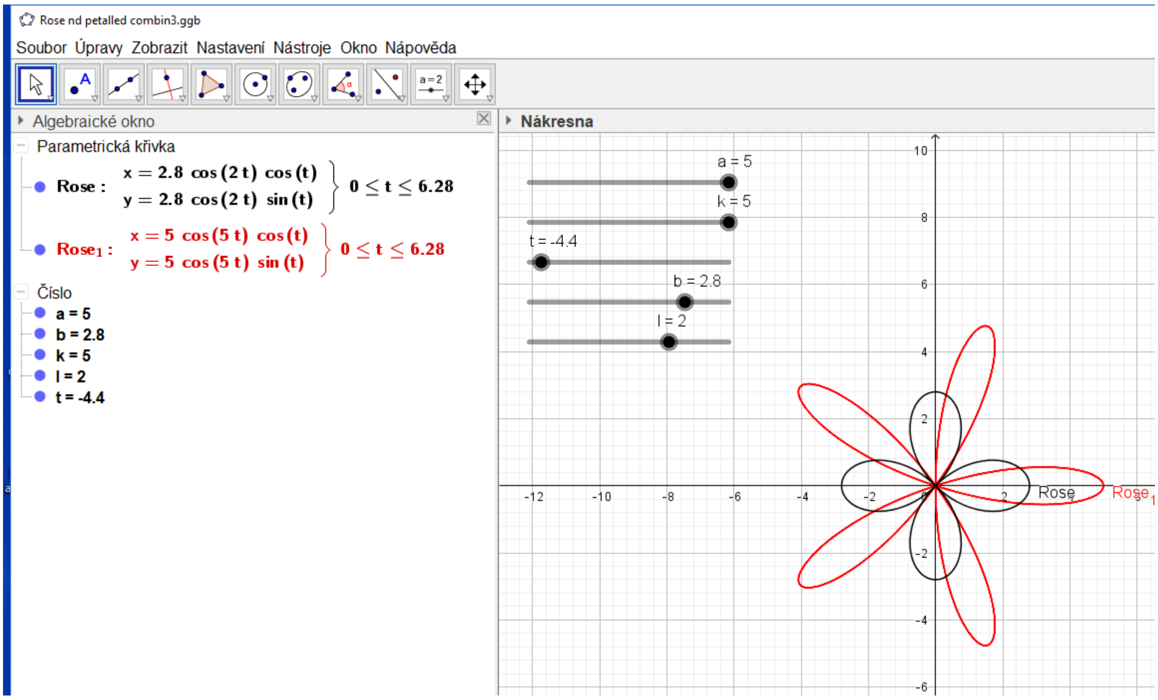


**Fig. 8.** The applet – two roses

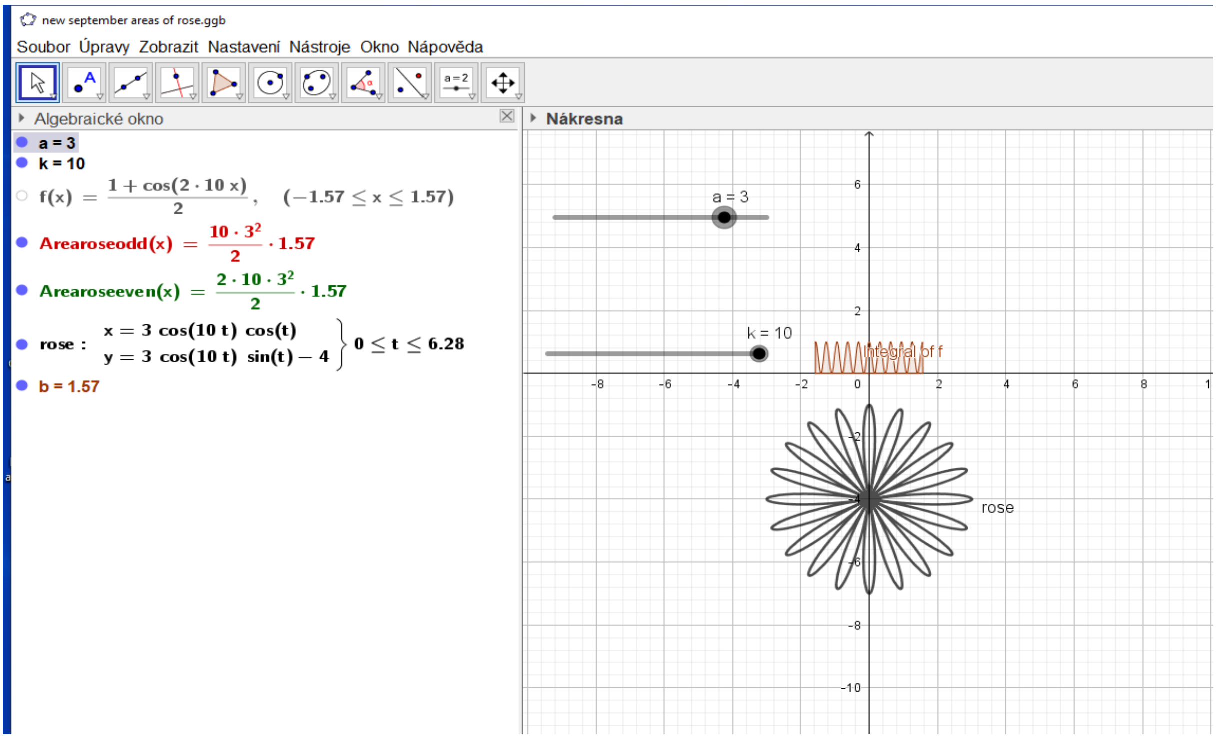


**Fig. 9.** The applet – the rose area

**CONCLUSION**

Rose curves (rondonea curves) were described by Guide Grandi in 1722 – see [7]. Today, these curves are very popular and many applications of them are possible to find (for example) in technical practice – see for example [6], [8]. Applications solved in the special dynamic geometric freeware are also very interesting for students. We can use it in many cases, not only in geometry. The above alternative method for calculation areas and volumes could be one of method, how to calculate areas of roses. This method were proved by the author [1]. If we are able to find the suitable parameterizations of the smooth or piecewise smooth closed curves, the calculations of areas (and of course volumes, generally in the *n*-dimensional space) will be easier because we calculate with integrals of fewer dimensions, i. e. for example, the triple integral transforms into the double one and in case of solids of revolution even into one-dimensional integral. Thus, this alternative formula for the volume can simplify final calculations. However, the process of the finding the parameterization proves to be a rather complicated problem in some cases. The paper also shows a possibility of the using a geometric software for practical applications.

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# THE LEAST-SQUARES AND GALERKIN METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

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**Abstract:** *There are other methods of discretization than the Galerkin method. These other methods can have in some cases formal and practical advantages. The aim of the paper is to compare the Galerkin method and the least-squares finite element method on selected problems of differential equations. In particular, the erosion transport equation is favorably solved by the least-squares finite element method compared to the Galerkin finite element method.*

**Keywords:** least squares method, Galerkin method, Rayleigh-Ritz method, finite element method, numerical solution.

## INTRODUCTION

The least-squares finite element method is a simple, efficient and robust technique for the numerical solution of partial differential equations. The method can solve a broad range of problems in fluid dynamics and electromagnetics with only one mathematical/computational formulation. It can be shown that commonly adopted special treatments, for example in computational fluid dynamics, such as upwinding, numerical dissipation, staggered grid, non-equal-order elements, operator splitting and preconditioning, edge elements, vector potential, and so on, are unnecessary. On these methods and more details are in [2].

## 1 FINITE ELEMENTS

The finite element method is one of the most general techniques for the numerical solution of differential equations. The method does not operate directly on the differential equations, but the continuous boundary and initial value problems are put into equivalent variational forms. The solution appears in the integral of a quantity over a domain. The integral can be broken up into the sum of integrals over a collection of subdomains called finite elements.

The method has some important features: It can be applied to domains of complex shape and with quite arbitrary boundary conditions. Finite elements can be placed anywhere in physical domains. Finite element analysts can add or delete elements without changing the global data structure. The clear structure and versatility of the method makes it possible to construct purpose software for application. In many cases it is possible to mathematically analyze and estimate the accuracy of finite elements.

According to the variational principle, the finite element method can be classified into three major groups: the Rayleigh-Ritz method, the Galerkin method and the least-squares method. More details on these methods can be found in [4].

## 2 THE RAYLEIGH-RITZ METHOD

The Rayleigh-Ritz method minimizes the total potential energy. The numerical solution has the best approximation property. It means that the difference between the finite elements solution and the exact solution is minimized with respect to a certain energy norm. Moreover, the Rayleigh-Ritz finite element method leads to symmetric and positive definite systems of linear algebraic equations. This method is successful and dominating computational technique in solid mechanics, heat transfer and static electromagnetics. But it is applicable only for equations with self-adjoint operators. The method is successful in application to problems governed by self-adjoint, second- or fourth-order elliptic diffusion-type equations.

## 3 THE GALERKIN METHOD

The Galerkin method is based on the principle of virtual work or the weighted residual form. This method has been considered a universal approach to construction of finite element schemes. But attempts to apply it to non-self-adjoint equations in fluid dynamics and other transport problems encounter serious difficulties, such as oscillations and instabilities of the solution and poor approximation of its derivatives.

Let us consider a mathematical problem defined by a set of partial differential equations in the form

$$Lu = f \quad \text{in} \quad \Omega, \tag{1}$$

$$Bu = 0 \quad \text{on} \quad \Gamma, \tag{2}$$

where $L$ is the linear differential operator, $B$ is the boundary operator, $u$ is the dependent unknown vector, $f$ is the force vector, $\Omega$ is the domain, and $\Gamma$ is the boundary of $\Omega$. The algebraic equation permitting a numerical solution is formed as a weighted residual

$$\int_\Omega v_i^T (Lu - f)\, d\Omega + \int_\Gamma \overline{v}_i^T Bu\, d\Gamma = 0, \tag{3}$$

where $v_i$ and $\overline{v}_i$ are suitably chosen test functions. The function is approximated by a set of unknown parameters $\widetilde{u}_j$ and trial functions $\varphi_j(x)$, where $x$ stands for the vector of independent variables.

$$u \simeq u_n = \sum_{j=1}^{n} \varphi_j\, \widetilde{u}_j. \tag{4}$$

## 4 THE LEAST-SQUARES METHOD

The least-squares finite element method based on simply minimizing the $L_2$ norm of the residuals of a first-order system of differential equations eliminates the drawbacks of the Galerkin method applied to non-self-adjoint equations and is receiving increasing attention. It can be said that for non-self-adjoint systems, such as arise in fluid mechanics and electromagnetics, the application of the least-squares finite element method is the right way to go.

The least-squares finite elements method is based on minimization of the residuals in a least-squares sense. There is minimized the following functional

$$I(u) = \int_\Omega (Lu - f)^2 \, d\Omega + \int_\Gamma (Bu)^2 \, d\Gamma \tag{5}$$

within the constraint of a given boundary condition (2). It means that the least-squares solution is calculated from the following variational identity

$$\int_\Omega (L\varphi_i)^T (Lu - f) \, d\Omega = 0. \tag{6}$$

The Galerkin formulation (3) is only conceptual without much practical meaning and thus in most cases needs further problem-dependent mathematical manipulation in order to obtain a realistic computational formulation for a particular problem. In contrast to the Galerkin formulation, the least-squares formulation (6) is a final mathematical as well as computational formulation for any problem.

The Galerkin mixed method brings sometimes more troubles then benefits, at least for second-order elliptic problems. The term ”mixed” means that both primal variables and dual variables are approximated.

First, the original simple minimization problem is turned into a difficult saddle-point problem. Different elements must be used to interpolate primal variables and dual variables. Often low-order elements must be employed for dual variables to satisfy the Ladyzhenskaya-Babuška-Brezzi condition, see e. g. [1].

Second, the Rayleigh-Ritz method leads to symmetric positive definite matrices, while the Galerkin mixed method produces non-positive definite matrices which have been hard to solve for large-scale problems.

## 5 AN ILLUSTRATIVE PROBLEM SOLVED BY DIFFERENT METHODS

Consider a basis $\{\varphi_1, \dots, \varphi_n\}$ and $V_n = span\{\varphi_1, \dots, \varphi_n\}$. Let us denote

$$\begin{array}{lll}
A^{(1)} = (\varphi_i, \varphi_j) & A^{(4)} = (\varphi_i', \varphi_j') & b^{(1)} = (f, \varphi_i) \\
A^{(2)} = (\varphi_i', \varphi_j) & A^{(5)} = (\varphi_i'', \varphi_j') & b^{(2)} = (f, \varphi_i') \\
A^{(3)} = (\varphi_i'', \varphi_j) & A^{(6)} = (\varphi_i'', \varphi_j'') & b^{(3)} = (f, \varphi_i'')
\end{array}$$

Let us consider the following boundary-value problem

$$Lu = f,$$

in particular

$$-\alpha u'' + \beta u' + \gamma u = f(x), u(0) = u_0, u(1) = u_1, x \in (0,1), \alpha, \beta, \gamma \text{ constants.} \tag{7}$$

We derive the system of algebraic equations corresponding to the problem (7) when some selected methods are applied.

### 5.1 The Galerkin method applied

We have

$$(Lu_n, \varphi_j) = (f, \varphi_j) \text{ for all } j,\ u_n = \sum_{i=1}^{n} v_i \varphi_i.$$

Then the system of algebraic equations is

$$Av = b,$$

where

$$A_{ji} = (L\varphi_i, \varphi_j) = -\alpha(\varphi_i'', \varphi_j) + \beta(\varphi_i', \varphi_j) + \gamma(\varphi_i, \varphi_j). \tag{8}$$

Well known properties of generalized derivatives are used in an appropriate Hilbert space and then $A_{ji}$ in (8) can be expressed as

$$A_{ji} = +\alpha(\varphi_i', \varphi_j') + \beta(\varphi_i', \varphi_j) + \gamma(\varphi_i, \varphi_j).$$

Thus

$$A = -\alpha A^{(3)} + \beta A^{(2)} + \gamma A^{(1)}, \quad b = b^{(1)}.$$

For sufficiently smooth functions there is possible to use $A^{(4)} = -A^{(3)}$ and then

$$A = \alpha A^{(4)} + \beta A^{(2)} + \gamma A^{(1)}, \quad b = b^{(1)}.$$

### 5.2 The least-squares method applied

Now we minimize the following expression

$$(Lu_n - f, Lu_n - f) = (Lu_n, Lu_n) - 2(f, Lu_n) + (f, f)$$

which can be considered as

$$v^T Bv - 2v^T c + F.$$

Then the system of algebraic equations is

$$Bv = c,$$

where

$$c_i = (f, L\varphi_i) = -\alpha(f, \varphi_i'') + \beta(f, \varphi_i') + \gamma(f, \varphi_i) = -\alpha b^{(3)} + \beta b^{(2)} + \gamma b^{(1)},$$
$$B_{ij} = (L\varphi_i, L\varphi_j) = (-\alpha\varphi_i'' + \beta\varphi_i' + \gamma\varphi_i, -\alpha\varphi_j'' + \beta\varphi_j' + \gamma\varphi_j).$$

After simple manipulation we get

$$B_{ij} = B_{ij}^{I} + B_{ij}^{II} + B_{ij}^{III} + B_{ij}^{IV},$$

where

$$
\begin{aligned}
B_{ij}^{I} &= \alpha^2(\varphi_i'', \varphi_j'') + \beta^2(\varphi_i', \varphi_j') + \gamma^2(\varphi_i, \varphi_j), \\
B_{ij}^{II} &= -\alpha\beta(\varphi_i'', \varphi_j') - \alpha\beta(\varphi_i', \varphi_j''), \\
B_{ij}^{III} &= -\alpha\gamma(\varphi_i'', \varphi_j) - \alpha\gamma(\varphi_i, \varphi_j''), \\
B_{ij}^{IV} &= \beta\gamma(\varphi_i', \varphi_j) + \beta\gamma(\varphi_i, \varphi_j')
\end{aligned}
$$

and then

$$B = \alpha^2 A^{(6)} + \beta^2 A^{(4)} + \gamma^2 A^{(1)} - \alpha\beta(A^{(5)} + A^{(5)^T}) - \alpha\gamma(A^{(3)} + A^{(3)^T}) + \beta\gamma(A^{(2)} + A^{(2)^T}).$$

### 5.3 The least-squares method with a substitution applied

Consider the substitution

$$z = u'$$

and then the problem (7) can be written as

$$-\alpha z' + \beta z + \gamma u = f, \text{ that is } K(z, u) = f.$$

There is minimized the following expression

$$(-\alpha z' + \beta z + \gamma u - f, -\alpha z' + \beta z + \gamma u - f) + (z - u', z - u')$$

which is

$$(K(z, u) - f, K(z, u) - f) + (z - u', z - u')$$

and then

$$(K(z, u), K(z, u)) - 2(K(z, u), f) + (f, f) + (z, z) - 2(z, u') + (u', u'). \tag{9}$$

Put $K_1(z) = -\alpha z' + \beta z$ and $K_2(u) = \gamma u$. Thus

$$K(z, u) = K_1(z) + K_2(u)$$

and expression (9) can be rewritten as

$$
\begin{aligned}
(K_1(z), K_1(z)) + 2(K_1(z), K_2(u)) &+ (K_2(u), K_2(u)) - 2[(K_1(z), f) + (K_2(u), f)] + \\
&+ (f, f) + (z, z) - 2(z, u') + (u', u') \qquad (10)
\end{aligned}
$$

We approach the approximation $u_n = \sum v_i \varphi_i$, $z_n = \sum w_i \varphi_i$. Then the minimization of the expression in (10) becomes the minimization of the following expression

$$w^T C_1 w + 2w^T C_2 v + v^T C_3 v - 2w^T d_1 - 2v^T d_2 + F + w^T C_4 w - 2w^T C_5 v + v^T C_6 v.$$

If we use the matrix form we get

$$(w^T, v^T) \begin{pmatrix} C_1 + C_4 & C_2 - C_5 \\ (C_2 - C_5)^T & C_3 + C_6 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} - 2(w^T, v^T) \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + F.$$

Then we have the system of equations

$$\begin{pmatrix} C_1 + C_4 & C_2 - C_5 \\ C_2^T - C_5^T & C_3 + C_6 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where

$$\begin{array}{lll} C_1 = -\alpha^2 A^{(4)} + \beta^2 A^{(1)} - \alpha\beta(A^{(2)} + A^{(2)^T}) & C_2 = -\alpha\gamma A^{(2)} + \beta\gamma A^{(1)} & C_3 = \gamma^2 A^{(1)} \\ C_4 = A^{(1)} & C_5 = A^{(2)^T} & C_6 = A^{(4)} \end{array}$$

and

$$d_1 = -\alpha b^{(2)} + \beta b^{(1)},\ d_2 = \gamma b^{(1)}.$$

## 6 SOLVING THE EROSION TRANSPORT EQUATION

Dynamic systems are expressed by differential equations. In hydrodynamics the systems are expressed by the partial differential equations of convection-diffusion-reaction scheme as symbolically follows

$$\underbrace{C\frac{\partial u}{\partial t}}_{\text{capacity}} = \underbrace{\nabla \cdot (D\nabla u)}_{\text{diffusion}} - \underbrace{\nabla \cdot (\vec{q}u)}_{\text{convection}} - \underbrace{\sum_{r=0}^{n} \lambda_r u^r}_{\text{reaction}}.$$

Some of these terms can be missing.

Differential equations can be naturally solved by various methods. The most standard way is Galerkin approach. Under certain oversimplified assumptions Galerkin discretization leads to the same discrete approximation as the standard Finite difference method. There could be no unwanted oscillations but no one can guarantee anything leading to such a lucky result.

### 6.1 Derivation of surface runoff model strong formulation

To derive control surface runoff equations we shall come out from the law of mass conservation which can be formulated by the following relation

$$\frac{\partial V}{\partial t} = -\nabla \cdot \vec{q} + S, \tag{11}$$

where $V$ represents the so called volume function, $\vec{q}$ is the volume flux and $S$ is a source term. The volume flux change can be described by the change of surface runoff depth $h_u$

$$\frac{\partial V}{\partial t} = \frac{\partial h_u}{\partial t}. \tag{12}$$

Then there is possible to describe the volume flux by a standard model

$$Q = \alpha h^m, \tag{13}$$

where $Q$ is the channel discharge, $h$ is the depth of water in the channel and $\alpha$ and $m$ are certain coefficients. There is possible to consider for $\vec{q}$ with approximation (13) the following formulation

$$\begin{aligned} \vec{q} &= \begin{pmatrix} q_x \\ q_y \end{pmatrix} = \begin{pmatrix} \alpha_x h_u^m \\ \alpha_y h_u^m \end{pmatrix}, \\ \vec{q} &= \vec{\alpha} h_u^m. \end{aligned} \tag{14}$$

The rate of volume change of elementary volume $V$ can be described by the following relation

$$\frac{\partial V}{\partial t} = \frac{\partial h_u}{\partial t}. \tag{15}$$

The source term $S$ in equation (11) can be formulated as

$$S(\vec{x}, t) = i(t) - q_{in}(\vec{x}, t), \tag{16}$$

where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $i(t)$ is the rainfall intensity and $q_{in}(\vec{x}, t)$ is the infiltrated amount.
Substituting (14), (15) and (16) into equation (11) we get an equation of kinematic wave for a generic catchment. The resulting equation has the following form

$$\frac{\partial h_u}{\partial t} = -\nabla \cdot (\vec{\alpha} m h_u^m) + i(t) - q_{in}(\vec{x}, t). \tag{17}$$

The equation contains a nonlinear term $h_u^m$, which can be replaced by

$$h_u^m = \bar{h}_u^{m-1} h_u, \tag{18}$$

where the term $\bar{h}_u$ corresponds to the value of unknown function $h_u$ in the previous iteration level. Nonlinear equation (17) can be further linearized

$$\frac{\partial h_u}{\partial t} = -m \bar{h}_u^{m-1} \nabla \cdot (\vec{\alpha} h_u) + \overbrace{i(t) - q_{in}(\vec{x}, t)}^{i_r(\vec{x}, t)}. \tag{19}$$

Equation (19) is a partial differential equation of the first order. Its solution is function $h_u(\vec{x}, t)$ which is the function searched.
Initial and boundary conditions there is possible to set simply

$$h_u(\vec{x}, t_0) = 0, \quad \forall \vec{x} \in \Omega, \tag{20}$$

$$h_u(\vec{x}, t) = 0, \quad \forall (x, t) = \Gamma \times t \in [0, T_{end}). \tag{21}$$

The solution of equation (19) with appropriate initial and boundary conditions thus represents a deterministic model indicating surface runoff genesis.

### 6.1.1 Variational solution of the kinematic wave equation

We shall discuss the possibilities for solving the problem (17).

**Space-time approximation by the Galerkin method.** Let us define

$$L(h_u) = -\alpha_x m h_u^{m-1}\frac{\partial h_u}{\partial x} - \alpha_y m h_u^{m-1}\frac{\partial h_u}{\partial y} + i_r(x,t) \tag{22}$$

$$= -m h_u^{m-1}(\alpha_x \frac{\partial h_u}{\partial x} + \alpha_y \frac{\partial h_u}{\partial y}) + i_r(x,t)$$

We solve the problem

$$\frac{\partial h_u}{\partial t} = L(h_u). \tag{23}$$

The Galerkin method means finding a function $h_u$ satisfying the equality

$$\int_0^T \int_\Omega \left( L(h_u) - \frac{\partial h_u}{\partial t} \right) v d\Omega dt = 0$$

and then the following equality

$$\int_0^T \int_\Omega \left( -\alpha_x m h_u^{m-1}\frac{\partial h_u}{\partial x} - \alpha_y m h_u^{m-1}\frac{\partial h_u}{\partial y} + i_r(x,t) - \frac{\partial h_u}{\partial t} \right) v d\Omega dt = 0 \tag{24}$$

holds for all test-functions $v$.

**Space-time approximation by the least-squares method.** The least-squares method means the minimization of a nonnegative functional. Then the global minimum of which will be a solution of (23).
Let us define

$$F(h_u) = \int_0^T \int_\Omega \left( -\alpha_x m h_u^{m-1}\frac{\partial h_u}{\partial x} - \alpha_y m h_u^{m-1}\frac{\partial h_u}{\partial y} + i_r(x,t) - \frac{\partial h_u}{\partial t} \right)^2 d\Omega dt \tag{25}$$

and for a boundary condition

$$G(h_u) = \int_0^T \int_\Gamma (h_u)^2 \, d\Omega dt. \tag{26}$$

Let us put

$$E(h_u) = F(h_u) + K * G(h_u) \tag{27}$$

where $K$ is a penalization constant and $\Gamma$ is a part of the boundary $\Omega$ where a homogeneous Dirichlet boundary condition is given. The problem of minimizing functional $E$ defined by (27) for functions from appropriate function spaces, expresses the space-time approximation by the least-squares method.

**Remark 1** *Both previous approaches share some common features. One of the advantages is the complete space-time approximation, but it is at the cost of adding one dimension (time) and thus the problem solved is substantially increased. One way to at least partially eliminate this deficiency is to use the recursive domain decomposition method described in [3].*

**A combination of the finite element method and the Euler method.** Denote $h_u^t(x) = h_u(x,t)$ for a fix $t$. Choose a time step $\tau$. In the problem (23) we replace the time derivative by the expression

$$\frac{\partial h_u}{\partial t} = \frac{h_u^{t+\tau} - h_u^t}{\tau}$$

and we get

$$\frac{h_u^{t+\tau} - h_u^t}{\tau} = L(h_u^t)\,. \tag{28}$$

Then the least-squares method means the minimization of the following functional

$$E^t(h_u^{t+\tau}) = F^t(h_u^{t+\tau}) + K * G^t(h_u^{t+\tau})\,,$$

where

$$F^t(h_u^{t+\tau}) = \int\limits_{\Omega} \left(h_u^{t+\tau} - h_u^t - \tau L(h_u^t)\right)^2 d\Omega$$

and

$$G^t(h_u^{t+\tau}) = \int\limits_{\Gamma} \left(h_u^{t+\tau}\right)^2 d\Omega\,.$$

**A combination of the least-squares method and the implicit Euler method.** We change the value of the right hand side in (28) and get

$$\frac{h_u^{t+\tau} - h_u^t}{\tau} = L(h_u^{t+\tau})\,. \tag{29}$$

Then the functional minimized is changed to the form

$$F^t(h_u^{t+\tau}) = \int\limits_{\Omega} \left(h_u^{t+\tau} - h_u^t - \tau L(h_u^{t+\tau})\right)^2 d\Omega\,.$$

Solutions are found in the form of the following approximations

$$h_u = \sum_{i=1}^{N} \gamma_i \psi_i(x,t)\,, \tag{30}$$

where functions $\psi_i$ are defined in $\Omega \times (0,T)$ and form a basis. Similarly

$$h_u^t = \sum_{i=1}^{n} \alpha_i \varphi_i \quad \text{and} \quad h_u^{t+\tau} = \sum_{i=1}^{n} \beta_i \varphi_i, \tag{31}$$

where $\varphi_i$ are defined in $\Omega$ and form a basis.

**Partial linearization.** Let us define

$$\tilde{L}(u,v) = -\alpha_x m v^{m-1}\frac{\partial u}{\partial x} - \alpha_y m v^{m-1}\frac{\partial u}{\partial y} + i_r(x,t),$$

$$L(h_u) = \tilde{L}(h_u, h_u)\,.$$

We have

$$\begin{aligned} F(h_u) &= \int_0^T \int_\Omega \left(-\alpha_x m h_u^{m-1}\frac{\partial h_u}{\partial x} - \alpha_y m h_u^{m-1}\frac{\partial h_u}{\partial y} + i_r(x,t) - \frac{\partial h_u}{\partial t}\right)^2 d\Omega dt \\ &= \int_0^T \int_\Omega \left(L(h_u) - \frac{\partial h_u}{\partial t}\right)^2 d\Omega dt \\ &= \int_0^T \int_\Omega \left(\tilde{L}(h_u, h_u) - \frac{\partial h_u}{\partial t}\right)^2 d\Omega dt\,. \end{aligned}$$

Denote

$$\tilde{F}(u,v) = \int_0^T \int_\Omega \left(\tilde{L}(u,v) - \frac{\partial h_u}{\partial t}\right)^2 d\Omega dt,$$

$$\tilde{E}(u,v) = \tilde{F}(u,v) + K * G(u).$$

Consider the following problem of a fix point:

$$h_u^{(i+1)} = argmin\ \tilde{E}(u, h_u^{(i)}) \tag{32}$$

where the minimization takes place via all admissible functions $u$. Solving equation (32) means looking for a minimum of a quadratic functional what leads to a system of linear equations with a symmetric positive definite matrix.

We minimize the functional

$$E^t(h_u^{t+\tau}) = F^t(h_u^{t+\tau}) + K * G^t(h_u^{t+\tau})\,,$$

where

$$F^t(h_u^{t+\tau}) = \int_\Omega \left(h_u^{t+\tau} - h_u^t - \tau L(h_u^{t+\tau})\right)^2 d\Omega,$$

$$\tilde{F}^t(u,v) = \int_\Omega \left(u - h_u^t - \tau \tilde{L}(u,v)\right)^2 d\Omega,$$

$$G^t(h_u^{t+\tau}) = \int_\Gamma \left(h_u^{t+\tau}\right)^2 d\Omega\,.$$

If we apply the partial linearization, we get

$$\tilde{E}^t(u,v) = \tilde{F}^t(u,v) + K * G^t(u)\,.$$

The problem leads to looking for a minimum of $\tilde{E}^t(u,u)$ and the minimum is desired $h_u^{t+\tau}$. Functional $\tilde{E}^t(u,v)$ can be written in the form

$$\begin{aligned}\tilde{E}^t(u,v) &= \mathbf{u^t}A(v)\mathbf{u} + K\mathbf{u^t}B\mathbf{u} - 2c^t(v)\mathbf{u} + d \\ &= \mathbf{u^t}(A(v)+KB)\mathbf{u} - 2c^t(v)\mathbf{u} + d\end{aligned}$$

where $\mathbf{u}$ is the vector of coordinates of $u$ with respect to the basis functions $\varphi_i$, the matrices $A(v)$ and $B$ symmetric positive semidefinite matrices in $\mathbb{R}^{n\times n}$.
The matrix $B = (b_{ij})_{i,j=1}^n$ has elements

$$b_{ij} = \int\limits_{\Gamma} \varphi_i\varphi_j d\Gamma\,.$$

The matrix $A(v) = (a_{ij}(v))_{i,j=1}^n$ has elements

$$a_{ij}(v) = \int\limits_{\Omega} \left[\varphi_i + \tau m v^{m-1}\left(\alpha_x\frac{\partial\varphi_i}{\partial x} + \alpha_y\frac{\partial\varphi_i}{\partial x}\right)\right]\left[\varphi_j + \tau m v^{m-1}\left(\alpha_x\frac{\partial\varphi_j}{\partial x} + \alpha_y\frac{\partial\varphi_j}{\partial x}\right)\right] d\Omega\,.$$

Vector $c$ has elements

$$c_i(v) = \int\limits_{\Omega} \left[h_u^t + i_r(x,t)\right]\left[\varphi_i + \tau m v^{m-1}\left(\alpha_x\frac{\partial\varphi_i}{\partial x} + \alpha_y\frac{\partial\varphi_i}{\partial x}\right)\right] d\Omega$$

and

$$d = \int\limits_{\Omega} \left[h_u^t + i_r(x,t)\right]^2 d\Omega.$$

Then to find a minimum of the functional $\tilde{E}^t$ for a fix $v$ means to solve the system of linear equations

$$(A(v)+KB)\mathbf{u} = c(v) \tag{33}$$

and thus the desired minimizing element is

$$u = \sum_{i=1}^{n} \mathbf{u_i}\varphi_i.$$

Approximations of the solution of erosion transport equation (19), (20) and (21) obtained by using the Galerkin Finite Element Method and the Least-squares Finite Element Method are displayed in the following Fig 6.1.1. The figure clearly shows that the Galerkin Finite Element Method exhibits unwanted oscillations, while the Least-squares Finite Element Method does not exhibit unwanted oscillations and behaves better. The situation is displayed for $t = 100$ s.

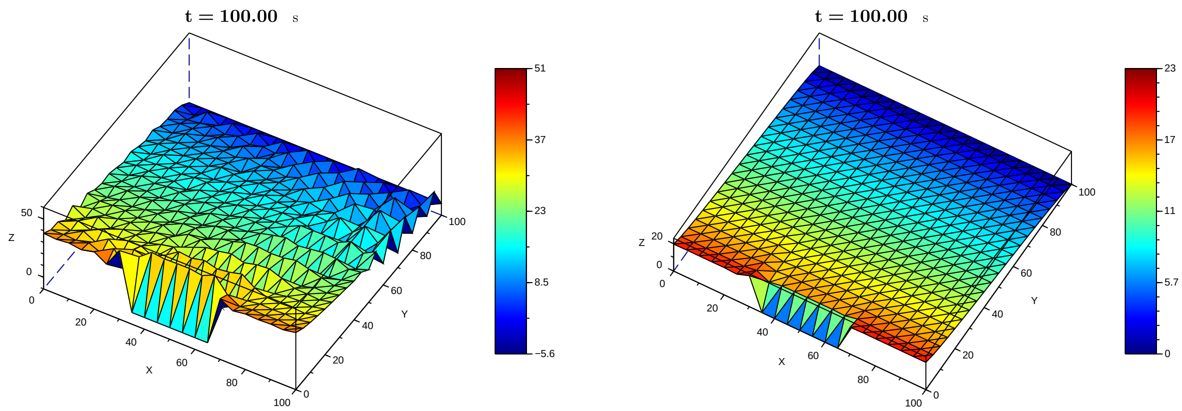


Fig. 6.1.1: Galerkin FEM solution (left) and Least-squares FEM solution (right)
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## CONCLUSION

The basic idea of the least-squares method for numerical solution of differential equations is known. However, the true power of the least-squares finite element method has not been exploited in all details for all types of problems. The idea of the least-squares finite element method is simple, but its theory is not so simple and is still evolving.
The authors' contribution lies in modeling a practical problem of surface runoff in hydrodynamics and involving the least-squares method to solve the erosion transport equation. The least-squares method was not appriori considered to be advantageous to use for solving such problems. It is demonstrated that the least-squares method is suitable for this problem, although the method does not have a natural physical interpretation. On the other hand, when applying the Galerkin method, although this method has a natural physical interpretation as a variational method, but unwanted oscillations may occur in approximations of the solution. The Galerkin method (or the finite difference methods) for convective problems yields uncertain performance, no theory can guarantee numerical results qualities. The least-squares method works fine with any generic catchment setup, even with non-physical crazy setup.

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# IDENTIFICATION OF ELECTRICALLY ACTIVE DEFECTS IN MODERN STRUCTURES BASED ON GALLIUM NITRIDE

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**Abstract:** *A compound semiconductor Gallium Nitride (GaN) is an ideal for high power and high-frequency applications since it is suitable to create the well-known two-dimensional gas. Nevertheless, even a minor concentration of electrically active defects is capable to decrease the quality of such semiconductors. An essential task in the continuous evolution of semiconductor structures based on modern compounds is the identification and detection of defect states. This contribution discusses the nature of defect states and describes the principles of the most frequently used defect characterisation method. A better understanding of the presence and the origin of defect states is a technical must, which is utilised to optimise and ensure a high-quality technological and fabrication process.*

**Keywords:** gallium nitride GaN, high electron mobility transistor HEMT, defect characterisation in semiconductors, high frequency and high-power structures.

## INTRODUCTION

In 1956 John Bardeen, Walter Houser Brattain a William Bradford Shockley were awarded the Nobel price of physics for the invention of the 20$^{th}$ century, the transistor [1]. That time nobody could forecast that this invention [2] will change the technical advances of our society to a level, that is capable to redefine the borders between fiction and reality. More intensive development in fields like electronics, micro and nanoelectronics and photonics made possible the debut of Silicon as the “king” of semiconductors. Todays need for higher quality, performance and effectivity with lower costs pushed Silicon to its physical borders, therefore new and new compounds are investigated as suitable replacements for example in the T = 200°C functional temperatures [3]. For instance, in case of a simple example such as mobile phones charged with 4.6 A and 20V, the current silicon technology does not allow the miniaturization in such a scale as needed; therefore new compounds for the new generation of switching structures need to be investigated [4, 5, 6]. GaN a wide bandgap, two-component semiconductor, and its desirable properties make this material a suitable candidate for high frequency and high-power applications (Fig. 1) [7]. Real benefits of GaN become visible in applications such as converters for electrical automobiles, portable batteries, and 5G networks. Improving efficiency of power electronic devices is crucial to reduce switching losses and lower the $CO_2$ emission. The world's total net electricity consumption as well as electricity generation is increasing day by day. It is predicted that it will double between years 2003 and 2030, from 14,781 billion kilowatt hours to 30,116 billion kilowatt hours, which is an average increase rate of 2.7% annually [8]. Electronic devices on GaN basis have the potential to reduce energy losses from 10% to 25% [9], therefore more effective as the silicon based ones. Another positive side of GaN is low or no dependency on high working temperatures and radiation hardness.

Although GaN probably will not entirely replace silicon in common applications, since Silicon is already widely used, technologies understood and mastered with a considerable high fabrication quality and low concentrations of defects.

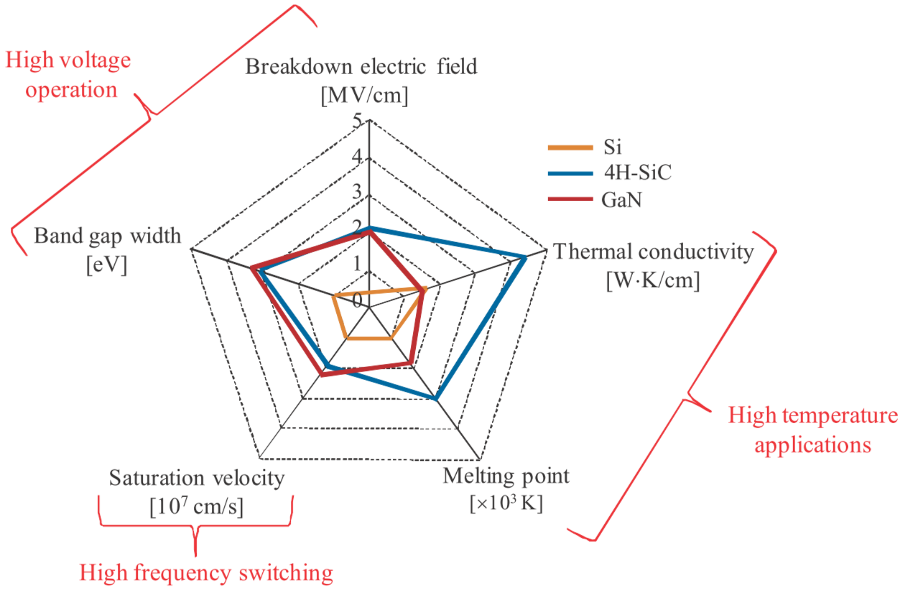



**Fig. 1.** Diagram of GaN, 4H-SiC and Si properties by field of application Source [10].

The aim of this contribution is to describe defect states of GaN by the most frequently used characterization tool, the Deep Level Transient Spectroscopy method. The article tends to evaluate actual structures developed and fabricated for 5G networks and to discuss possible effects of defect states on this material and application.

## 1. GALLIUM NITRIDE IN THE SEMICONDUCTOR INDUSTRY

The single crystal of gallium nitride first saw the light a day in 1969, when two scientists Maruska and Tietjen in the United States were able to grow it on a sapphire substrate [11]. Despite this first promising result, there was a great disappointment, because scientists were unable to find a suitable method for P-type doping. However, a significant historical breakthrough in the research and development of gallium nitride occurred two decades later in 1989 when a Japanese scientific team headed by Professor Isamu Akasaki of Nagoya University succeeded in producing the first gallium nitride diode with a PN junction. This success was transferred towards the development of a white electroluminescent diode, which is still an essential element of all LCDs.

Nowadays GaN has found applications in almost every area of the semiconductor industry, and without GaN-based elements we cannot imagine devices such as mobile phones, radars, traffic lights, large-screen displays, and others [12]. Despite this significant test, the research of this material is still progressing and all leading research centres tend to better understand this unique material and improve the fabrication quality [13, 14, 15]. Among the worlds recognised workplaces, the Institute of Electronics and Photonics of the FEI STU in Bratislava is also included, an excellent example of this is the visit of Prof. Hiros Amman, Nobel Prize winner for GaN diodes (Fig. 2).

**Fig. 2.** Nobel price winner for blue LED Hirosi Amano (second from left) on his visit at our university (the Institute of Electronics and Photonics), September 19th, 2018.
Source: own

## 2. DEFECTS IN SEMICONDUCTORS

A material defect is a disruption of the ideal regular arrangement of atoms in the semiconductor crystal. In many cases, it is only a non-active structural fault; however, in others it can also have a significant influence on electrical properties. While an ideal semiconductor can be described by energy bands separated by the forbidden gap, the presence of an electrically active defect is able to create an energy level precisely in this gap. These extra levels can capture electrons and hold them for a long period of time, hence defects are often referred to as traps. Typical examples of defect states are admixture atoms, excess atoms in the lattice (interstitials), missing atoms in the crystalline lattice (vacancies), or alignment disorders (dislocations) [16, 17]. A real semiconductor compound can contain traps of various origins, mostly created by unintentional contamination during crystal growth or further processing [18].

In order to understand the generation of defects in semiconductor structures, the diagnostics of semiconductor materials must be an integral part of the research activity. Emphasis is placed mainly on processes that can induce individual defect states and the impact of these on the properties of final semiconductor structures. Due to their properties like automation, availability, sensitivity, non-destructive character, capacitance and current measurement methods are ranked among the most important processes of semiconductor material diagnostics. The principle of the given measurement methods is based on the monitoring of charge changes in the depletion region.

One of the classical methods of characterization is Deep Level Transient Spectroscopy (DLTS) [19]. This method is based on sensing the capacity changes after applying a voltage pulse on the investigated structure. The method follows the relaxation of the system to its original state, which is strongly influenced by the trapping of charges as well as their leakage. The obtained capacitive responses are used to construct Arrhenius dependencies, from which we can determine the basic parameters of traps: activation energy and trapping cross-section. These parameters represent a unique picture of the defect state and can serve as a fingerprint to identify the origin of the defect state.

## 3 TRANSISTORS WITH HIGH ELECTRON MOBILITY

High Electron Mobility Transistors (HEMTs) of the AlGaN/GaN compound has attracted considerable attention, due to its beneficial properties in 5G networks, high power and high frequency applications [18, 20]. This favourable feature is ensured by the presence of the 2 Dimensional Gas 2DEG in the structure, which is formed between the two different bandgap layers, hence at the interface of AlGaN and GaN. 2DEG forms with a high density, over $1\times10^{13}$ $cm^{-2}$, originating from spontaneous and piezoelectric polarization fields as well as from the large conduction band offset, and the electron saturation velocity as high as $2\times10^{7}$ cm/s. The ultimate goal of the actual research is to increase the effectivity of HEMTs in the gigahertz range and at the same time to reduce fabrication costs [21].

One of the important problems of GaN HEMTs is heat dissipation, which is generated by the current flow. The newest substrate suitable for GaN is considered to be silicon carbide SiC, which provides the needed heat transfer and electrical properties. A more advanced investigation showed that the temperature conductivity could be increased by up to 30% by the addition of epitaxially grown isotopic clean SIC layer ($^{28}Si$ + $^{12}C$) isotopes) on the top of the substrate [20]. Nevertheless, heat conduction depends on many other factors such as the presence of defect states such as dislocations in the active region of the transistor [10]. The mismatch between the crystal grid (3.5%) of SiC and GaN also induces mechanical stress that can be compensated, e.g. by an AlN thin layer.

Another issue is the leakage current through the SiC substrate. By this phenomenon, the current flows not int the emitter as it should be, rather through the substrate. This unwanted behaviour can be corrected by additional AlGaN/GaN layers acting as a potential barrier for the leakage current. In addition, this structural setup has beneficial effects on mechanical stress [22].

Reducing production costs is a major challenge in current GaN research. Without finding solutions, mass production and greater commercial application of these structures are not possible. In this respect, it is necessary to replace the expensive SiC semi-insulating substrate with cheaper ones, such as N doped SiC. This material is already widely used in power and LED industries where the thin semi-insulating SiC layer is epitaxially grown [23].

All the above-mentioned technological steps of fabrication are capable of generating electrically active defect states, thereby have a limiting effect on the electrical properties and the quality of the final structures.

## 4 ILLUSTRATION OF DEFECT IDENTIFICATION

To illustrate the identification and measurement of electrically active defect states in GaN three test samples A, B, C are examined. The structures were prepared by MOCVD Metalorganic vapour-phase epitaxy in III-V Lab France. Fig. 3 shows the schematic view of these HEMT samples where the interfaces between AlGaN an GaN, hence the 2DEG is clearly visible. All the structures have minor differences; hence a qualitative comparison should be possible: Structure A as a standard referent structure only newest technological optimization used, while B and C were grown on an N type SiC substrate with a semi-insulating SiC layer on the top. This layer in the case of sample C was an isotopic clean SiC with additional AlGaN/GaN layers included. For all samples ohmic contacts were prepared by TiAlNiAu, and the gate electrode is from NiPtAu to provide a Schottky barrier.

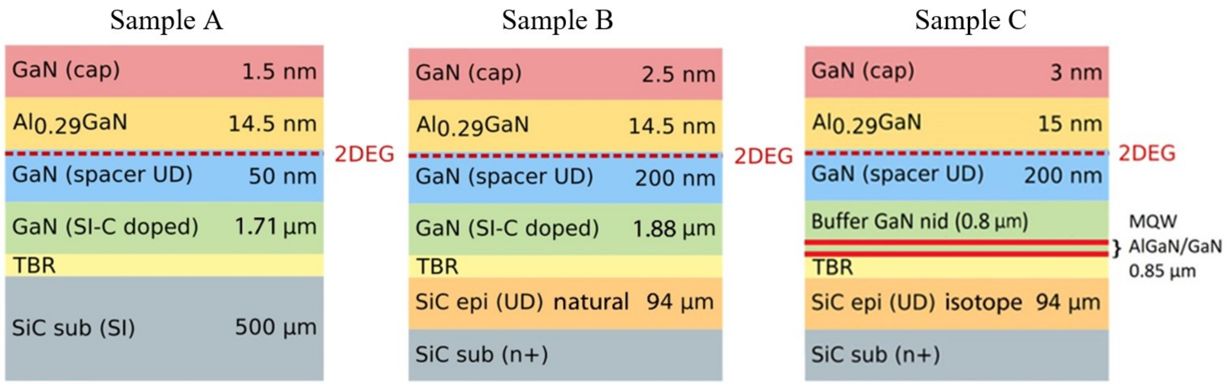



**Fig. 3.** Comparison of investigated HEMT structures.
Source: own

Firs electrical measurements are shown in Fig. 4. C-V and I-V characteristics of all 3 samples showed a quality technological process with low defect concentrations. Reproducibility of these measurements is high even after many cycles of measurements under voltage stress between temperatures 80K to 540K. None of these had effect on electrical properties such as threshold voltage, leakage current, or the quality of the Schottky gate contact that performs the rectifying function. The most visible character of the curves can be observed for sample C where the leakage current was significantly lower. This result can be contributed to the already discussed addition of AlGaN/GaN.

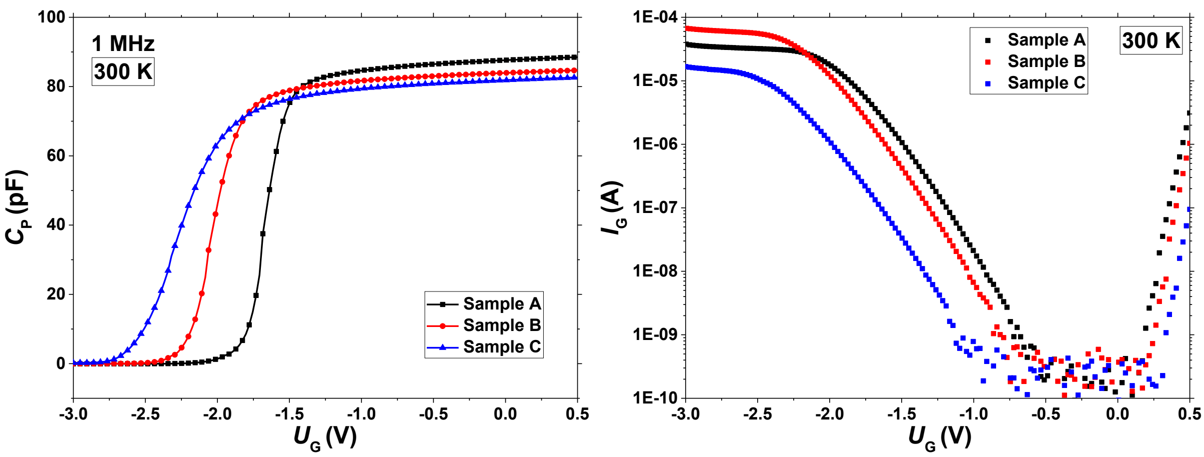
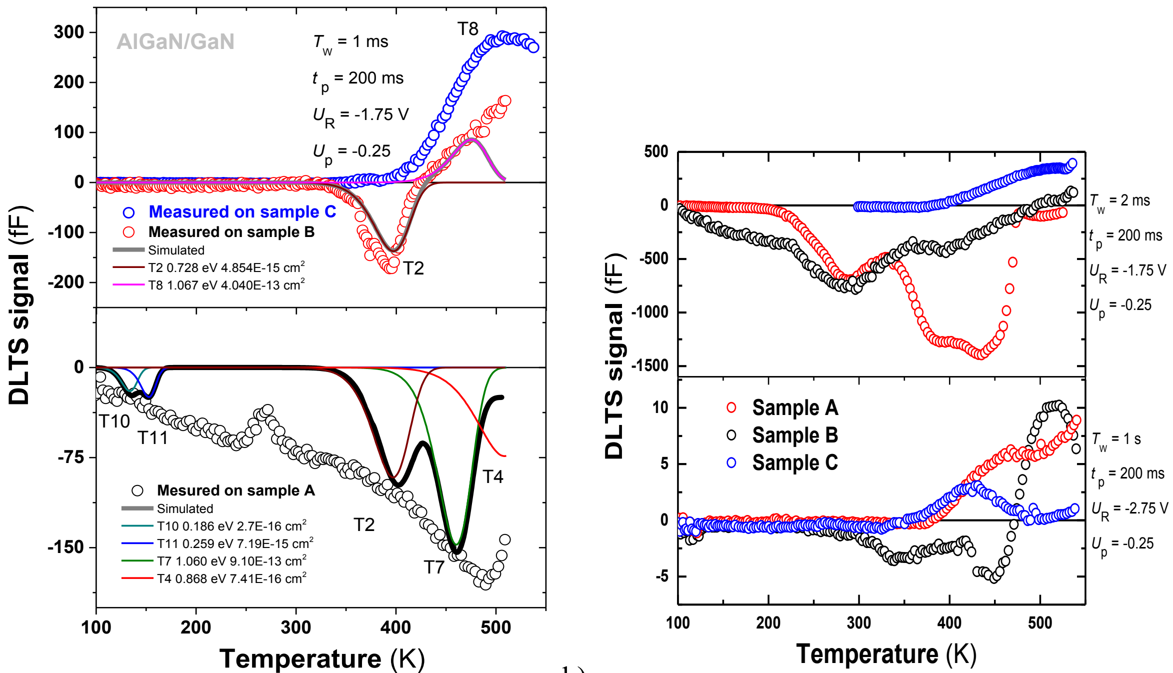



**Fig. 4.** Comparison of C-V a I-V characteristics of GaN/AlGaN/GaN/SiC HEMT structures
Source: own

C-V and I-V characteristics are important to define the input parameters of the DLTS method: reverse voltage $U_R$, filing voltage $U_P$, period width $T_w$, and filling pulse width $t_P$. To be able to evaluate these samples as much as possible, a wide variety and many sets of DLTS measurements were needed to be carried out, to scan the whole sample and retrieve spectra for a reliable evaluation (Fig. 5). Eleven deep energy levels were identified based on Arrhenius curves, which were calculated by the obtained spectra (Fig. 6), and output trap parameters were compared with literature [18, 24, 25].




a) b)

**Fig. 5.** a) DLTS spectrums with simulated defect states A and C, b) DLTS spectrums of A, B and C samples at various measurement conditions (reverse voltage $U_R$, filing voltage $U_P$, period width $T_w$ and filling pulse width $t_P$)
Source: own

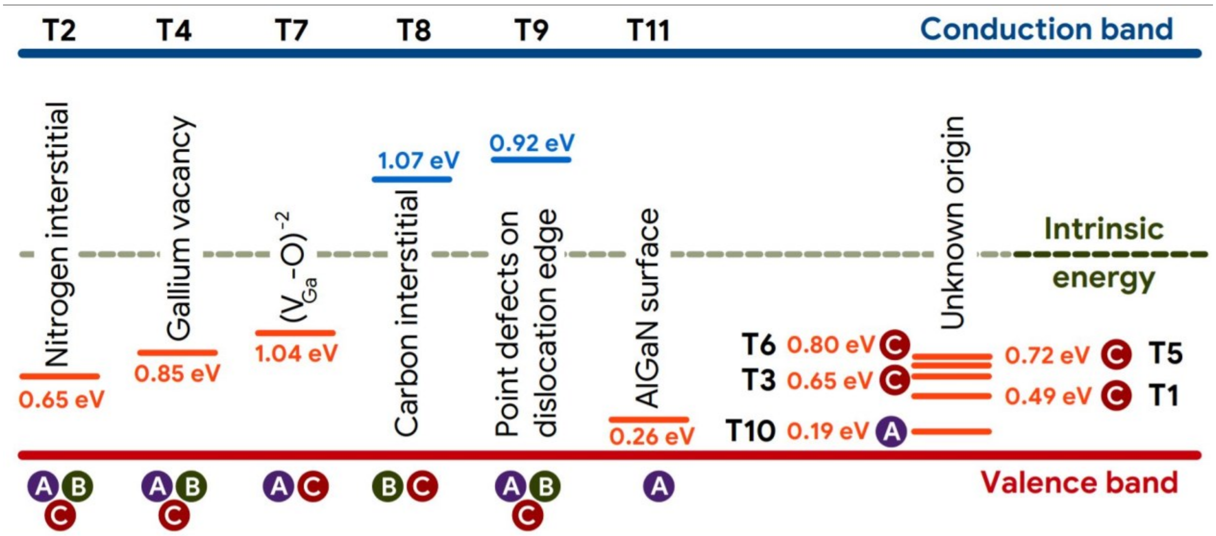



**Fig. 6.** Deep energy levels in GaN/AlGaN/GaN/SiC HEMT structures with activation energies and possible origins.
Source: own

Fig. 5a shows the measured DLTS spectra of samples A and B with defect simulations. Each maximum or spectrum peak indicates the presence of a defect state. Fig. 5b shows the comparison of all samples at matching experimental conditions, while in Fig. 6 all identified defects states were assessed in the band diagram.

As shown, the forbidden gap indicates many deep energy levels. Three of these were found in all samples: nitrogen interstitial (nitrogen atoms outside of the ideal crystal grid), gallium vacancy (gallium atoms are not occupying their ideal state leaving empty locations) and point defects on the edge of dislocation (complex defect state in a form of disarrangement of atoms in the crystal grid). The identified defect states are native, hence they are typical for GaN. The concentration can be reduced by the growth process optimisation.

### 4.1 Assessment of results

The provided samples of GaN/AlGaN/GaN/SiC HEMT structures were evaluated by C-V, I-V and the DLTS method. The results indicated sample C shows the most suitable properties, where the AlGaN/GaN barrier and the isotopic substrate significantly improved the characteristics. Despite such a huge structural change, no significant degradation was identified, and the characteristics stayed intact even after temperature and voltage stress measurements.

## CONCLUSION

Progress of humankind depends on continuous evolution of technology and the improvement of the quality of life. In recent decades, silicon-based structures have improved their efficiency, increased speed and have been miniaturized to their physical limits. However, to advance, new materials that are capable to open new possibilities and solve previously unknown problems need to be investigated. The potential of gallium nitride to save 10 to 25% of energy consumption provides sustainability for years to come, and it is, in fact, the future of electronics. Therefore, it is necessary to raise public awareness of new materials, progressive technologies, and to inspire the young generation to be actively involved in science and research.

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# ACTIONS OF GROUPS AND HYPERGROUPS OF ARTIFICIAL TIME VARYING NEURONS

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**Abstract:** *The contribution is based on certain analogy between descriptions of differential equations including actions ordinary differential operators and concepts of formal time-varying neurons. Above mentioned concept let authors describe properties of structures and hyperstructures time-varying neurons with fresh view of point to perceptron structure. There is presented the algebraic approach to systems of artificial neurons as a main topic. Moreover we describe actions of group and hypergroups of artificial neurons which is relatively new point of view in the theory of artificial neural networks.*

**Keywords:** Neural network, transposition hypergroups, linear ordinary differential operators, groups of neurons.

## INTRODUCTION

The contribution is based on certain analogy between descriptions of differential equations including actions ordinary differential operators and concepts of formal time-varying neurons. As it is mentioned in the dissertation [14] neurons are the atoms of neural computation. Out of those simple computational units all neural networks are build up. The substanding representative of the famous school of Otakar Borůvka František Neuman([16, 17, 18]) wrote: "Algebraic, topological and geometrical tools together with the methods of the theory of dynamical systems and functional equations make possible to deal with problems concerning global properties of solutions by contrast to the previous local investigations and isolated results." Influence of mentioned ideas is a certain motivating factor of our investigations.

So, we consider linear ordinary differential operators of the form $L_n = \sum_{k=0}^{n} p_k(x)D^k$, where $D_k = \frac{d^k}{dx^k}$, $p_k(x)$ is a continuous function on some open interval $J \subset \mathbb{R}$, $k = 0, 1, \ldots, n-1$, $p_n(x) \equiv 1$, i.e. $L_n(y) = 0$ which is a linear homogenous ordinary differential equation of the form:

$$y^{(n)}(x) + \sum_{k=0}^{n-1} p_k(x)y^{(k)}(x) = 0. \tag{1}$$

Some ideas and approaches treated e.g. in [4, 5, 6, 8] are transformed in a certain sense into investigations of structures of neurons. From the biological point of view it is advisible to use an integrative propagation function. And therefore convenient choice would be to use the weighted sum of the input $f(w,x) = \sum_i w_i x_i$, that is the activation potential equals to the scalar product of input and weights. In fact, the most popular propagation function since the dawn of neural computation, however it is often used in the slightly different form:

$$f(w,x) = \sum_i w_i x_i + \Theta, \tag{2}$$

Symbols of sums in the above formulas is usual in the corresponding literature (see e.g. [7, 9, 14, 15, 22, 23, 24, 25, 26, 29]). The special weight $\Theta$ is called bias. Applying $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$ as the above activation function yields the famous perceptron of Rosenblatt. In that case the function $\Theta$ works as a threshold. Besides $(2)$ there are, of course, many other possible propagation functions. If $(2)$ is supplemented with the identity as activation function and real-valued domains are given a real linear neuron $\mathsf{y} = \sum_i w_i x_i + \Theta$ is obtained. This real linear neuron can be seen as an example of a Clifford neuron. The literature devoted to the mentioned topic is e.g. [2, 3, 7, 9, 14, 15, 22, 23, 24, 25, 26, 29].

## STRUCTURES OF ARTIFICIAL NEURONS

We can consider a certain generalization of classical artificial neurons mentioned above consisting in such a way that inputs $x_i$ and weight $w_i$ will be functions of an argument $t$ belonging into a linearly ordered (tempus) set $T$ with the least element $0$. As the index set we use the set $\mathbb{C}(J)$ of all continuous functions defined on an open interval $J \subset \mathbb{R}$. So, denote by $W$ the set of all non-negative functions $w : T \to \mathbb{R}$ forming a subsemiring of the ring of all real functions of one real variable $x : \mathbb{R} \to \mathbb{R}$. Denote by $Ne(\vec{w}_r) = Ne(w_{r1}, \ldots, w_{rn})$ for $r \in \mathbb{C}(J)$, $n \in \mathbb{N}$ the mapping

$$y_r(t) = \sum_{k=1}^{n} w_{r,k}(t) x_{r,k}(t) + b_r \tag{3}$$

which will be called the artificial neuron with the bias $b_r \in \mathbb{R}$. By $\mathbb{AN}(T)$ we denote the collection of all such *artificial neurons*. Neurons are usually denoted by capital letters $X$, $Y$ or $X_i$, $Y_i$, nevertheless we use also notion $Ne(\vec{w})$, where $\vec{w} = (w_1, \ldots, w_n)$ is the vector of weights. We suppose - for the sake of simplicity - that transfer functions (activation functions) $\varphi$, $\sigma(or\ f)$ are the same for all neurons from the collection $\mathbb{AN}(T)$ or the role of this function plays the identity function $f(y) = y$. Now, similarly as in the case of the collection of linear differential operators above, we will construct a group and hypergroup of artificial neurons. Denote by $\delta_{ij}$ the so called Kronecker delta, $i, j \in \mathbb{N}$, i.e. $\delta_{ii} = \delta_{jj} = 1$ and $\delta_{ij} = 0$, whenever $i \neq j$. Suppose $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in \mathbb{AN}(T)$, $r, s \in \mathbb{C}(J)$, $\vec{w}_r = (w_{r1}, \ldots, w_{r,n})$, $\vec{w}_s = (w_{s1}, \ldots, w_{s,n})$, $n \in \mathbb{N}$. Let $m \in \mathbb{N}$, $1 \leq m \leq n$ be a such an integer that $w_{r,m} > 0$. We define

$$Ne(\vec{w}_r) \cdot_m Ne(\vec{w}_s) = Ne(\vec{w}_u), \tag{4}$$

where

$$\vec{w}_u = (w_{u,1}, \ldots, w_{u,n}) = \big(w_{u,1}(t), \ldots, w_{u,n}(t)\big), \tag{5}$$
$$w_{u,k}(t) = w_{r,m}(t) w_{s,k}(t) + (1 - \delta_{m,k}) w_{r,k}(t), t \in T$$

and, of course, the neuron $Ne(\vec{w}_u)$ is defined as the mapping $y_u(t) = \sum_{k=1}^{n} w_k(t)x_k(t) + b_u,\ t \in T,\ b_u = b_r b_s.$ Further for a pair $Ne(\vec{w}_r),\ Ne(\vec{w}_s)$ of neurons from $\mathbb{AN}(T)$ we put $Ne(\vec{w}_r) \leq_m Ne(\vec{w}_s),\ w_r = (w_{r,1}(t), \ldots, w_{r,n}(t)),\ w_s = (w_{s,1}(t), \ldots, w_{s,n}(t))$ if $w_{r,k}(t) \leq w_{s,k}(t),\ k \in \mathbb{N},\ k \neq m$ and $w_{r,m}(t) = w_{s,m}(t),\ t \in T$ and with the same bias. Evidently $(\mathbb{AN}(T), \leq_m)$ is an ordered set. A relationship (compatibility) of the binary operation "$\cdot_m$" and the ordering $\leq_m$ on $\mathbb{AN}(T)$ is given by this assertion analogical to the above one. Proofs of following assertion can be found e.g. in [7, 9].

***Lemma* 1.** *The triad* $(\mathbb{AN}(T), \cdot_m, \leq_m)$ *(algebraic structure with an ordering) is a non-commutative ordered group.*

Denoting $\mathbb{AN}_1(T)_m = \{Ne(\vec{w}); \vec{w} = (w_1, \ldots, w_n),\ w_k \in \mathbb{C}(T),\ k = 1, \ldots, n,\ w_m(t) \equiv 1\}$, we get the following assertion:

**Proposition 1.** *Let* $T = \langle 0, t_0) \subset \mathbb{R},\ t_0 \in \mathbb{R} \cup \{\infty\}$. *Then for any positive integer* $n \in \mathbb{N},\ n \geq 2$ *and for any integer* $m$ *such that* $1 \leq m \leq n$ *the semigroup* $(\mathbb{AN}_1(T)_m, \cdot_m)$ *is an invariant subgroup of the group*$(\mathbb{AN}(T)_m, \cdot_m)$.

If $m,\ n \in \mathbb{N},\ 1 \leq m \leq n-1$, then a certain relationship between groups $(\mathbb{AN}_r(T)_m, \cdot_m)$, $(\mathbb{LA}(T)_{m+1}, \circ_{m+1})$ is contained in the following proposition:

**Proposition 2.** *Let* $t_0 \in \mathbb{R},\ t_0 > 0,\ T = \langle 0,\ t_0) \subset \mathbb{R}$ *and* $m,\ ,n \in \mathbb{N}$ *are integers such that* $1 \leq m \leq n-1$. *Define a mapping* $F:\ \mathbb{AN}_n(T)_m \to \mathbb{LA}_n(T)_{m+1}$ *by this rule: For an arbitrary neuron* $Ne(\vec{w}_r) \in \mathbb{AN}_n(T)_m$, *where* $\vec{w}_r = (w_{r,1}(t), \ldots, w_{r,n}(t)) \in [\mathbb{C}(T)]^n$ *we put* $F(Ne(\vec{w}_r)\,) = L(w_{r,1}, \ldots, w_{r,n}) \in \mathbb{LA}_n(T)_{m+1}$ *with the action :*

$$L(w_{r,1}, \ldots,\ w_{r,n})y(t) = \frac{d^n y(t)}{dt^n} + \sum_{k=1}^{n} w_{r,k}(t)\frac{d^{k-1}(t)}{dt^{k-1}},\ y \in \mathbb{C}^n(T). \tag{6}$$

*Then the mapping* $F: \mathbb{AN}_n(T)_m \to \mathbb{LA}_n(T)_{m+1}$ *is a homomorphism of the group* $(\mathbb{AN}_n(T)_m,\ \cdot_m)$ *into the group* $(\mathbb{LA}_n(T)_{m+1},\ \circ_{m+1})$ (concerning this group see [5]).

## TRANSPOSITION HYPERGROUPS OF NEURONS

Now, using the construction described in the above Lemma 1. we obtain the final transpozition hypergroup (called also non-commutative join space, cf. e.g, [6, 11, 19, 21]). Notice that constructionsbased on Ends Lemma (cf. [8]) are also treated in [19, 20, 21]. Denote by $\mathbb{P}(\mathbb{AN}(T)_m)^*$ the power set of $\mathbb{AN}(T)_m$ consisting of all nonempty subsets of the last set and define a binary hyperoperation $*_m : \mathbb{AN}(T)_m \times \mathbb{AN}(T)_m \to \mathbb{P}(\mathbb{AN}(T)_m)^*$ by the rule

$$Ne(\vec{w}_r) *_m Ne(\vec{w}_s) = \{Ne(\vec{w}_u);\ Ne(\vec{w}_r) \cdot_m Ne(\vec{w}_s) \leq_m Ne(\vec{w}_u)\} \tag{7}$$

for all pairs $Ne(\vec{w}_r),\ Ne(\vec{w}_s) \in \mathbb{AN}(T)_m$. More in detail if $\vec{w}(u) = (w_{u,1}, \ldots, w_{u,n}),\ \vec{w}(r) = (w_{r,1}, \ldots, w_{r,n}),\ \vec{w}(s) = (w_{s,1}, \ldots, w_{s,n})$, then $w_{r,m}(t)w_{s,m}(t) = w_{u,m}(t),\ w_{r,m}(t)w_{s,k}(t) + w_{r,k}(t) \leq w_{u,k}(t)$, if $k \neq m,\ t \in T$. Then we have that $(\mathbb{AN}(T)_m,\ *_m)$ is a non-commutative hypergroup. The above defined invariant (termed also normal) subgroup $(\mathbb{AN}_1(T)_m,\ \cdot_m)$ of the group

$(\mathbb{AN}(T)_m, \cdot_m)$ is the carried set of a subhypergroup of the hypergroup $(\mathbb{AN}(T)_m, *_m)$ and it has certain significant properties. Using certain generalization of methods from [4] we obtain after investigation of constructed structures this result:

Let $T = \langle 0, t_0) \subset \mathbb{R}$, $t_0 \in \mathbb{R} \cup \{\infty\}$. Then for any positive integer $n \in \mathbb{N}$, $n \geq 2$ and for any integer $m$ such that $1 \leq m \leq n$ the hypergroup $(\mathbb{AN}(T)_m, *_m)$, where
$\mathbb{AN}(T)_m = \{Ne(\vec{w}_r);\ \vec{w}_r = (w_{r,1}(t), \ldots, w_{r,n}(t)) \in [\mathbb{C}(T)]^n,\ w_{r,m}(t) > 0,\ t \in T\}$ is a transposition hypergroup (i.e. a non-commutative join space) such that $(\mathbb{AN}(T)_m, *_m)$ is its subhypergroup. A certain generalization of the formal (artificial) neuron can be obtained from expression of linear differential operator of the *n*-th order. Consider function $u : J \to \mathbb{R}$, where $J \subseteq \mathbb{R}$ is an open interval; input are derived from the function $u \in \mathbb{C}^n(J)$ as it follows: Inputs $x_1(t) = u(t), x_2 = \frac{du(t)}{dt}, \ldots, x_n(t) = \frac{d^{n-1}(t)}{dt^{n-1}}$, $n \in \mathbb{N}$. Further the bias $b = b_0\frac{d^n u(t)}{dt^n}$. As weights we use the continuous function $w_k : J \to \mathbb{R}$, $k = 1, \ldots, n-1$.

Then formula

$$y(t) = \sigma\left(\sum_{k=1}^{n} w_k(t)\frac{d^{k-1}u(t)}{dt^{k-1}} + b_0\frac{d^n u(t)}{dt^n}\right) \tag{8}$$

is a description of the action of the neuron $Dn$ which will be called a formal(artificial) differential neuron. This approach allows to use solution spaces of corresponding linear differential equations. Used terms and proofs of auxiliary assertions can be found in literature [5, 6].

Now consider an arbitrary but fixed chosen neuron structure $Ne_0 \in \mathbb{AN}_n(T)_m$ and denote

$$Ct(Ne_0(\vec{w}_r)) = \{Ne(\vec{w}_r) \in \mathbb{AN}_n(T)_m; Ne(\vec{w}_r) \circ Ne_0(\vec{w}_r) = Ne_0(\vec{w}_r) \circ Ne(\vec{w}_r),$$

that $Ct(Ne_0(\vec{w}_r))$ is submonoid of the group $\mathbb{AN}_n(T)_m$ called the centralizer of the neuron structure $Ne_0(\vec{w}_r$ within the group $\mathbb{AN}_n(T)_m$.

Define a binary hyperoperation

$$\bullet : Ct(Ne_0(\vec{w}_r)) \times Ct(Ne_0(\vec{w}_r)) \to \mathcal{P}(Ct(Ne_0(\vec{w}_r))$$

$$\text{by } Ne(\vec{w}_r) \bullet Ne(\vec{w}_s) = \{Ne_0^n(\vec{w}_r) \circ Ne(\vec{w}_s) \circ Ne(\vec{w}_r); n \in \mathbb{N}_0\},$$

where $Ne_0^n(\vec{w}_r)$ is the n-th iteration (or the n-th power) of $Ne_0(\vec{w}_r)$ within the group $\mathbb{AN}_n(T)_m$. Considering the binary relation $\sigma_0 \subset Ct(Ne_0(\vec{w}_r)) \times Ct(Ne_0(\vec{w}_r))$ defined by $Ne(\vec{w}_r)\sigma_0 Ne(\vec{w}_s)$ if and only if $Ne(\vec{w}_s) = Ne_0^n(\vec{w}_r) \circ Ne(\vec{w}_r)$ for some $n \in \mathbb{N}_0$ we get without any effort that $(Ct(Ne_0(\vec{w}_r)), \sigma_0)$ is a quasi-ordered group. Indeed, the relation $\sigma_0$ is evidently reflexive and transitive.
Moreover, if $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in Ct(Ne_0(\vec{w}_r))$, $Ne(\vec{w}_r)\sigma_0 Ne(\vec{w}_s)$ and $Ne(\vec{w}_u) \in Ct(Ne_0(\vec{w}_r))$ are arbitrary operators then there exists a non-negative integer $m \in \mathbb{N}_0$ such that $Ne(\vec{w}_s) = Ne_0^m(\vec{w}_r) \circ Ne(\vec{w}_r)$. Then $Ne(\vec{w}_u) \circ Ne(\vec{w}_s) = Ne(\vec{w}_u) \circ Ne_0^m(\vec{w}_r) \circ Ne(\vec{w}_r) = Ne_0^m(\vec{w}_r) \circ Ne(\vec{w}_u) \circ Ne(\vec{w}_r)$ and $Ne(\vec{w}_s) \circ Ne(\vec{w}_u) = Ne_0^m(\vec{w}_r) \circ Ne(\vec{w}_r) \circ Ne(\vec{w}_u)$, thus we have $(Ne(\vec{w}_u) \circ Ne(\vec{w}_r))\sigma_0(Ne(\vec{w}_u) \circ Ne(\vec{w}_s))$ and $(Ne(\vec{w}_r) \circ Ne(\vec{w}_u))\sigma_0(Ne(\vec{w}_s) \circ Ne(\vec{w}_u))$. Since $Ct(Ne_0(\vec{w}_r))$ is a group $(Ne(\vec{w}_r)) \in Ct(Ne_0(\vec{w}_r))$ implies $Ne^{-1}(\vec{w}_r)) \in Ct(Ne_0(\vec{w}_r)))$, we obtain that $(Ct(Ne_0(\vec{w}_r)), \circ, \sigma_0)$ is a quasi-ordered group. Notice that for $Ne(\vec{w}_r)$, $Ne(\vec{w}_s) \in Ct(Ne_0(\vec{w}_r))$ such that $Ne(\vec{w}_r)\sigma_0 Ne(\vec{w}_s)$ we have $Ne^{-1}(\vec{w}_s)\sigma_0 Ne^{-1}(\vec{w}_r)$. By the Lemma on Ends (see the [3] or [4, 5, 13, 14, 15] defining on $(Ct(Ne_0(\vec{w}_r))$ the hyperoperation:

$$Ne(\vec{w}_r) \bullet Ne(\vec{w}_s) = \sigma_0(Ne(\vec{w}_s) \circ Ne(\vec{w}_r)) =$$

$$= \{Ne(\vec{w}_u) \in Ct(Ne_0(\vec{w}_r); (Ne(\vec{w}_s) \circ Ne(\vec{w}_r))\sigma_0 Ne(\vec{w}_u)\}$$

we obtain that $(Ct(Ne_0(\vec{w}_r), \bullet)$ is a non-commutative hypergroup. Now suppose that all considered structures of artificial neurons have all coefficients of the class $\mathbb{C}^\infty$. In what follows we denote by $Un(Ct(Ne_0(\vec{w}_r))$ the carrier (i.e. the underlying set) of the monoid $Ct(Ne_0^0(\vec{w}_r)$. As usually $Un(Ct(Ne_0(\vec{w}_r))^*$ means the free monoid of finite nonempty words (including the empty word corresponding to the identity neuron $Id$) formed by neurons from the set $Un(Ct(Ne_0(\vec{w}_r))$. Denote

$$\mathbb{P}_{Ne_0(\vec{w}_r)} = \{(Ne_0(\vec{w}_r) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f); f \in \mathbb{C}^\infty(T),$$
$$Ne_1(\vec{w}_r), \ldots Ne_m(\vec{w}_r) \in Un(Ct(Ne_0(\vec{w}_r))^*\}$$

and denote by $\mathcal{M}(\mathbb{P}_{Ne_0(\vec{w}_r)})$ the triad $(\mathbb{P}_{Ne_0(\vec{w}_r)}, Ct(Ne_0(\vec{w}_r), \delta_{Ne_0(\vec{w}_r)})$, where the action or transition function

$$\delta_{Ne_0(\vec{w}_r)} : \mathbb{P}_{Ne_0(\vec{w}_r)} \times Ct(Ne_0(\vec{w}_r) \to \mathbb{P}_{Ne_0(\vec{w}_r)}$$

is defined by the equality

$$\delta_{Ne_0(\vec{w}_r)}((Ne_0(\vec{w}_r) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r))(f), Ne(\vec{w}_u)) =$$
$$= (Ne_0(\vec{w}_r) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r))(f)$$

for any function $f \in \mathbb{C}^\infty(T)_m$ and any neuron $Ne(\vec{w}_u) \in Ct(Ne_0(\vec{w}_r)$. Then we obtain

**Theorem 1.** *Let $t_0 \in \mathbb{R}$, $t_0 > 0$, $T = \langle 0, t_0) \subset \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$. Let $\mathbb{AN}_n(T)_m$ be the above constructed quasi ordered group of neurons n-th order $Ne(w_0, w_1, \ldots, w_{n-1})$, $0 < w_0(x), x \in T$. Let $Ne_0(\vec{w}_r) \in \mathbb{AN}_n(T)_m$ be arbitrary but fixed chosen neuron. then the triad*

$$\mathcal{M}(\mathbb{P}_{Ne_0(\vec{w}_r)}) = (\mathbb{P}_{Ne_0(\vec{w}_r)}, Ct(Ne_0(\vec{w}_r), \delta_{Ne_0(\vec{w}_r)})$$

*is a multiautomaton (in other words it is an action of the hypergroup $(Ct(Ne_0(\vec{w}_r), \bullet)$ on the phase set $\mathbb{P}_{Ne_0(\vec{w}_r)})$, i.e. the transition function $\delta_{Ne_0(\vec{w}_r)} : \mathbb{P}_{Ne_0(\vec{w}_r)} \times Ct(Ne_0(\vec{w}_r) \to \mathbb{P}_{Ne_0(\vec{w}_r)}$ satisfies the Generalized Mixed Associativity Condition (GMAC).*

More information about term GMAC can be found in [4], page 102 or in [6], definition 2.5 page 61.

*Proof.* Suppose $f \in \mathbb{C}^\infty(T), Ne_1(\vec{w}_r), \ldots, Ne_m(\vec{w}_r), Ne(\vec{w}_s), Ne(\vec{w}_u) \in Ct(Ne_0(\vec{w}_r)$ are arbitrary artificial neurons. Then we have

$$\delta_{Ne_0(\vec{w}_r)}(\delta_{Ne_0(\vec{w}_r)}((Ne_0(\vec{w}_r) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f), Ne(\vec{w}_u) =$$
$$= \delta_{Ne_0((\vec{w}_r)}((Ne_0(\vec{w}_r) \circ Ne(\vec{w}_s) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f), Ne(\vec{w}_u) =$$
$$= (Ne_0(\vec{w}_r) \circ Ne(\vec{w}_u) \circ Ne(\vec{w}_s) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f) \in$$
$$\in \{(Ne_0^{k+1}(\vec{w}_r) \circ Ne(\vec{w}_u) \circ Ne(\vec{w}_s) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f), k \in \mathbb{N}_0\} =$$
$$= \{\delta_{Ne_0((\vec{w}_r)}((Ne_0(\vec{w}_r) \circ Ne(\vec{w}_x) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f),$$
$$Ne(\vec{w}_x) \in \{(Ne_0^k(\vec{w}_r) \circ Ne(\vec{w}_u) \circ Ne(\vec{w}_s); k \in \mathbb{N}_0\}\} =$$
$$= \delta_{Ne_0((\vec{w}_r)}((Ne_0(\vec{w}_r) \circ Ne_1(\vec{w}_r) \circ \cdots \circ Ne_m(\vec{w}_r)(f), Ne(\vec{w}_s) \bullet Ne(\vec{w}_u)).$$

Consequently the Generalized Mixed Associativity Condition is satisfied. □

In this connection in the paper [10] there are also mentioned the so called *P-hypergroups* due to T. Vougiouklis - cf. [27, 28].

## CONCLUSION

Theory of ANN (Artificial Neural Networks) is the very large and deep part of methods of Artificial intelligence and there are considered as the important part of the Theory of Systems. Various types of actions of groups and hypergroups of artificial neurons belong to study of structures motivated by actual theory of hyperstructures and their application. Investigation of group and hypergroups of artificial neurons is based on structures of ordinary differential operators which are parts of large theory treated by various authors as J. Diblík, J. Baštinec, Z. Šmarda, Z. Piskořová, G. Vážanová, contributions of which are presented in former proceedings of conference MITAV([1, 12, 13]). These considerations allows new view and insight into classical broadened applied parts of the artificial neural networks theory. This contribution is devoted to actions of hyperstructures of artificial neurons, in fact to automata without outputs with structured inputs with state sets formed by words of artificial neurons. This is a certain generalization of structures derived from groups of differential operators. It is well-known fact that applications of neural networks is in focus of science and technology in present time. The algebraic approach using structures presented in our contribution is not contained in older literature devoted to this topics.

**Author Contributions:** Contributions of both authors of this paper are equal.

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# ON REGULARITY CONDITIONS OF MATRIX TRANSFORMATIONS

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**Abstract:** *The aim of this article is to show that the conditions, which guarantee the regularity of transformation matrix are independent. The individual cases via examples are illustrated, i.e. if any of the conditions are not satisfied then the transformation matrix is not regular.*

**Keywords:** matrix transformation, sequence, Banach spaces.

## INTRODUCTION

Regular matrix transformations in the monograph [4] are studied. In papers [2], [3], [5], [6] and [7] we can find the generalization of regular matrix transformations of sequences of Banach space and some interesting properties about their structure. In [7] was proved a theorem to give a sufficient and necessary condition for regularity of transformation matrix $A = (a_{nk})$ for the sequences of Banach space. This theorem generalizes the situation in [4], where the conditions for sequences of real numbers were stated. Thus we can use the matrix transformation of sequences from the "bigger" sets.

Let $(X, \|\cdot\|)$ denotes an arbitrary Banach space. Denote elements of Banach space by $\alpha, \beta, \ldots$. The zero element of $X$ denote by $\theta$ and its unit element by $\epsilon$ such that $\|\epsilon\| = 1$.

The sequence $\alpha = (\alpha_n)$, where $\alpha_n \in X$ for all $n = 1, 2, \ldots$ *converges* to $\beta \in X$ if and only if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$, such that for all $n > k$ implies $\|\alpha_n - \beta\| < \varepsilon$. The sequence $\alpha = (\alpha_n)$, where $\alpha_n \in X$ for all $n = 1, 2, \ldots$ is called *Cauchy sequence* if and only if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$, such that for all $i, j > k$ the inequality $\|\alpha_i - \alpha_j\| < \varepsilon$ holds. It is well known that in Banach space every Cauchy sequence converges. (see e.g. [1], [6]).

Let $A = (a_{mn})$ $(m, n = 1, 2, \ldots)$ be an infinite matrix of real numbers. A sequence $\alpha = (\alpha_n)$, where $\alpha_n \in X$ for all $n = 1, 2, \ldots$ is said to be *A-limitable* (limitable by a method $(A)$) to an element $a \in X$, if $\lim_{m\to\infty} \beta_m = a$, where $\beta_m = \sum_{n=1}^{\infty} a_{mn}\alpha_n$. If a sequence $\alpha = (\alpha_n)$ is $A$-limitable to the element $a$, we write A-$\lim_{n\to\infty} \alpha_n = a$. The method $(A)$ defined by the matrix $A$ is said to be *regular* if $\lim_{n\to\infty} \alpha_n = a$ implies A-$\lim_{n\to\infty} \alpha_n = a$ (see [6], [7]). If the method $(A)$ is regular then the matrix $A$ is called *regular transformation matrix* or in short *regular matrix*.

For the sequences of elements of Banach space we have the following theorem.

**Theorem A.** *Let $A = (a_{mn})$ be an infinite matrix of real numbers. The sequence*

$$\beta_m = \sum_{n=1}^{\infty} a_{mn}\alpha_n$$

*converges to* $a$ *for* $m \to \infty$ *and* $\alpha_n \to a$ *if and only if the following conditions hold:*

*a)* *there is a constant* $K$ *such that* $\sum_{n=1}^{\infty} |a_{mn}| \leq K$ *for every* $m = 1, 2, \ldots,$

*b)* *for every* $n = 1, 2, \ldots$, $\lim_{m\to\infty} a_{mn} = 0$,

*c)* $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = 1$.

*Proof.* See [7]. □

The infinite matrix $A = (a_{mn})$ of real numbers is regular if and only if it satisfies each condition of previous theorem (see [4]).

## 1 RESULT

In this section we give some examples of matrices which fulfill only two conditions of Theorem A. Moreover we give examples of convergent sequences and we will show that the transformed sequences with respect to the matrix do not converge to the same element. As standard examples of Banach spaces we will use either the space of all bounded sequences of real numbers denoted by $\ell_\infty$ endowed by supremum norm i.e. $\|x\| = \sup_i |x_i|$ or the space $(\mathbb{R}, \|\cdot\|)$, where $\|x\| = |x|$.

**Example 1.** Define the matrix $A = (a_{mn})$ as follows

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \frac{1}{3} & \frac{1}{9} & \cdots & \frac{1}{3^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m+1} & \frac{1}{(m+1)^2} & \cdots & \frac{1}{(m+1)^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The condition a) of Theorem A is satisfied, since

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \quad \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}, \quad \ldots, \quad \sum_{n=1}^{\infty} \frac{1}{(m+1)^n} = \frac{1}{m}, \quad \ldots.$$

It is enough to choose $K = 1$.

Similarly, the condition b) of Theorem A is satisfied, since for every $n = 1, 2, \ldots$ we have

$$\lim_{m\to\infty} a_{mn} = \lim_{m\to\infty} \frac{1}{(m+1)^n} = 0.$$

Finally, we show that the condition c) of Theorem A is not satisfied, because

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{1}{(m+1)^n} = \lim_{m\to\infty} \frac{1}{m} = 0 \neq 1.$$

Let we choose the sequence $\alpha = (\alpha_n) = (\epsilon, \epsilon, \dots, \epsilon, \dots)$ such that $\|\epsilon\| = 1$. Clearly $\alpha$ converges to $\epsilon$. Calculate the sequence $\beta = (\beta_n)$ as the image of sequence $\alpha$ with respect to the matrix $A$,

$$\beta = A\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \frac{1}{3} & \frac{1}{9} & \cdots & \frac{1}{3^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m+1} & \frac{1}{(m+1)^2} & \cdots & \frac{1}{(m+1)^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \epsilon \\ \epsilon \\ \vdots \\ \epsilon \\ \vdots \end{pmatrix} = \left(\epsilon, \frac{1}{2}\epsilon, \frac{1}{3}\epsilon, \dots, \frac{1}{m}\epsilon, \dots\right).$$

It is easy to see that the sequence $\beta$ converges to the zero element $\theta$ of the Banach space. Thus we showed that without the condition c) of Theorem A the regularity of matrix $A$ is not satisfied.

**Example 2.** Define the matrix $A = (a_{mn})$ in the following way

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The condition a) of Theorem A is satisfied, because for every $m = 1, 2, \dots$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \leq K,$$

where for example we can choose $K = 1$.

Similarly, the condition c) of Theorem A is satisfied too, since

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = \lim_{m\to\infty} \sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{m\to\infty} 1 = 1.$$

Finally, the condition b) of Theorem A is not satisfied, because

$$\lim_{m\to\infty} \frac{1}{2} = \frac{1}{2}, \quad \lim_{m\to\infty} \frac{1}{4} = \frac{1}{4}, \quad \dots, \quad \lim_{m\to\infty} \frac{1}{2^n} = \frac{1}{2^n}, \quad \dots.$$

Therefore $\lim_{m\to\infty} a_{mn} \neq 0$ for every $n = 1, 2, \dots$.

Choose the sequence $\alpha = (\alpha_n) = \left(\frac{2}{1}, \frac{4}{3}, \frac{8}{7} \dots, \frac{2^n}{2^n-1}, \dots\right)$. Clearly $\alpha_n \in \mathbb{R}$ for all $n = 1, 2, \dots$ and $\alpha$ converges to $1$. Then the image of sequence $\alpha$ with respect to the matrix $A$ is the sequence

$$\beta = A\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \frac{2}{1} \\ \frac{4}{3} \\ \vdots \\ \frac{2^n}{2^n-1} \\ \vdots \end{pmatrix}$$

$$= \left(1 + \frac{1}{3} + \frac{1}{7} + \dots + \frac{1}{2^n - 1} + \cdots, 1 + \frac{1}{3} + \frac{1}{7} + \dots + \frac{1}{2^n - 1} + \cdots, \dots\right).$$

We get a constant sequence $\beta$ for which all terms are the same $1+\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{2^n-1}+\cdots$. Denote by $s$ the sum of this series. Then the sequence $\beta$ converges to $s$ and it is well known that $1 < s < 2$. Therefore the sequences $\alpha$ and $\beta$ do not have the same limit. Hence we showed that without the condition b) of Theorem A the matrix $A$ is not regular.

**Example 3.** In this example, we investigate the last case i.e. the conditions b) and c) hold, but the condition a) of Theorem A does not. Denote by $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$. This alternating series converges relatively and its sum equals to $\ln 2$. Let

$$A = \begin{pmatrix} \frac{1}{s} & -\frac{1}{2s} & \frac{1}{3s} & -\frac{1}{4s} & \cdots & \frac{(-1)^{n+1}}{ns} & \cdots \\ 0 & \frac{1}{s} & -\frac{1}{2s} & \frac{1}{3s} & \cdots & \frac{(-1)^n}{(n-1)s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The condition b) of Theorem A is satisfied, since in each column, starting with a certain row the terms are zeros, i.e. for every $n = 1, 2, \ldots$

$$\lim_{m\to\infty} a_{mn} = 0.$$

The condition c) of Theorem A is satisfied too, because the sum of each row can be written as

$$\frac{1}{s}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots\right) = 1.$$

Therefore we have

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = \lim_{m\to\infty} 1 = 1$$

for each $m = 1, 2, \ldots$. The condition a) of Theorem A is not satisfied, because

$$\sum_{n=1}^{\infty} |a_{mn}| = \sum_{n=1}^{\infty} \left|\frac{1}{s} . \frac{(-1)^{n+1}}{n}\right| = \frac{1}{s} \sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{s} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

for every $m = 1, 2, \ldots$. There does not exist any constant $K$ such that $\sum_{n=1}^{\infty} |a_{mn}| \leq K$.

To show that the matrix $A$ transforms a convergent sequence $\alpha = (\alpha_n)$ into a sequence with different limit, we choose an another matrix $A'$. Put $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ and then

$$A' = \begin{pmatrix} \frac{1}{S} & -\frac{1}{\sqrt{2}S} & \frac{1}{\sqrt{3}S} & -\frac{1}{\sqrt{4}S} & \cdots & \frac{(-1)^{n+1}}{\sqrt{n}S} & \cdots \\ 0 & \frac{1}{S} & -\frac{1}{\sqrt{2}S} & \frac{1}{\sqrt{3}S} & \cdots & \frac{(-1)^n}{\sqrt{n-1}S} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{S} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously the matrix $A'$ satisfies conditions b) and c) of the Theorem A and the condition a) does not. Since

$$\sqrt{n} < n \Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

By using the comparison test, we get that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges too. Let $\alpha = (\alpha_n) = \left(1, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{4}}, \dots, \frac{(-1)^{n+1}}{\sqrt{n}}, \dots\right)$. Clearly $\alpha_n \in \mathbb{R}$ for all $n = 1, 2, \dots$ and $\alpha$ converges to $0$. Then the image of sequence $\alpha$ with respect to the matrix $A'$ is the sequence

$$\beta = A'\alpha = \begin{pmatrix} \frac{1}{S} & -\frac{1}{\sqrt{2}S} & \frac{1}{\sqrt{3}S} & -\frac{1}{\sqrt{4}S} & \cdots & \frac{(-1)^{n+1}}{\sqrt{n}S} & \cdots \\ 0 & \frac{1}{S} & -\frac{1}{\sqrt{2}S} & \frac{1}{\sqrt{3}S} & \cdots & \frac{(-1)^{n}}{\sqrt{n-1}S} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{S} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{2}} \\ \vdots \\ \frac{(-1)^{n+1}}{\sqrt{n}} \\ \vdots \end{pmatrix}.$$

Then

$$\beta_1 = \frac{1}{S} \sum_{k=1}^{\infty} \frac{1}{k}, \quad \beta_2 = -\frac{1}{S} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+1}}, \quad \dots, \quad \beta_n = \frac{(-1)^{n+1}}{S} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+n-1}}, \quad \dots.$$

From the Leibniz criterion the alternating series $S$ converges, therefore $\frac{1}{S}$ is a constant. The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\sqrt{k+n-1}}$ diverges for every $n = 1, 2, \dots$ i.e.

$$\beta_n = \begin{cases} +\infty, & n \text{ is odd,} \\ -\infty, & n \text{ is even.} \end{cases}$$

It means that the sequence $\beta$ does not exist. The condition a) guaranties the existence of transformed sequence. We showed that without the condition a) of Theorem A the matrix $A'$ is not regular.

Finally we will give an example of regular matrix. Denote by $\ell_2$ the set of all sequences of real numbers $x = (x_i)_{i=1}^{\infty}$, where $\sum_{i=1}^{\infty} x_i^2 < +\infty$. The norm is define as $\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$. It is well known that $(\ell_2, \|\cdot\|)$ is Banach space. We choose a sequence and transform it into a sequence via regular matrix. We show that the limit of the sequence and transformed sequence is the same.

**Example 4.** Let $\alpha^{(n)} = \left(\alpha_i^{(n)}\right)$ is a sequence of elements in the space $\ell_2$ defined as follows

$$\alpha^{(1)} = \left(\frac{1}{1^2}, 0, 0, \dots\right), \quad \alpha^{(2)} = \left(0, \frac{1}{2^2}, 0 \dots\right), \quad \dots, \quad \alpha^{(n)} = \left(0, \dots, 0, \frac{1}{n^2}, 0 \dots\right), \quad \dots.$$

Clearly $\alpha^{(n)}$ converges to the sequence $(0, 0, \dots)$ according to the norm $\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$. Let $A = (a_{mn})$ be a matrix such that

$$a_{mn} = \begin{cases} \frac{1}{m}, & n \leq m, \\ 0, & n > m. \end{cases}$$

This type of matrix is called *Cesàro matrix* and it is well known that Cesàro matrix satisfies all conditions of Theorem A. Hence, it is regular matrix.

Let we now calculate the sequence $\beta^{(m)} = \sum_{n=1}^{\infty} a_{mn}\alpha^{(n)} = A\alpha^{(n)}$. We get the sequence

$$\begin{aligned}
\beta^{(1)} &= \left(1\frac{1}{1^2}, 0, 0, \dots\right), \\
\beta^{(2)} &= \left(\frac{1}{2}\frac{1}{1^2}, \frac{1}{2}\frac{1}{2^2}, 0 \dots\right), \\
&\vdots \\
\beta^{(m)} &= \left(\frac{1}{m}\frac{1}{1^2}, \frac{1}{m}\frac{1}{2^2}, \dots, \frac{1}{m}\frac{1}{m^2}, 0, \dots\right), \\
&\vdots
\end{aligned}$$

The sequence $\beta^{(m)}$ converges to the sequence $(0, 0, \dots)$, because

$$\begin{aligned}
\left\|\beta^{(m)} - 0\right\| &= \sqrt{\left(\frac{1}{m}\frac{1}{1^2}\right)^2 + \left(\frac{1}{m}\frac{1}{2^2}\right)^2 + \dots + \left(\frac{1}{m}\frac{1}{m^2}\right)^2} \\
&= \sqrt{\frac{1}{m^2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \dots + \frac{1}{m^4}\right)} \\
&\leq \frac{1}{m}\sqrt{\frac{\pi^4}{90}},
\end{aligned}$$

which for $m = 1, 2, \dots$ converges to zero. We showed that the sequences $\alpha^{(n)}$ and $\beta^{(m)}$ have the same limit.

## CONCLUSION

In this article, we have shown via examples the independence of regularity conditions of the matrix transformation. Therefore the conditions of Theorem A are independent. By omitting any condition from Theorem A the matrix $A = (a_{mn})$ becomes non-regular.

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# TRAINING FOR PRIMARY SCHOOL TEACHERS IN USING SERVICE PLICKERS TEACHING MATHEMATICS

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**Abstract:** *The article is devoted to the study of the problem of the training of primary school teachers for the use of information technology (IT) in the process of teaching mathematics, in particular, in terms of the use of various online services for teachers. The peculiarities of development of cognitive processes of modern primary schoolchildren are described, and, based on this, the necessity of using IT in mathematics lessons has been substantiated. The results of the diagnostics of the state of teacher training for the implementation of IT are analyzed. The necessity of systematic training of teachers for the use of IT is revealed. Teachers' learning technology is developed to create interactive exercises in mathematics, including through the online services Plickers.*

**Keywords:** information technology, gadgets, primary school teacher, digital generation, cognitive processes, interactive exercises, Plickers.

## INTRODUCTION

Due to the development of digital technology gadgets have become central to the life of modern humanity, and have become a necessary means of effective functioning in the world. Information technology has greatly expanded human capabilities for instant access to the right information, and most importantly, IT provides it with access to learning, and affects the quality of perception, facilitating understanding of learning information.

## 1 BACKGROUND TO USE IT TEACHING MATHEMATICS OF PRIMARY SCHOOLCHILDREN

### 1.1 Features of the development of cognitive processes of primary schoolchildren - representatives of the digital generation

Modern primary schoolchildren are the representatives of the digital or, as they say, the online generation. From birth they are surrounded by the world of modern technologies, without which they don`t imagine their existence [1]. Little children are good at using all kinds of gadgets, finding the necessary cartoons, games, etc. on the Internet. Thus, the idea is gradually formed that all the necessary information can be found instantly (Google knows everything), so it makes no sense to specifically work on memorizing it. Obviously, the digital world is negatively influencing the progress of the cognitive processes of modern students, they have a worse memory than non-digital generation representatives, they remember, for the most part, not the essence of information, but the way of finding it.
Virtual world - computer games, cartoons - offer the child high levels of stimulation, action.
Of course, traditional teaching, in the real world, compared with the virtual reality, is given to the student boring and uninteresting. The dynamic picture that the virtual world doesn`t

require the child to focus on, therefore, today's students lose their attention and get tired quickly under traditional learning in the real-physical world. Consequently, traditional methods of teaching, firstly, don`t satisfy the students' need for action in the virtual environment, and secondly, they don`t take into account the peculiarities of the course of cognitive processes of contemporary schoolchildren.

Scientists note that even today's primary schoolchildren have a predominantly visual-figurative thinking that requires the illustrating of educational material. In addition, representatives of the digital generation are visual; they better perceive the new information in a well-known form, through the visual channels of perception. Also, digital children need a visual schematization of learning actions using a system of arrows and parentheses that facilitate the establishment of relationships and imply the performance of certain operations [2]. This means that the path of learning must be based on the act of real things (the action in material form for P. Halperin), to an act with substitutes for real things - drawings, schemes that, on the one hand, illustrate the way in which the action is performed, and on the other hand - configuring the pupil to perform all the operations of the guiding principle of action (performance of action in the materialized form for P. Halperin). Thus, the proposals of Rickers scientists regarding the expediency of submitting educational information in graphical, possibly electronic forms [3] are taken into account. In this context, it should be noted that the predominance of visual perception affects the deterioration of auditory memory. Free access to a huge amount of information and a quick transition from topic to topic, which is typical of hypertext, causes a reflexive, superficial appreciation of ideas (P. Briggs). Scientists even introduce a term like computer surfing, which manifests itself in the viewing of a huge number of websites, and the page's browsing takes no more than two seconds. Obviously, this approach to perception makes it impossible to comprehend a text or a task, the search for answers is carried out, in most cases, by a simple overview of the options.

Scientists note that computer surfing is like a protective reaction, since children are saturated with information. At the same time, they need more and more information they don`t try to understand or remember.

Images, visual associations and verbal minimalism are attributes of the present. The transition to symbols and emoticons is a natural result of communication with the help of modern gadgets, where you need to use a small amount of words or pictures to express your thoughts or describe a particular situation. Modern children have such a phenomenon as "Clip-Thinking", that is, the inability to verbalize thoughts and withdrawal from the culture of the written word into patterns, drawings and symbols, as well as the inability to focus on the subject for a long time [4]. The feature of clip (NET) thinking is the habit of using hypertext in which thoughts do not form sequential structures, but are associated associatively; the deterioration of analytical and synthetic thinking, the violation of the process of analyzing phenomena, the inability to comprehend information, to distinguish even the opposite statements; loss of ability to perceive bulky texts; a habit for short messages that don`t require focus, concentration of attention, tracking plot lines [5].

The child's activity in the virtual world is essentially communication, on its own, where it can change the settings, respectively, in their own capabilities, for example, in a computer game to set the appropriate level at which one can win. Quite often, modern schoolchildren identify schooling with computer games, that is, they need to perform a certain number of tasks in a qualitative and timely manner in order to get an assessment and move to a different level. As you know, all gamers want to become winners, and in school students are ready to complete their training assignments either on their own or in a team with classmates. Despite the desire to win victory in the affair, on the path to this victory, monotonous training work causes disciples in the students, their own mistakes give rise to anger. Therefore, when developing the teaching methods of students, this need needs to be taken into account and pupils must be successful in carrying out their learning work.

Despite the fact that modern children live in the days of Messenger and have many friends in social networks, they are not ready for live communication with representatives of any generation. The desire to always be a winner, the inability to build productive friendly relationships leads to the child's readiness to solve all misunderstandings by force (L. Kondratenko, L. Minilova) [5].

Today, the digital generation is already studying in primary school, and many teachers are paying attention to changes in the development of cognitive processes of modern schoolchildren. It is already obvious that traditional techniques are no longer effective for students of the digital generation, and the search for new techniques is mostly happening at random, without a detailed study and taking into account the features of the current generation of schoolchildren. Consequently, the teacher needs to use innovations in the process of teaching primary schoolchildren who meet the needs and take into account the opportunities of modern students, in particular, the desire to act in a virtual environment through IT.

The problem of using IT in the educational process of general educational institutions of Ukraine is considered by M. Haran, M. Zhaldak, N. Morze, O. Ovcharuk, I. Smirnova and others. Research by Tamim, Higgins, Li and Ma, Cheung & Slavin, Demir and Basol, and Chauhan, Slavin and Lake, Rakes have shown that the use of IT has a positive effect on the results of teaching mathematics of primary schoolchildren [6].

The problem of using IT in the teaching process, in particular mathematics in primary school, is briskly debated both in academic circles and in professional community of teachers. Scientists and teachers have divided into two camps: the first one - those who believe that IT has a positive impact on the development of higher mental functions (Small and Vorgan) [7] and use IT in the education of junior pupils, the second - those who for certain the reasons do not use IT and believe that modern gadgets badly affect the development of children, for example, Spitzer [8]. However, the results of most studies indicate the prevalence of the positive impact of digital technology on the modern student than the negative.

So, a leading scientist, Angela M Fish, along with colleagues, shows that children using home digital devices have a higher level of cognitive development than children who don`t have a home computer [9]. Similar results were obtained by another group of researchers at Jackson, Witt, Games, etc. that children using the Internet have higher academic performance than children who don`t use the Internet [10].

During the last decade, researchers from DeBell and Chapman have been researching the positive effects of digital technology on the development of the visual intelligence: the ability to simultaneously control a few visual stimuli, visualization of spatial relationships [11], pattern recognition, visual memory [12], metacognitive planning processes, search strategies and information evaluation [13].

As a result of the research, scientists G. Soldatova, A. Vishneva, S. Chigarkova concluded that moderate use of digital technologies in teaching can effectively influence the cognitive development of primary schoolchildren, provided that the time of study and recreation is properly planned [14].

Consequently, we can conclude that in order to meet the needs of students, in order to take into account the peculiarities of their cognitive processes, and therefore to organize efficient training of representatives of the digital generation, the teacher must use IT as it is already obvious that computers will become part of the experience of early childhood of many young generations in the future (Angela M Fish). Thus, research on the use of IT in the teaching of mathematics by primary school teachers is relevant, which correlates with the readiness of teachers to use all sorts of digital means.

## 1.2 Diagnostics of the preparation of primary school teachers for the introduction of IT at mathematics lessons

Experimental work was carried out on the basis of the Izmail State University of Humanities (Ukraine, Odessa region, Izmail). The pilot questionnaire was attended by 30 teachers of primary schools in the Odessa region - students of advanced training courses at the Izmail State University of Humanities.

The effectiveness of mastering the content of discipline "Mathematics teaching methods" is provided at the expense of the intensification of the educational process, the presentation of information in various forms and the use of it in the form of an interactive dialogue, which is provided thanks to IT [15].

To diagnose the basic knowledge and attitudes of primary school teachers to the introduction of IT at mathematics lessons in primary school, we have developed an appropriate questionnaire with different types of questions that related to general information about respondents and certain practical skills.

As a result of the analysis of teachers' answers about their own age, we have set the average age of a sample of primary school teachers - 42 years. In particular, it's obvious that all teachers are representatives of the non-digital generation.

One of the survey questions offered teachers a five point scale to assess their ability to possess IT. As a result of analysis of questionnaires, we found that all (100%) teachers have a computer and have experience in using IT in their professional activities. 24 teachers estimated that they had skills in the Internet by 5 points (80%), while the rest of teachers faced difficulties (20%); 27 teachers answered that they can work with e-mail by 4 points (90%) and 3 points - 3 (10%). Work with file system 3 respondents rated 5 points (10%), 9 teachers for 4 (30%), 9 teachers. 3 points (30%), 6 taught. 2 points (20%) and 3 teachers for 1 point (10%). Work in text editor Microsoft Word 3 teachers (10%) rated by 5 points, 12 teachers (40%) - 4 points, 9 teachers (30%) for 3 points and 6 teachers (20%) by 2. On the ability to create presentations we received the following distribution of students: 5 points - 3 teachers (10%), 4 points - 6 (20%), 3 points - 6 (20%), 2 points - 9 (30%) and 1 point - 6 (20%). On the development of the ability to process the graphic image, we received the following data: 5 points - 6 (20%), 4-12 (40%), 3-9 (30%), 2 points - 3 (10%); the ability to perform calculations using the spreadsheet: 5 points - 3 (10%), 4 - 6 (20%), 1 point - 21 taught. (70%); Ability to install the necessary software: 5 points - 3 taught. (10%), 4 to 6 (20%), 3 to 6 (20%), 2 to 15 (50%); blogging ability: 5 points - 3 teachers (10%), 4 -3 (10%), 3-6 (20%), 2 - 15 (50%) and 1- 3 (10%). Significantly worse self-appraisal results for teachers regarding the ability to work in professional online networks: 5 points - 3 teachers (10%), 4-3 (10%), 3 - 3 (10%), 2 -6 (20%) and 1 - 15 (50%); The ability to create Web sites for 4 points was valued by 3 teachers (10%) and by 1 point - 27 (90%). Study results are presented in Figure 1.

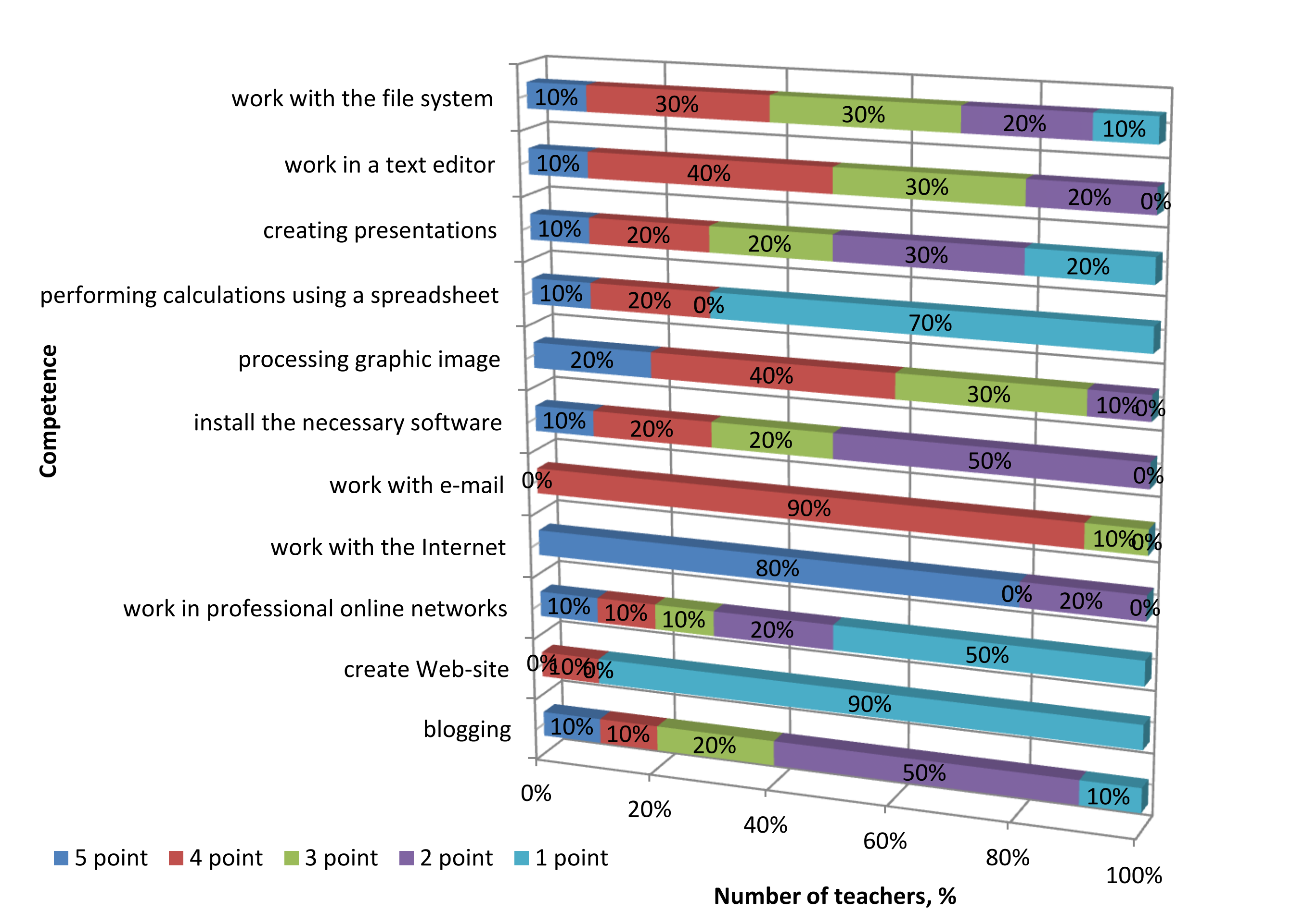


**Fig. 1.** Self-assessment by teachers of the level of IT possession
Source: own

Analyzing the answers to these questions, we can conclude that teachers have certain knowledge and skills of IT, but most are at a low level.

At the same time, we were interested in the place where these skills and abilities were acquired. 3 respondents indicated that IT knowledge and skills were gained during the study process at the university (10%). 15 teachers acquired IT skills through participation in seminars, methodological meetings, trainings (50%) and 12 teachers - independently (40%).

The next step was to answer the teachers' questions about the use of publicly accessible Web-resources with educational materials. Respondents listed the following sites: https://skvor.info/, http://www.yrok.net.ua/, http://interactive.ranok.com.ua, My Test, Google services and others. Yes, we determined what means the IT teachers use in teaching mathematics for primary schoolchildren. However, for the training of digital representatives, this list of resources is quite limited.

We were also interested in the issue of teachers' understanding of the need to use IT in the training of modern primary schoolchildren. Thus, 27 teachers (90%) think that it is expedient to use educational games, 21 (70%) - multimedia presentations, 24 (80%) - interactive exercises, 21 (70%) - educational videos and audio recordings, 24 (80%) - training programs, 18 (60%) - tests and quizzes. Consequently, most teachers agree that it is necessary to introduce IT in the education of modern students.

We were also interested in the understanding of the teachers about the expediency of using gadgets in the process of teaching mathematics. It is positive that 21 teachers (70%) think it

necessary to use gadgets that allow them to create interactive tasks using all kinds of on-line services. At the same time, the majority of respondents indicated that they lacked information on the peculiarities of teaching mathematics of primary pupils using modern gadgets.
The results of the study coincide with Cabero and Barroso's assertions that IT skills in young teachers are better than their older colleagues [16]. But regardless of age, any skills can be gained, most importantly; teachers have a desire and willingness to do so. Interesting Siddiq research with colleagues, it concluded that although older teachers feel that they are less competent in using IT, they are more aware of the benefits of using it in the learning process [17].
Despite the fact that the experience of teaching mathematics with primary schoolchildren with the help of IT in teachers is insignificant or even absent in some. All teachers (100%) have expressed a desire to participate in the teaching of creating interactive mathematical exercises with the help of the Internet service Plickers.

## 2 TRAINING PRIMARY SCHOOL TEACHERS TO CREATE MATHEMATICAL INTERACTIVE EXERCISES WITH PLICKERS

### 2.1 Theoretical and practical bases for training primary school teachers to use IT at mathematics lessons

The purpose of our study is to train teachers to use IT, including all kinds of on-line services. In this context we consider Web 2.0. It should be noted that the term Web 2.0 was introduced in 2004 by the American company O'Reilly Media to designate the second generation of communities and services based on the web platform. Web 2.0 provides users with the ability to create and distribute their own content on the World Wide Web [18]. We are interested in the features of the Web 2.0 service, which can be used in the teaching of mathematics of primary schoolchildren [19]. In particular, this platform offers a free Plickers program, developed by Emmy Nolan, president and founder of the company with the same name.
Plickers offers a real-time evaluation of the results of certain tasks, and therefore, teachers can immediately learn the answers of students using their own smartphones. The basic idea behind this new learning resource is that the company says: "We believe that deep learning can happen when we measure our progress and use data as a starting point and not just as a final measure" [20].
The Plickers mobile app is based on the IOS and Android operating system and is based on the traditional wireless review system, with traditional controls or clickers being replaced with paper-based print cards.
To create interactive exercises in the Plickers service, the teacher needs to take the following steps: go to https://plickers.com/ and register and be sure to link your account with email in case of password recovery. When you open the service, the user immediately finds the site menu - a list of hyperlinks in its sections: New set, Resent, Your library, Reports, Scoresheet, Classes.
The next step is to create a class. Teachers need to go to the Classes section and add their students. When entering pupils' names, Plickers automatically assigns them a corresponding card (clicker). This program is very convenient for classroom assessment with a large number of students, as the standard set of clickers - 40 and extended - 63. The teacher prints them and gives each student his card. It should be noted that a teacher can simultaneously register and work with several classes at a time.
Next, the teacher needs to create the content, that is, create one or a series of interactive exercises by selecting the New set section for this. The program offers the constructor an issue that the teacher has to fill (Fig. 2).

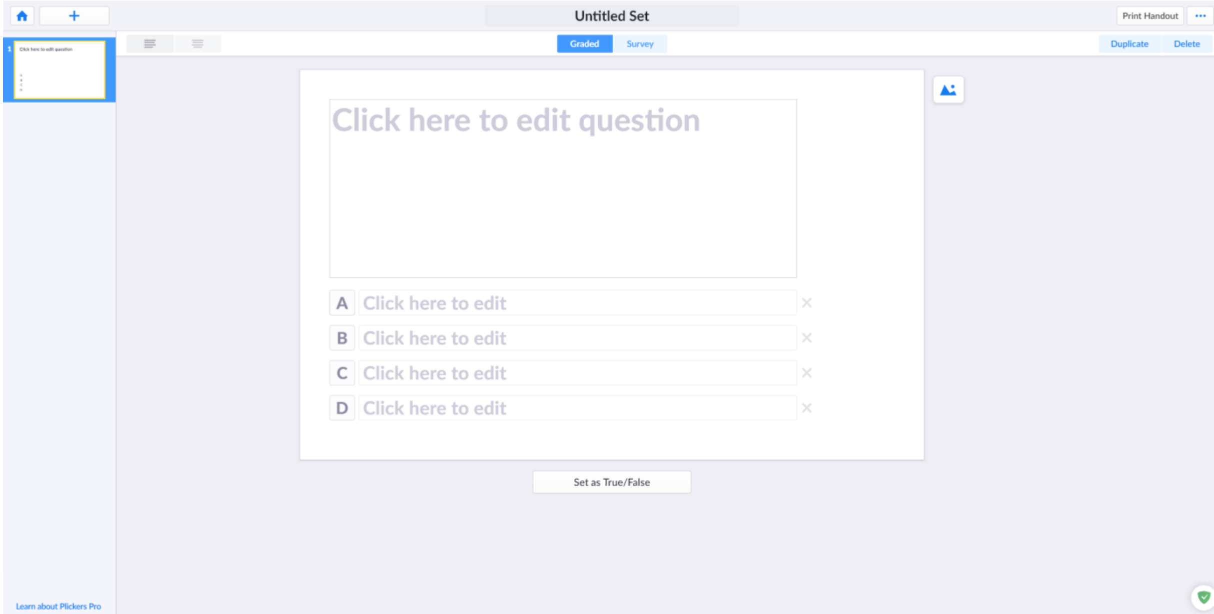


**Fig. 2.** Construction of questions of the service Plickers
Source: own

There are two variants of the questions: with four variants of answers (Fig. 3) and "true / false" (Fig. 4). The teacher chooses the appropriate designer and fills in the corresponding fields, always indicating the correct answer. To illustrate the tasks of Plickers, you can turn an image or GIF into an issue. The next step is to save the teacher's exercises.

The Plickers service allows the primary school teacher to create interactive math exercises. The tasks may include any educational material from the initial course of mathematics: the numbering of integer non-negative integers and ordinary regular fractions, arithmetic operations of addition, subtraction, multiplication and division with integer integers, magnitudes, plot mathematical problems, as well as algebraic and geometric propaedeutics. Here are examples of mathematical tasks created on this platform (Fig. 3, 4).

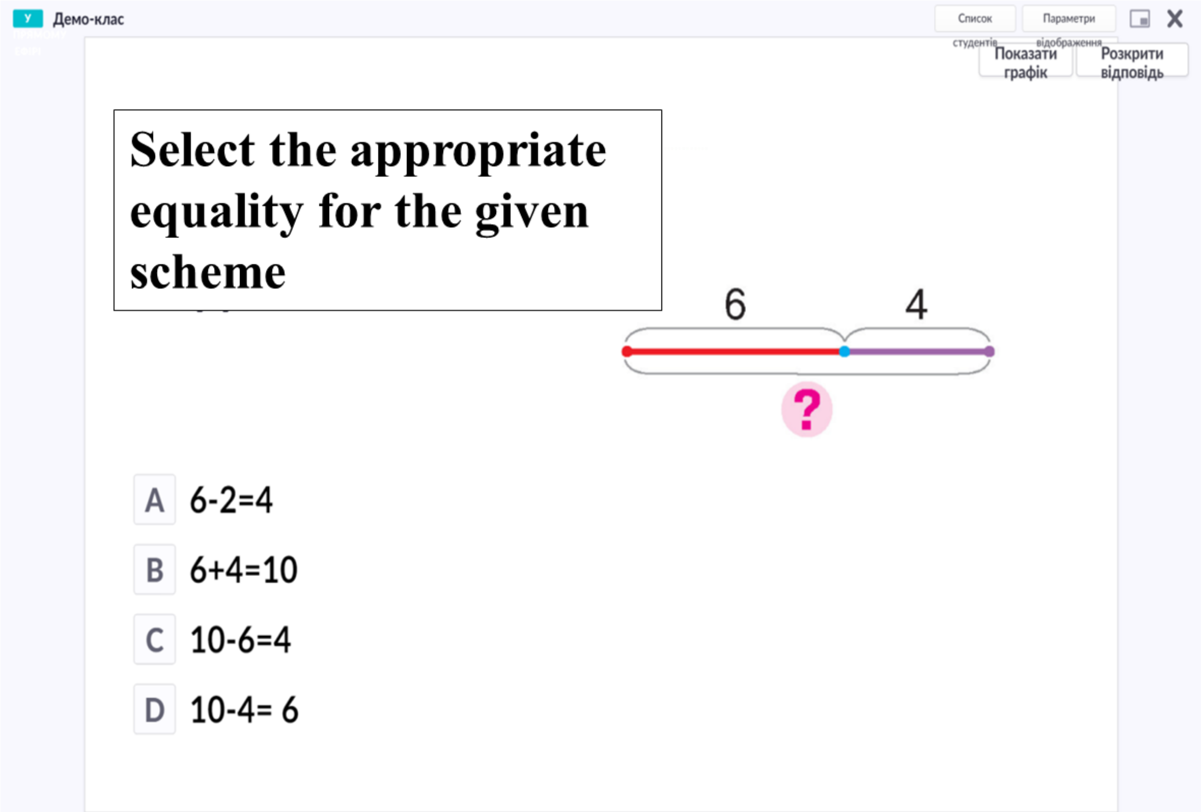


**Fig. 3.** The interactive exercise created using the Plickers
Source: own

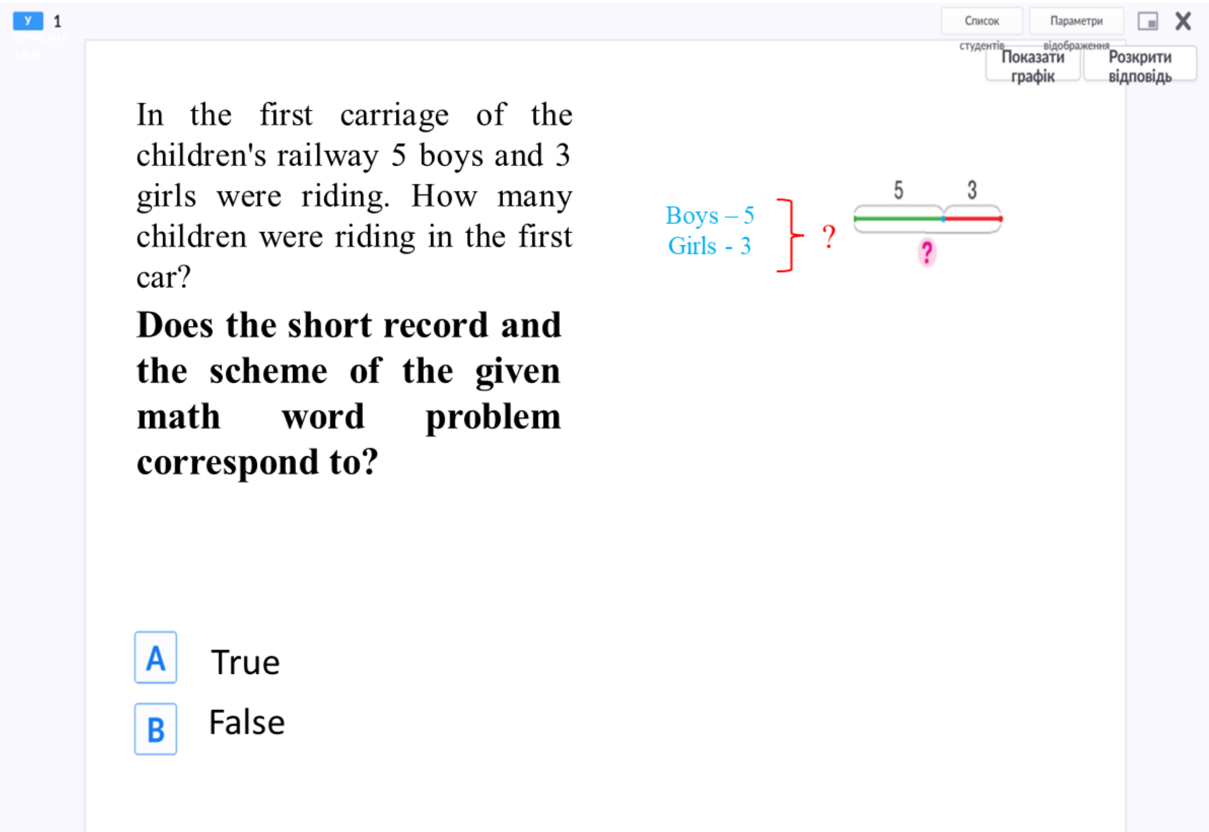


**Fig. 4.** The interactive exercise created using the Plickers
Source: own

In the section Your library you can find all the exercises created by the teacher. Also, he has the opportunity to modify his own exercises or create new series of exercises for the corresponding class. All exercises and series of exercises the teacher places in the appropriate folders, and then by the name of the desired folder, he can find it in the search bar.
If you select the Demo classes section, we will now go over to the contents of the class and report history. A teacher can click any of the classes in his list and go to the screen with detailed data on the success of the classroom children, a list of reports from the already played sets, and including sets or queues of future questions.
To start working with a class, the teacher needs to make sure that he added the content (one task or series) to the queue of the class. Next you need to select Now Playing on the web site. When using a mobile device, we choose a class and educational content for playback. After the teacher has selected a question or task that students should answer or resolve, the Plickers website is automatically updated and a corresponding question with a set of answers appears on the screen. Students have to solve the problem and select the answer they submit, which they consider to be correct.
In order to scan the students' answers, the teacher must log in to the mobile application and website using the same account. In the mobile app, he should touch the circle icon to open the Plickers scanner and instantly scan student cards; and for this the teacher needs to go back and keep the device in such a way as to capture more cards at a time.
To complete the scan, you need to click on the red icon on the mobile device and then all intermediate results will be available to him. The last step by choosing the appropriate function on the device, the teacher demonstrates the correct answer to the whole class. Then students are offered the following task or question.
It should be noted here that in our opinion, despite the advantages of this service, which allow the teacher to simultaneously control the performance of certain tasks by all class students, nevertheless, the presentation of the problem with the possible variants of answers is not a form that is directly interested in the students. In our view, younger students, while performing interactive exercises, the dynamic activity is more interesting - the movement, the cohesion of individual parts - the conditions and the correct answer, and the receipt of information about the correctness of the problem at once. These interactive exercises can be created on the Learning Apps platform, but this platform doesn`t provide such a service as collecting information for the teacher about doing the tasks of the students.
The advantage of the Plickers service is the ability for the teacher to work with the summary results in the Reports section. In addition, in the Scoresheet section, the teacher can monitor students' progress by choosing a range of dates and a training topic to view the student outcomes. Plickers uses color coding to indicate task execution level. For example, if a student completed a task by 100% - 85%, then the program will choose a dark green color for the fill; 84% - 70% - light green, 69% - 60% - orange, below 60% - red, and if there is no answer at all, then gray. The program allows you to export all the results in Excel, which will allow the teacher to further work with the data.
The disadvantages of using the Plickers service at math lessons in primary school are a limited number of designs, namely two, with which you can create an interactive exercise. That is, there is a high probability that a student may randomly choose any answer option and may be correct. In addition, the disadvantages of creating test exercises using this platform is the impossibility of filing tasks with one, but with a few correct answers, tasks for setting the sequence of steps. However, this service is useful to the teacher in terms of the possibility of receiving instant information about the progress of work on the tasks of students of the entire class, to make a final cut of the task of the class, to monitor the success. Also, this service can

be used by primary school teachers occasionally to diversify educational tasks in mathematics at the final stage of skills development.

### 2.2 Approaches to teaching primary school teachers to create math interactive exercises with the help of Plickers

Realizing the advantages and disadvantages of the Plickers service, our study aimed to prepare primary school teachers to use the Plickers service in the process of teaching mathematics students of grades 1-4. At the first stage of the study, we have demonstrated to the teachers the ready interactive exercises created on this service; teachers mastered working with them as users. In particular, when studying questions on the mathematics teaching methods at primary school, the control over the course of the process took place with the use of this service, and teachers could learn not only the algorithm of work on the platform Plickers, but also to evaluate the formulation of tasks, design responses. It was also useful to perform various roles through playing situations on this platform - "teacher-group of students". Thus, at the first stage, teachers acquired the experience of both the role of the executor of tasks, and in the situation of control over the work of the performers.

The second step was to familiarize teachers with the features, structure and capabilities of the Plickers service, and most importantly, the algorithm for working with it. In the course of the conducted experiment, lectures were held on topics: "Opportunities of Plickers" and "Creating Interactive Exercises for Teaching Mathematics of Primary Schoolchildren Using the Service of Plickers". It should be noted that in the second lecture attention was paid to the requirements for the formulation of answer variants in the test, in particular, all answers should be meaningful, the length of responses should be the same, etc.

The third stage of experimental work involved the practical training of teachers for the creation of interactive exercises and the implementation of an immediate control over the implementation of these exercises. Most teachers easily mastered the technique of creating interactive exercises with the help of the Plickers service, but many difficulties arose during the organization of "live broadcast", that is, playback-simulation of the situation of work on created exercises in the class with the implementation of the control function of the work of each class "pupil" and evaluation of created exercises.

At the fourth stage, teachers had to apply the acquired knowledge, skills and abilities of working with the service Plickers, in the execution of individual projects - their own mathematic interactive exercises for students of grades 1-4 in the textbooks S.O. Skvortsova and O.V. Onopriienko [21, 22] and conducting a live broadcast with the class. Most teachers (65%) have coped with this task and created their own collection of interactive math exercises for primary schoolchildren and acquired instant assessment skills with Plickers. 35% of teachers had difficulty in conducting a live broadcast.

Teachers' performance of individual projects was evaluated according to the following criteria: methodological, technical and aesthetic. Indicators of the methodological criterion were the correctness of the methodological design of the task, the completeness of observance of the guiding basis of action, the clarity of the instruction for the implementation of the task for children, the manufacturability of the actions with the elements of the task. The technical criterion was characterized by the indicators: the optimality of the choice of platform capabilities for the task implementation, the correctness of the settings, dynamism. The indicators of the aesthetic criterion were: external attractiveness (brightness, curiosity and contemporaneity of images, the location of the plane of the elements of the task), observance of the color gamut, which causes positive emotions in children.

The discovery of these indicators by methodological and technical criteria made it possible to characterize the levels of development of interactive tasks using the service Plickers:
- low: there are fragmentary knowledge and ability to create interactive exercises; the teacher can customize the work of the students on the platform over the exercises, but has difficulty in conducting a live broadcast, monitoring the process of solving educational tasks and assessing the results of student achievements.
- average: has some theoretical knowledge about the service and is able to create his own model practice, but has difficulties in setting up the service during live broadcasting and collecting current results in a series of training tasks.
- sufficient: the user knows the features of the service and is able to create standard exercises using images and possesses the knowledge and skills of setting up the service during the live broadcast, collecting current and final results and evaluating them.
- high level: the teacher's knowledge is deep, strong system; the teacher is able to create interactive exercises and modify them according to needs, to organize in practice a live broadcast of a series of educational tasks and further work with the current and final results of students' learning achievements.

According to the results of the assessment of the individual projects of primary school teachers, 65% of teachers have mastered the method of creating interactive exercises with the help of Plickers service at the sufficient level. 35% of teachers achieved a high level of mastery of the Plickers service. Such results can be explained by the fact that the service Plickers is quite difficult at work. But with the detailed study of this service and the experience gained with it, Plickers will become your helper in working with the class.

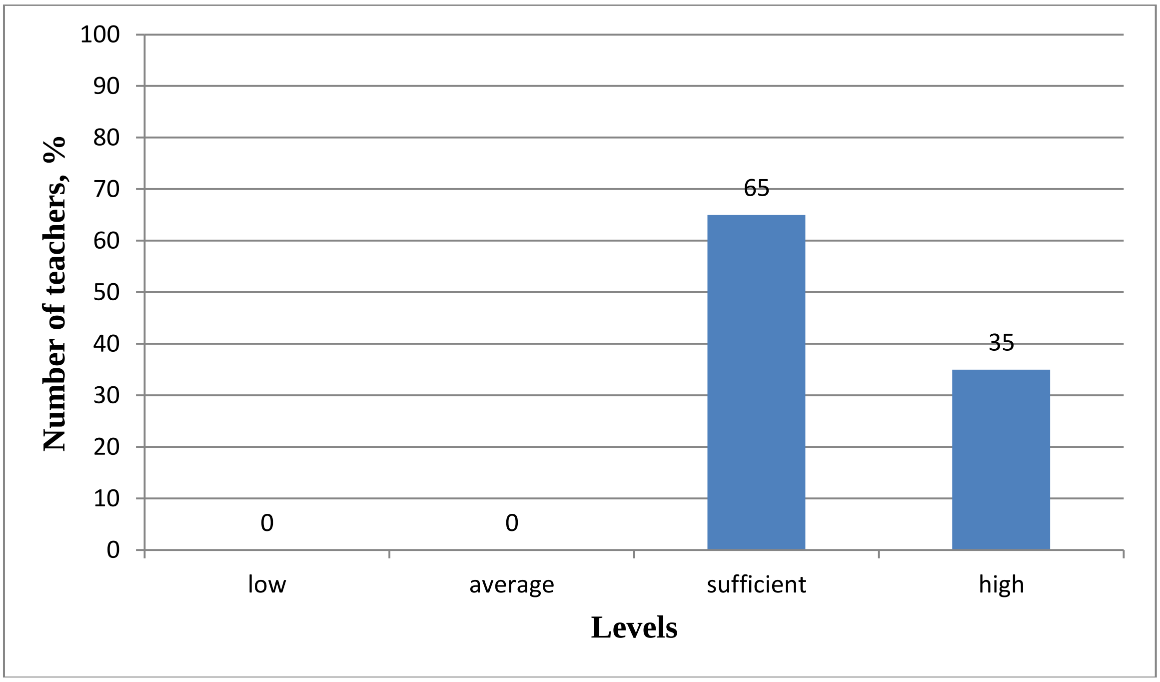


**Fig. 5.** Results of individual project assessment of primary school teachers using the Plickers
Source: own

## CONCLUSION

IT is an integral part of the training of modern schoolchildren - representatives of the digital generation. The study confirms that most primary school teachers are representatives of the non-digital generation, which as a result is not sufficiently understood by modern information technology education.

In the course of experimental work, the state of preparation for the use of IT by primary school teachers was investigated. It has been found that most teachers have basic knowledge, skills, IT skills and 100% readiness to improve them, in particular, to learn how to create interactive exercises with the help of Plickers.

We created a teacher training system to create interactive exercises with the Plickers service, featuring the following stages: demonstration of ready interactive exercises, familiarization with the capabilities of the Plickers service and algorithm for working with it, practical training for creating interactive exercises, individual teacher training and "live broadcast". Unlike L. Shevchenko, M. Sablina, O. Kutnyak and others who explore the problem of using IT in math studies in the context of using Web 2.0 services, in particular Learning Apps, H5P, Plickers, Flippitty, Kahoot, Triventy and others. We focus on the training of primary school teachers to use these services, by developing the technology of training, the criteria and indicators for evaluating its results. Researchers, in particular V. Krykun, who advised to use the Plikers resource in the learning process, do not specify the peculiarities of using this service in mathematics lessons, in particular in primary school, to which our research was directed.

As a result of experimental training, 65 % of teachers learned the ability to create interactive exercises and organize work in the classroom with the help of Plickers.

Prospects for further research we see in the creation of a bank of interactive exercises in mathematics; primary school teachers and students "013 Primary Education" with other online resources and online learning services for mathematics, namely Moodle, Google Forms, Webanketa, Online Test Pad, GoConqr, PlayBuzz, Baamboozle, Kahoot!, Triventy, Socrative, Quizalize, ProProfs, Purpose Games, H5P, Flippitty. In the study of the possibilities of composition of various services in order to diversify the use of IT at mathematics lessons in primary school, the formulation of requirements for the use of a particular service at a certain stage of the mathematics lesson, the definition of the types of students' activities, types of mathematical problems, their didactic purpose.

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# NEUROPSYCHOLOGICAL PRINCIPLES OF COMPUTATIONAL ACTIVITY OF YOUNGER STUDENTS

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**Abstract:** *The article is devoted to the research of computing activity of junior pupils taking into account their neuropsychological peculiarities. The place of computing activity in the system of higher mental functions is determined. It is shown that the necessary prerequisite for the development of higher mental functions is the aging of various brain structures. The following basic areas of the brain that are involved in the calculations are defined: occipital, parietal and frontal areas of the cerebral cortex. It has been established that various compositions of these zones are used to perform computations, which in turn influences the choice of methods of reasoning. It has been experimentally proved that in the process of computing activity, young learners apply different methods of computing to the same tasks.*

**Key words:** computing activity, arithmetical actions, junior pupils, neuropsychological features, higher mental functions, cerebral cortex, brain area.

## INTRODUCTION

In the early 2000s, the European education community updated its work on defining the list of basic skills and sub skills needed by young people to live in a modern society. That was said in 2001 at the meeting of the European Council in Stockholm in the report "On Concrete Future Objectives of Education and Training Systems, 2001". Further discussion of the problem of basic skills and sub skills needed by young people to live in a modern society was held at the meeting of the European Council in Barcelona in 2002. It should be noted that among basic skills, along with reading and writing literacy, there is also computing, i.e. the computing skill, as indicated in the Communication from the European Commission to the European Council and the European Parliament "Efficiency and equity in European education and training systems" (Efficiency and Equity in European Education and Training Systems, September 8, 2006) [1, p. 51].

At the same time, the countries of the European Union have actively participated in the study of the nature of key competences in order to effectively use them in the education systems of the Member States of the DeSeCo project. An important study in this area was the study 'Key Competencies. Developing Concept in General Compulsory Education', held in 2002 by the European Commission. In the report to this study, the key competences that a secondary school has to form are united into seven groups, with the ability to count down listed in the first group of key competencies [1, p. 138].

Consequently, the reports indicate that computing competence and computing skill are recognized by the world community as the key, since a modern person, despite the availability of all kinds of electronic means, has to repeatly perform oral calculations every day, analyze and evaluate the results of computing done with the help of gadgets, as well as to find a more rational way of reasoning. Computing competence manifests itself in the ability to perform qualitative computing, and its basis is the development of human computing skills and sub skills [2, p. 13 – 16].

## 1 COMPUTING ACTIVITY. COMPUTING SKILL AND SUB SKILL

Based on P.Y. Halperin's definition of activity as a process that is systematically or episodically restored by an "activist" and leads to a certain result [3, p. 28-41], computing refers to a process that results in the execution of arithmetical actions between numbers. In elementary school of Ukraine, students learn the arithmetic actions of addition, subtraction, multiplication and division, so the computing activity of a junior student is aimed at finding the result of arithmetical actions of addition, subtraction, multiplication and division (entirely or with the remainder) of nonnegative numbers within a million.

In the works of psychologists J. Piaget [4], P. Halperin [5], N. Menchinskaya [6], V. Davydova [7] and N. Nepomnyashchaya [8], the complex process of formation of the concept of number and computational operations is presented in ontogenesis. This process in the early stages of child development is of visual-active character and reduces to the enumeration of elements in the space field. Then, the computing activity acquires a linguistic character, and at the next stage of formation, this function is carried out in an ideal plan. Speech acts, on the one hand, are means of expressing a complex system of knowledge, and on the other hand, the organizer of the computing activity. Gradually, operations that take place "in thought" are curtailed and replaced by abstract arithmetic thinking. Consequently, the actions and operations that carry out the computing activity are mental actions, which are the result of the transformation of external material actions with objects or their substitutes into the plan of perception, beliefs and concepts.

In our study, computing skills are understood as the ability to successfully perform computing activity as a result of the learned way of action, based on knowledge of its theoretical foundations and the skills of performing actions and operations that make up the calculation. The computing act is a system of operations, the successive execution of which allows a person to get the correct result in performing arithmetic operations. Based on the content of the computing activity, namely, the number of transactions it consists of, as well as variations of conditions, and as a consequence, the choice of a particular system of operations, computing skills are classified into simple and complicated. Simple skills include the successful performance of computing activity, which consists of a small number of actions / operations that do not provide variation conditions – branching. Complicated skills are characterized by more actions / operations and, depending on the fulfillment of a certain condition, provide a certain system of operations, i.e. contain branching. Simple skills in performing arithmetic operations are subject to automation, and can be executed by a person instantly, supposedly by the formula. Complicated skills consist of operations that need to be automated, i.e, learned at the sub skill level, and based on the variation of the conditions – branches, it is obvious that in general such a system of operations cannot be fully automated.

Confirmation of our position on the existence of simple and complicated computing skills we find in the works of the 2002 Nobel Prize winner D. Kaneman. Describing the process of thinking, as that in which one conscious thought comes from another and is the result of brain work that produces an impression, premonition and helps to make a decision, the scientist examines the processes of fast and slow thinking. Fast thinking turns out to be a quick outcome with minimal mental effort, and slow – requires effort and order. According to the psychologist, to the fast thinking as a result of the formed sub skill belong oral calculations in the "Ten" concentric (simple skills according to our classification of computing skills). While written calculations, since they require effort and compliance with a certain order, the need to contain a lot of intermediate information in memory, D. Kaneman believes, are an example of slow thinking (complicated skills in our classification of computing sub skills) [9, p.24].

*The purpose of the article* is to present the results of the study of the computing activity of the pupils of the 2nd grade of elementary school at the implementation of addition and subtraction of numbers within 100, analysis of the results grounded on the data of neuroscience and, based on this, the formulation of the neuropsychological foundations of constructing a method for the formation of computing skills and sub skills in junior schoolchildren.

It has been several years since students – representatives of the digital generation – joined the elementary school. The development of their cognitive processes and metacognitive skills is influenced by the digital environment in which they are from an early age. Representatives of the digital generation have deteriorated memory, significantly reduced amount of attention, observed clips thinking, etc. As you know, the basis of cognitive processes is the human brain. Therefore, for the construction of effective teaching methods, in particular the formation of computing skills and sub skills, you need to know how the human brain functions when performing computing activities. Thus, one of the basics in designing and modeling the learning process should be the psychological and neurophysiological features of modern children, in our study – elementary school students.

## 2 NEUROPSYCHOLOGICAL BASES OF THE COMPUTING ACTIVITY OF THE YOUNG LEARNER

For successful computing, on the one hand, there should be readiness of brain structures, and on the other hand, demand from them. That is, in order to perform calculations, the child must develop certain areas of the brain involved in the computing activity, and on the other hand, computing activities develop these zones. The development of the brain of the child goes in two directions – the functional development and development of the neural apparatus (M. Bezrukykh). According to M. Bezrukykh, the most significant changes in the transformation of the neural apparatus occur in 5-6 years, and then in 9-10 years; from 5 to 6-7 years, there is a key reorganization of the morphological structure of the functional organization of the brain.

Neurosciences have systems of brain regions that can participate in the computational activity that is formed in cognitive activity. Cognitive activity of a person is a conscious activity directed on the knowledge of the surrounding reality by means of mental processes of perception, thinking, memory, attention and speech. Therefore, we are interested in the age-old peculiarities of the course of cognitive processes in junior pupils, the basis of which is the ripening of certain areas of the brain. Let us turn to the data of neuroscience and age psychology.

Cognitive activity of a person rests on the joint work of an entire system of areas of the cerebral cortex, located at the boundary of the occipital, temporal and posterior-central cortex. The activity of these zones is necessary for the successful synthesis of visual information, for the transition from the level of direct visual synthesis to the level of symbolic processes, for operating systems of numbers and abstract relations. The harmonious work of these zones is necessary for the transformation of visual perception into abstract thinking, mediated by internal schemes and for preserving the gained experience in memory [10, p.104-105]. The temporal, parietal and occipital areas of the cerebral cortex provide the perception and

processing of tactile, auditory and visual stimuli introduced into the memory system [11, p. 384-393].
Population studies conducted at the Institute of Age Physiology of the Russian Academy of Sciences (RAS) under the leadership of M. Bezrukykh convincingly prove that at the junior school age, the brain system responsible for receiving and processing external stimuli passes to another level of functioning (Age Physiology): 1) at 6 y.o. there are changes in the organization of the system of perception; namely, conditions are created for in-depth perception of objects, operation of a large number of features; 2) some delay in the explanations seen, is due not to the primary deficit of visual perception, but to the slow selection of words; 3) children up to 6-7 y.o. show difficulty in perceiving and interpreting storylines, especially serial pictures. Processes associated with remembering occur on the combination of the temporal, frontal and with the involvement of the occipital part of the brain [12, p. 85]. In the part of the brain called the hippocampus the integration of memories takes place, the understanding of the world (thinking, emotions) is the search system of memory [13, p. 89]. Studies conducted under the leadership of M. Bezrukykh convincingly prove that at the junior school age there is a shift in the memory system to a different level – from the immediate memorization inherent in preschoolers, to memorization mediated by specific semantic tasks. And this requires the development of new methods of memorizing on the basis of comprehension of the material, rather than its formal repetition [14, p.380].

Reception, processing (coding) and synthesis of information obtained from various analyzers and apparatuses are provided by the temple, lower-parietal and frontal area of the cerebral cortex. These areas develop programs for the most complex behaviour and control, including computing [10, p. 17]. According to the data of the Institute of Age Physiology of the RAS, at 3-6 y.o., the formation of stable cognitive attention takes place, the number of features that are allocated increases, as well as the amount of attention; at 6-8 y.o., there is the formation of mechanisms of arbitrary electoral attention, while emotional involuntary attention is gradually replaced by arbitrary cognitive; at 7-8 y.o., unsatisfactory attention prevails, but unsatisfactory and arbitrary attention still has the features of immaturity, and, finally, at 9-10 y.o. unsatisfactory attention is organized on the type of adult (M. Bezrukykh). The focus of the child not only on the incentives that are directly attractive to him/her but also on the more abstract, disconnected characteristics of the environment, its information component is observed. At the same time, up to 7-8 y.o., an arbitrary activity organized with the help of attention is easily pushed by activities that directly interest the child.

In the upper part of the brain, which is called the middle forefrontal bark, such complex mental processes as thinking, imagination and action planning [13, p. 51] occur. Self-control and executive functions are provided by the part of the brain behind the forehead [15, p. 117]. The prefrontal sections of the cerebral cortex are responsible for the ability of the person, making informed decisions, behaviour and skills [10,p.126; 13 p. 53]. It should be noted that the prefrontal areas of the cerebral cortex ripen in the later stages of ontogenesis (at 4-8 y.o.) [10, p. 126]; their development lasts up to 25 y.o. [13].

In the frontal areas of the cerebral cortex, there are complex syntheses of external and internal information that regulate, program, and control over the course of action, as well as decision-making in certain situations, especially when it is necessary to apply a certain method of action, in particular for calculations [10, p. 384; 16, p. 46; 12, p. 85].

The results of the research conducted under the leadership of M. Bezrukykh show that although at 6-7, 9-10 y.o. the mechanisms of selective attention and organization of activities are improved, it is only at the age of 9-10 that possible arbitrary, purposeful activity of the child takes place – the activity in which he/she can formulate the goal itself! Scientists observe problems with self-regulation in students of grades 1-3; at this age, the child badly regulates his/her own volitional efforts, because for this purpose the frontal areas of the cerebral cortex, which are formed up to 9- 10 y.o., must be formed. The activity of the frontal sections of the brain involves the development of internal speech, which is accompanied by the child's thinking (L. Vygotsky, A. Leontiev, O. Luria) [10, p. 124-125]. In the sections that are located in the area of the frontal lobes in front of the central gyrus at the border of the parietal, occipital and temporal parts, the information obtained is combined with emotions and memories, and there are complex analytical and synthetic mental functions that are responsible for thinking, decision-making and planning [12, p. 85]. The parietal-temple-hindhead zone becomes mature up to 8-12 y.o. and provides the development of speech, writing and computing [16, p. 46]. The junior student's type of thinking is visual-imagery thinking, which is related to his/her emotional sphere; the basis of imagery thinking is visual perception, and the means – an image. Scientists observe the main feature of children's thinking – to perceive everything specifically, literally, not the formation of the ability to rise above the situation and understand its general and/or abstract content (M. Bezrukykh).

The studies have established that the left cerebral hemisphere is responsible for logical thinking and for the perception of the tables of addition and multiplication and reference schemes [13, p. 20], and the parietal-occipital sections of the left hemisphere are involved in the perception of signs: numbers, signs of arithmetic actions, and auxiliary signs [10, p. 236]. In the right brain hemisphere, emotional perception, which is associated with emotions that arose in the process of performing calculations (joy, success, satisfaction from achieving the goal), turns into memories in which the left hemisphere makes sense [13, p. 26-31].

Scientists note the absence of a clear hemisphere specialization, which manifests itself in the nature of the brain providing for verbal activity. In adults, when solving the visually proposed verbal problem, functional associations of nerve centers involved in speech activity are localized in the left hemisphere. While in children of 7-8 y.o., the structures of both hemispheres are generally and uniformly involved in this process. Up to 9-10 y.o., with increased involvement of frontal areas speech processes become more selective. Language is the basis of thinking of the child. Features of speech activity at the junior school age determine the specifics of mental operations. For 7-8 y.o. imagery thinking is typical; it is based on the achievement of a certain degree of maturity of visual perception, and the means of such thinking is an image. With the development of mechanisms of speech activity, the child acquires the ability to allocate with the help of verbal-logical thinking the essential characteristics of objects and phenomena hidden from direct perception. Thus, at the younger school age, the functional abilities of the child significantly increase [14, p. 382].

Thus, today scientists know certain areas of the brain responsible for certain functions, including for calculations. But, at the same time, one must understand that in the normal conditions, when solving any problem, the entire brain works, and the combination of brain sections when solving the same problem can be quite different in different people (Swab Dick, A. Semenovich, T. Chernigovskaya). Consequently, it is impossible to state that mathematical activity, in particular computing, is provided by the functioning of the left hemisphere of the brain only. While performing any task or solving mathematical problems, the whole human brain works!

For complex intellectual functions, there are no “centers” that would produce them, but in the implementation of each of them, certain parts of the brain play a particularly significant role. Particularly important for intellectual activity are the particles of the third frontal lobe, the lower parietal and partly temple, since their damage gives the most serious violations of higher mental functions. Proceeding from the fact that in solving the computational problem the whole human brain works, we are still interested in the question of which zones, blocks of the brain can participate in computing activity. There are studies of the correspondence of certain areas of the brain to the performance of certain functions, including computations, but these zones, brain sections, are established in the presence of violations in their execution (O. Luria, L. Tsvetkova, A. Semenovich, S. Kotiagina, E. Grishina, T. Gogberashvili).

The persistent difficulty in mastering reading, writing, and counting T. Akhutina relates to the development of the operation of programming, regulation and control of arbitrary actions performed by the frontal parts of the brain (prefrontal cortex, frontal lobe); the processing of auditory and kinesthetic information relates to the activation of the temple and parietal sections, mainly the left hemisphere; operations of processing of visual and visual-spatial information are realized mainly by the back of the right hemisphere (parietal lobe) [17]. Consequently, computing is a complex multilevel process of the higher human nervous activity. Scientists at Stanford University have established that in the human brain there is a special center of computing, which is localized at the joint of the parietal and occipital lobes of the dominant hemisphere. However, this center of computing is closely connected with other areas of the cortex, without which it cannot fully function. Proceeding from the fact that various computational systems are used in the computing activity – optical, spatial, somato-spatial, locomotive, etc., then the brain’s basis of computation is the joint work of the occipital, parietal and frontal sections of the brain, which we will call ‘the system of brain sections that provide computing activity’ [18, p. 20].

The visual recognition of the digits is due to the work of the visual area of the cortex of the occipital particle, the auditory – due to the functioning of the auditory center of the cortex, which is localized in the upper temple lobe. Recognition of digits and the implementation of computations are carried out with the participation of memory and require a certain concentration of attention. The solving of the simplest mathematical problems occurs under the control of the frontal cortex sections responsible for the ability to perceive abstract concepts and to carry out targeted activities.

The work of the occipital lobe of the cerebral cortex is connected with visual perception; occipital-temple sections are responsible for the processing of visual information; intraparietal frustum – for comparative analysis and calculations; parietal sections – for the processing of spatial information. These two areas are associated with prefrontal cortex, which is responsible for leadership functions, attention, cognitive activity, and motor skills.

Recent studies prove that different people use different brain zones to perform the same higher mental functions, in particular, the implementation of computations [19]. Thus, studies by Swab Dick found that for oral calculations the Chinese use slightly different areas than those that are used by English-speaking people of the Western world. Both groups use Arabic numerals for calculations; however, English-speaking people in the processing of numbers are more likely to use language systems, while the Chinese are turning to visual-motor systems more quickly. This is due to the fact that the Chinese are learning graphic signs from birth [20, p. 56].

According to D. Medina – a molecular biologist-evolutionist who is engaged in the problems of brain development and the genetics of mental disorders, director of the Center for Brain Research at Pacific Seattle University, head of the Department of Bioengineering Medical School at the University of Washington – whatever functions are not attributed to individual parts of the brain, there are no two people with the same brain organization, who solve the problem equally and treat it in the same way at the sensory level. Scientists have discovered 14 separate sections that are responsible for various aspects of human intelligence scattered throughout the brain, like cognitive magic dust. These sections form the basis of the theory of P-FIT (Parietal-Frontal Integration Theory). When the P-FIT sections were observed during the depression of a person in meditation, different people used a variety of combinations of these sections to solve complex problems [15, p. 104].

In general, any higher mental function, including computing, is provided by the integrative activity of the entire brain. In the process of development of a child and as a result of his/her exercises, functional structure of the process changes, and the formation of activities at subsequent stages can rely on an already existing system of collaboratively working zones [10, p. 78]. Proceeding from the fact that calculations belong to higher mental functions, they are complex functional systems and cannot be localized in the narrow zones of cortex, but should cover complex systems of collaboratively working zones, each of which contributes to the implementation of complex mental processes and which can be located in different parts of the brain [10, p. 77]. D. Kaneman holds the same opinion and notes that there is no specific section in the brain in which all computing activity would be concentrated [9]. This thesis is based on the concept of the systemic structure of complex psychological processes suggested by O. Luria, in accordance with which each form of conscious activity is always a complex functional system and is based on the joint work of the three blocks of the brain: energy – regulation of tone and wakefulness; receiving, processing and storing information; programming, regulating and controlling complex forms of activity; each of these blocks contributes a lot to the implementation of the mental process as a whole [10, p. 126].

## 3 THE STUDY OF COMPUTING ACTIVITY OF ADDITION AND SUBTRACTION WITHIN A HUNDRED OF SECOND GRADE LEARNERS

With the aim of establishment of peculiarities of computing activity, we studied 60 learners of the 2nd grade of the Ivano-Frankivsk 3-level school 11 with intensive teaching of English (Ukraine). At the preparation stage, we examined class journals and interviewed teachers to find out how successful in mathematics their learners were. At the main stage of the experiment, we observed the computing activity of the learners, as well as interviewed them.

The schoolchildren were offered the task "To calculate the viral in the perfect way for you, commenting every step of the computing: 64 + 17, 38 – 26, 45 + 34, 55 – 38". The third-year students of the specialty 013 "Elemantary education" of Vasyl Stefanyk Precarpathian National University conducted the research. Two students worked with every learner: one interviewed the learner, and the other kept the minutes, taking note of the questions, the learner's answers and the time spent on solving the problem, as well as explanations.

The evaluation of the results was carried out against the following criteria: the correctness of the calculations; the time spent on the computing; the method chosen for the computing, the completeness and reasonableness of explanations, variability – possibility to use another way of computing. The results are demonstrated on the diagrams below.

Diagram 1 presents the distribution of learners according to the correctness of the calculations. The results obtained are the following: 21 learners (35%) solved the problem without any mistakes; 26 learners (43.3%) made 1 mistake; 13 learners (21.7%) made 2 or more errors (among them 3 schoolchildren (5%) made mistakes in all the problems) (in Fig. 1.). The mistakes were of the following nature: when adding with the transition through a dozen, decomposition into convenient addends was done wrongly; when subtracting through a dozen with the use of period subtraction from the result of calculation with the dozen of numbers, learners subtracted the result of calculation with the units of the numbers; occasional mistakes made when adding and subtracting with no transition through a dozen.

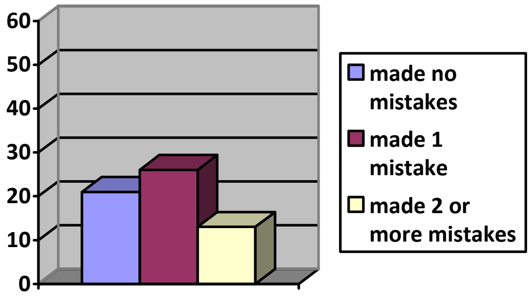


**Fig. 1. The distribution of learners according to the correctness of the calculations. Source: own**

The results of our study correlate with the results of the neuropsychologist L. Tsvetkova, who considers addition and subtraction through a dozen very difficult actions. The complexity of computing can be explained by the fact that such calculations can be done only indirectly, including a number of additional ways. In this case, computing becomes a mental activity, which includes in its structure several consecutive operations based on the knowledge of the digit structure of the number, the ability to appropriately replace the number with the sum, perform intermediate operations, and store interim results in the operating memory. Moreover, all these actions should take place on the basis of a stable overall program of activities. In operations of subtraction, an important factor is the preservation of spatial representations that allow the subject to set in the intermediate operations the desired direction of computing, which is expressed in addition or subtracting intermediate results. For example, when subtracting 18 from 46 in some cases it is necessary to subtract an intermediate result (subtraction by parts based on the rule of subtracting the sum from the number), and in others – add it (the way of rounding, which is based on the dependence of the difference from the change of the subtrahend) [18, p. 17].

Thus, when performing addition and subtraction within 100, a complex of sections of the brain are used that are responsible for perception (occipital and temple sections), processing of visual and auditory information (occipital, temple and parietal sections), memorization (occipital, temple and frontal lobe), analytical-synthetic thinking (occipital, temple, parietal sections and frontal lobe), decision making and planning (occipital, parietal sections and frontal lobe), self-control over execution and executive functions and internal speech (prefrontal cortex). Consequently, as a result of the analysis of the correctness of the calculations, we can indirectly judge if these areas of the brain function according to the age. Thus, in 21 students (35%), the occipital, temple and parietal sections of the brain, as well as frontal lobe and the forefront cortex, in the complex, function effectively for the performance of computing activity; 36 students (60%) have certain weaknesses, and 3 students (5%) may

experience certain violations caused by the disturbances in the functioning of a particular section of the brain.
According to the time spent on the computing, we got the following distribution of the learners: 3-5 min. – 19 learners (31.7 %), 6-8 min. – 32 learners (53.3 %), 8-10 min. – 9 learners (15 %); the results are demonstrated in the diagram below (in Fig. 2.).

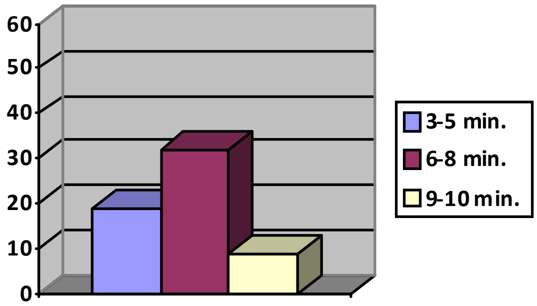


**Fig. 2. The distribution of learners according to the time spent on the computing**
**Source: own**

In the group of learners who have correctly completed all the tasks, the computing lasted from 4 to 7 minutes, which is explained by the fact that the students reasonably approached the tasks, even when all the operations that compose the calculation were mastered, they all resorted to expounded considerations, as demanded by the researcher; in addition, they were ready to use different computing techniques. From this, we can indirectly conclude that the front-cortex, which is responsible for self-control, executive functions, regulation and internal speech of the child, at this age – 7-8 years is actively developing. It was interesting to note that in a group of learners who had made 2 or more errors, there were cases of very fast completion of tasks. This indicates superficiality, unreasonableness and unconsciousness of the mental activity, which may be due to the developmental delay in the pre-frontal cortex and frontal lobes of the brain.

For cases of adding and subtracting numbers with the transition through a period (64 + 17, 55 - 38), learners could apply the following computing methods: addition / subtraction through a period; addition / subtraction by parts; addition / subtraction based on the rule of adding / subtracting the number to / from the sum; addition and subtraction by rounding. The distribution of students by choice of the most convenient method for computing is shown in Diagram 3.

Of the known computing methods (rounding; addition / subtraction through a period; addition / subtraction to / from the number of the sum; addition / subtraction of the sum to / from the number), the most commonly used methods were addition / subtraction through a period (17 learners – 28.3%) and rounding (12 pupils – 20%). The other 31 students (51.7%) used different methods (addition / subtraction through a period, rounding, subtraction from the number of the sum, subtraction from the sum of the number) (in Fig. 3.)

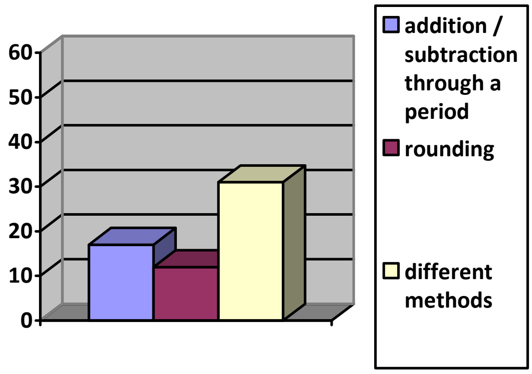


**Fig. 3. The distribution of learners by choice of computing methods**
**Source: own**

Most learners – 28,3% (17 students) chose the method of addition / subtraction through a period, since it consists of simpler actions by structure. This suggests that, although the frontal lobes of the brain are actively developing at this age, yet their potential is not sufficient for self-regulation, building a plan of action that should provide logical considerations, as in the case of using the method of rounding when the learner adds / subtracts not a number on condition, but a close to it round number, and then, understanding the dependence of the result of addition / subtraction on the change of one of the addends / subtrahends, on the contrary, subtracts / adds the difference between the round number and the number that was to be added / subtracted. The method of rounding for cases of addition and subtraction of two-digit numbers with the transition through a period was chosen by 12 students (20%). We are satisfied with this result, because it indicates the quality of the processes of analysis, synthesis, planning and self-control, which are particularly influenced by the development of the pre-frontal cortex and frontal lobes of the brain. To the question "Why did you choose exactly this way?" the learners answered: "So I'll do this calculation faster," "I like this way more", "When I calculate this way, I make fewer errors, or I'm not mistaken at all", "I use it in my everyday life", "In this case, it is better to use this very method". The learners' answers to the asked questions showed emotions (positive and negative), which depended on the success or failure of applying a particular computing method.

The next criterion was the completeness and reasonableness of learners' explanations. When evaluating the completeness of explanations, account was taken of the student's pronunciation of all operations that comprise the action of the computing, the literacy of mathematical speech, the logical sequence of the performed operations and their reasonableness. For example, calculating the value of the sum of numbers 64 and 17, their reasoning was as follows: 1) "I write down the first addend 64 in the form of the sum of period addends 60 and 4; 2) I write down the second addend 17 in the form of the sum of the period addends 10 and 7; 3) I add dozens to dozens: 60 + 10 = 70; 4) I add units to units: 4 + 7 = 11; 5) I add the results obtained: 70 + 11 = 81." At the same time, in the case of addition through a period, 15% of the learners (9 students) demonstrated the wrong solutions, with errors occurring at the stage of adding single-digit numbers with a transition through a period. Individual students recorded only intermediate results (record in a chain) and submitted the result. Most of the respondents illustrated the intermediate results schematically and with the help of visual images (showing the split into period and convenient addends with the help of arrows; the rounding method was illustrated by circling a certain component), which indicated the formation of imagery thinking but not the formation of sections of the brain that are responsible for the analytical-synthetic thinking.

By doing the computing, the learners explained the course of their reasoning. It should be noted that only a part – 15% (9 students) demonstrated literate mathematical speech, and the explanations of 56.7% (34 students) did not show clarity, although they followed the logical sequence of operations; 28% (17 students) did the action correctly but their reasoning was not expounded; they did not explain the course of reasoning. On request to explain the progress of the computing, individual students experienced difficulties in selecting the desired words and used schemes for explanation. This situation indicates that the temple-parietal-occipital sections of the brain have not yet been formed and the visual-imagery thinking was present, which is connected with the emotional sphere. The results of the study of the learners according to this criterion are presented in Diagram 4.

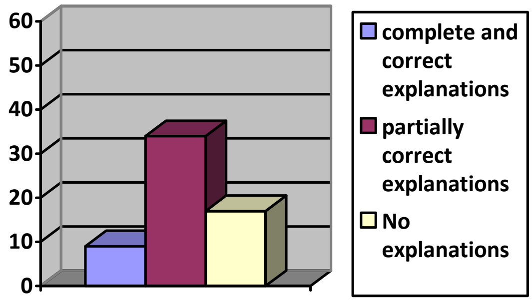


**Fig. 4. The distribution of learners according to the completeness of explanations of the computing**
**Source: own**

The next criterion was variability, that is, the student's ability to switch to another way of reasoning, using another method of computing. 31 learners (51.7%) showed their ability to perform calculations in a number of ways, justifying the feasibility of their use; 17 students (28.4%) showed awareness of different methods of computing but had some difficulties in choosing another method and in implementing it; 12 pupils (20%) did not find it possible to use another way of reasoning. For example, from 51.7% of the learners who chose the method of addition and subtraction through a period, 23.3% (14 students), at the request of the researcher to perform computing in other ways, suggested the solution with the help of adding by parts; 28.4% (17 students) – by the rounding method, which indirectly indicates the good functioning of the occipital, parietal and frontal lobe sections of the brain in the process of the computing activity. (in Fig. 5.)

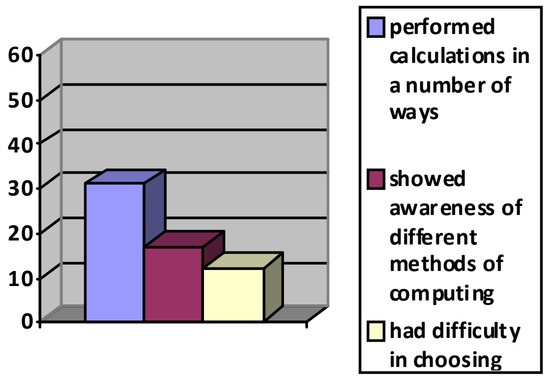


**Fig. 5. The distribution of students according to the ability to switch to another method of computing**
**Source: own**

Consequently, a larger number of pupils in order to perform calculations use, depending on the practicability, different methods of computing and different interpretations of their explanations (verbal and visual), and the set of convenient methods of computing in children is different. Such results showed the presence of individual neurophysiologic differences of the interviewed children. Concern was caused by 5% of learners who made mistakes in all the tasks but could neither explain the implementation of actions nor justify the choice of steps in calculations. In our opinion, this indicates a disadvantage in the functioning of certain areas of the brain (the interior of the parietal section of left hemisphere, which is responsible for quantitative evaluation) and may be the result of operating dyscalculia.

## CONCLUSION

For learning, including the development of computing skills, the child's brain readiness is needed, i.e. it requires that certain areas of the brain that are involved in cognitive activities be mature. However, there is also a reciprocal link – learning contributes to the development of the brain, namely the development of those areas that are involved in solving a math problem. Thus, computing activity, on the one hand, requires a certain level of development of the complex of brain sections of the student – occipital, parietal and frontal lobes, and on the other hand – activates these areas of the brain.

At the same time, one must realize that the frontal areas of the brain are formed up to 10-11 y.o., therefore younger schoolchildren may have problems with planning of activity, control over its course, self-regulation, which negatively affects the quality of the computing activity. In addition, we consider important the thesis of neurophysiologists that the dispersion of individual differences among elementary school students may be up to two years. This means that one learner can demonstrate, for example, analytical-synthetic skills already in elementary school, and another, even in middle school, will dominantly use imagery thinking. However, normative documents of primary education require the achievement of certain results, among which – the formation of computing skills. Proceeding from this, experts in the didactics of mathematics are tasked with the development of such methods of teaching computing activity, which will allow achieving normative results. Therefore, in the process of forming computing skills, it is necessary to take into account the advice of neuroscientists specializing in the correction of the problems of the functioning of certain brain sections, in particular acalculus – the disturbance of the computing activity caused by the defeat of certain areas of the brain involved in calculations.

So, we find relevant the advice of L. Tsvetkova, a student of A. Luria, on the most optimal teaching methods that lead to a sustainable restorative effect of the method, which allows a learner to reproduce in an expanded form the internal structure of the computing activity by taking outside individual operations, the successive execution of which can lead to the performance of the computing activity.

Written in the desired sequence operations will make a program that manages the outside of the course of the restoration of the impaired action and allows you to control this course [18, p. 8]. Dividing an action into a series of successive operations enables the learner, even in the presence of violations in certain areas of the brain, to perform computing activities already at the very beginning of training. To the syllabi created in a restorative education, it is necessary to put forward a number of requirements: selectivity in the content of the program, the sequence in the implementation of operations, multiple repetitions of the program in the student's learning process, the support of external supplementary aids. All this creates conditions for a high degree of activity and independence of the student in overcoming defects [18, p. 9]. This idea is fully in line with the requirements for the formation of mental actions, which ensure the efficiency of the formation of skills and abilities suggested by L. Friedman [21], and with the training according to the theory of phased formation of mental actions offered by P. Halperin [22]. According to which, in the first stages of teaching, the student is provided with a complete and correct orientation of the action – its plan, the algorithm. Then, the action is performed in a material form with objects, or in a materialized form – with their substitutes, expanded with the fixation of all its components. When the students have mastered the composition of the method of computing with the help of external regulators – memos and/or reference notes, and are able to explain the performed operations aloud, the action is performed in the form of external speech as fully expanded. At the next stage, the learner begins to skip auxiliary operations, calling out only the main ones – the action goes into the form of a loud speech to oneself. And, finally, the action is minimized and automated; it is performed as if by the formula in the mental plane. If to follow these steps when computing, we can expect that the teacher will achieve the goal, even in the case of minor violations of the functioning of learners' certain brain sections. Thus, one can expect that the percentage of students who make many errors in calculations will significantly decrease.

The results of the study of the computing activity of learners of the 2nd grade, which testify to the use of different methods of calculation by students, confirm the practicability and necessity of forming different methods of computing in elementary school learners. Even the method of rounding, which requires a certain level of development of analytical and synthetic activity, and therefore the maturity of the occipital, parietal and frontal lobe areas of the brain that are not well developed in all children under the age of 7-8, it is advisable to introduce and teach students computation by rounding one of the addends or subtrahends. This position is consistent with the thesis of specialists in neuroscience that the brain develops only when a person solves a new and complicated task for himself.

The prospects for further research we see in the development of a methodology for the formation of computing skills in elementary school students based on the consideration of neurophysiological features of modern children in order to correct defects in the course of cognitive processes and stimulate the development of certain sections of the brain.

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# GAMES IN TECHNICAL EDUCATION

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**Abstract:** *Games have a significant impact on the educational process. A game design elements implementation brings many benefits for an effective and enjoyable learning process. The presented paper describes an effort in the popularisation of sciences and technologies, using the web and virtual educational games, designed, developed and implemented as a result of many years of authors experience in this field. Focus is put on digital games as part of the training, individual and team projects of future engineers. These educational tools are also discussed in the context of popularisation materials of primary and secondary school students, to increase their interest in science-oriented education.*

**Keywords:** Educational games, web technology, Virtual Reality, technical education, popularisation of science and technology.

## INTRODUCTION

From the moment we are born, we instantly begin to observe, learn and develop our senses and thinking. Learning to play, or learning by playing is a natural process of our youth, in which educational games come handy [1]. After all, who would not want to play, and naturally gain new knowledge without any major effort?

Nowadays, access to information has become increasingly faster, supporting the development of science and technology, and particularly the field of information and communication technologies (ICT). These benefits were brought to us by the Internet since 1990 [2], along with rapidly evolving hardware such as laptops, desktops and mobile devices. These gadgets gradually enter all areas of our lives, including the entertainment industry and education. In recent years the hardware power made possible to commercially introduce 3D, augmented and virtual reality (VR) technologies, and there has been a growing interest since.

Compared to traditional 2D environments, the 3D environment adds a spatial dimension, in which the user is visually represented as an avatar, and can move and interact directly with that environment [3]. Many VR systems have the potential to enable the user to experience and discover phenomena, which would not be possible in the real world. VR makes possible to achieve a unique connection between the user and a computer-generated environment. In connection with education, this technology represents a more effective way to acquire skills [4] and to study complex topics [5]. Nevertheless, the availability of information in our digital age, and the fact of how we generate new knowledge in an unimaginable rate, many studies show that students knowledge base in many fields is declining. Our industry needs more and more educated, creative and adaptive professionals who can work with progressive technologies, but the interest of the young generation to study science and technology subjects is still decreasing [6]. Statistics in the electronics sector is particularly bad. The number of applicants of our university has been declining for several years, at the same time up to 40% of students fail to accomplish subjects such as mathematics [7] and physics [8], or they are not motivated enough because the curriculum does not meet their expectations.

To solve this tender, there has to be changed in our attitude towards new applicants; hence we need to realise who our students are. Today's youth, including all age categories, could be characterised as the "new generation" or the so-called "Digital-natives", "Millennials" or the i-Generation [9].

Beginning of a new wave of information and communication technologies dates way back to 1982; hence, all people growing up since this year could be affected by a new way of thinking [10]. They have emerged in preferring interactivity over passivity, internet over television, constant online existence and continuous contact to dozens of friends. Digital natives expect the same technology in their educational process as they experience at home - computers, internet, applications, social networks, web 2.0, mobile phones, tablets, video games, access to information ranging from leading professional lectures by professors from reputable universities to complete ballast. Media that could be harnessed to meet these expectations and to attract young generations interest are web and virtual reality educational games. Digital solutions, that involve the newest trends in hardware, appeal to a very wide group in terms of age and experience. They are mainly designed to help users to understand concepts, bring the subject closer, and develop problem-solving skills while playing games. In fact, these educational tools have enormous potential to support learning through a number of well-established cognitive, emotional and motivational mechanisms [11]. A clear educational goal, coupled with game-play, increases interest and motivates students to a self-studying attitude. The attractiveness of learning by playing [12], the gradual increase in the difficulty, hints for completing tasks, and positive feedback are among the key criteria in creating games [13].

Therefore, authors of this paper relate educational digital games as the most effective tools that have the potential to make the learning process more interesting, entertaining, interactive, as well as encouraging learners in active participation [14] and engagement. These solutions allow immediate direct feedback and, are capable to increase the motivation of current and future students on science and technology subjects [15]. Moreover, gathered experience in implementation of digital games in the educational process show that children, pupils and students have a positive reaction, are more confident in understanding and solving tasks on their own, they react to games as a more attractive form of learning, as well as are promoting joint learning through competition and cooperation [16].

This paper aims to present our many years of experience in the design, development and implementation of web and virtual educational games in the educational process, and the popularisation of science and technology by this media.

## 1 DESIGN AND DEVELOPMENT OF EDUCATIONAL GAMES

Our continuous aim is to raise awareness of children and youth about principal phenomena in electrics and electronics by educational games developed using the web and VR based technologies. Although, there were many famous and high-quality educational games developed and the numbers are continually growing, only rarely it is possible to find games in the field of electronics that are not focused on the virtualisation of circuitry wiring and device stacking but more on the fundamental physics and inner phenomena that needs to be understood. Primarily this is the field where we see the added value since virtualisation of an actual circuitry will never entirely replace real-life experiences gained in "handmade" exercises, more the visualisation of not visible phenomena can bring a deeper understanding of electronics. Practical skills gained in experimental laboratory experiments are irreplaceable in the shaping of future engineers.

### 1.1 Target groups

The development and practical utilisation of educational games of this contribution involved two different target groups.

The first group included students of their 3rd year of the bachelor study program, Applied Informatics as a subject of bachelor projects and 1st year of engineering study programs Applied Informatics and Electronics and Photonics as a subject of team projects. In such way not only programming skills needed for development were involved but also knowledge about electronics and photonics and the cooperation between theoretical and software development-oriented team members. Thereby, the direct outcome of these projects is an educational game, that was developed by a group of thinkers who work and cooperate on different levels. Programmers prepare their ideas and game visualisations, while electronics students discuss a specific concept or topic and its associated levels, scenario, or game storyboard. Joint meetings are welcomed as these enrich both study fields through the different tasks they address together. By examining this constellation, we can see that students are very open; there is no criticism of their mutual thoughts. The atmosphere is productive, creative and most of the time relaxed. The feedback from students is very positive. In such the atmosphere students are aware that every idea is allowed and welcomed, so they contribute to a joint problem-solving.

The second group involved student of elementary and secondary schools, as a target group of testers and players, in the context of popularisation of science and technology and as feedback from the end-user and its experience. The target group regularly attended seminars called "Electronics around us". They were actively involved not only in the game development process but as players and as the role of testers. Their comments and suggestions helped us to fine-tune several game functionalities, so these meet the essential criteria of quality educational games. Feedback was very positive (Fig. 1).

**Fig. 1.** VR educational game: Planet of solar panels. Students are testing the game.
Source: own

## 1.2 Requirements of a high-quality educational game

Although many educational games are freely available and adhere to different templates, there is no standardised concept for a high-quality educational game. From a psychological point of view, it is appropriate to make a game as simple as possible, so that the player can immediately understand what to do and how to react [17]. It is not recommended to use too many text fields, many input options, and long explanatory texts since it can lead to reduced interest. The problem is often associated with the ambiguity of task assignments, and adjustment of game levels. The game leads the user to complete all tasks and educational goals. It is essential to be successful for the user. The concept of an educational game in the field of electrical engineering and electronics should have clear and specific steps that the player must take to achieve an educational goal. The main idea is to acquaint the player with basic facts and principles they encounter in everyday life. Interactivity is always a key factor to attract the player's attention. This is achieved by various visual effects that constantly maintain the user's attention and help by handling the game and various functional elements. These are the appropriate range of colours, highlighting visible objects to be manipulated with, switches, movements, linkable elements, draggable elements, animated and dynamically changing manipulation and control elements.

Main theme and how the gameplay levels are designed is also a conceptual need. Mostly higher levels require more effort and are usually harder to accomplish. This intensity change and the goal to accomplish all the tasks necessary to advance in the game is mostly a standard concept; however, in case of VR, it is recommended to last around 15 minutes. The time frame ensures a sufficient concentration even of an unexperienced user and reduces the negative effects of a VR display [18]. The VR technology involves the plater in high-level action, it occupies all our senses and can help to more effectively memorize the subject compared to classical study textbooks. This higher interaction is the most crucial factor appreciated by the new generation students.

The goal of each educational game should be to provide an interactive source of information to the target group in a playful way so that players gain as much knowledge as possible about an actual topic. It should get players highly involved in the game, to be ready to interact as part of a story. Only this way is it possible to increase their interest and change their mindset about difficult technological topics.

## 1.3 Developed educational games

Since the beginning of our activities, around 10 educational games have been created in the field of electronics and electrical engineering. After developing, implementing and testing final solutions, games were listed among freely available educational materials on the eLearn central open (http://uef.fei.stuba.sk/moodleopen/).

Some examples of these are illustrated in Fig. 1 to 5. The scene of VR educational game "Planet of LED" are shown in Fig. 1. The input insight and one of the tasks of educational web game for two players: Secrets of electronics and photonics are in Fig. 2. The second scene and a task of educational Web game: Secrets of electronics are in Fig. 3. The menu and Silicon structure of VR educational game: Planet of unipolar transistors are displayed in Fig. 4. The task and the last scene of VR educational game: Planet of LED are presented in Fig. 5.

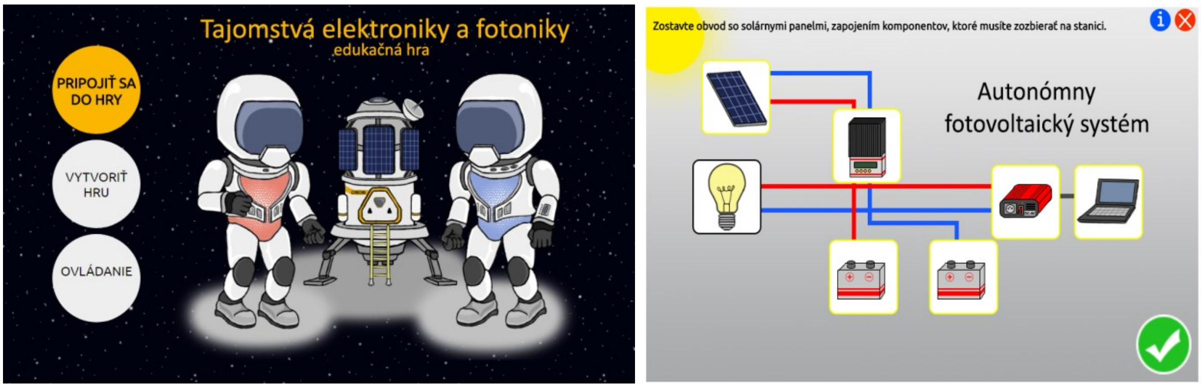


**Fig. 2.** Web educational game: Secrets of electronics and photonics.
Source: own

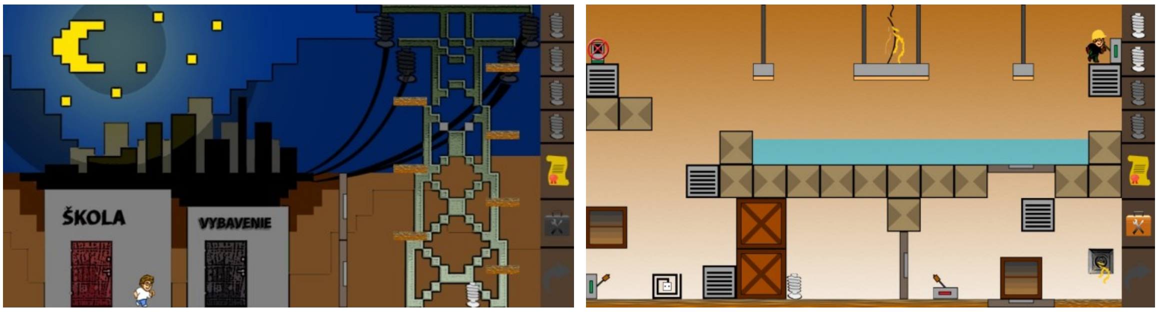


**Fig. 3.** Web electronical game: Secrets of electronics.
Source: own

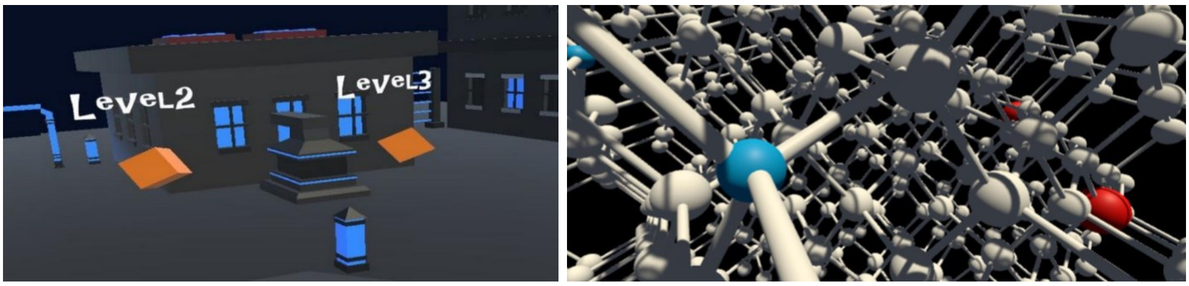


**Fig. 4.** VR educational game: Planet of unipolar transistors
Source: own

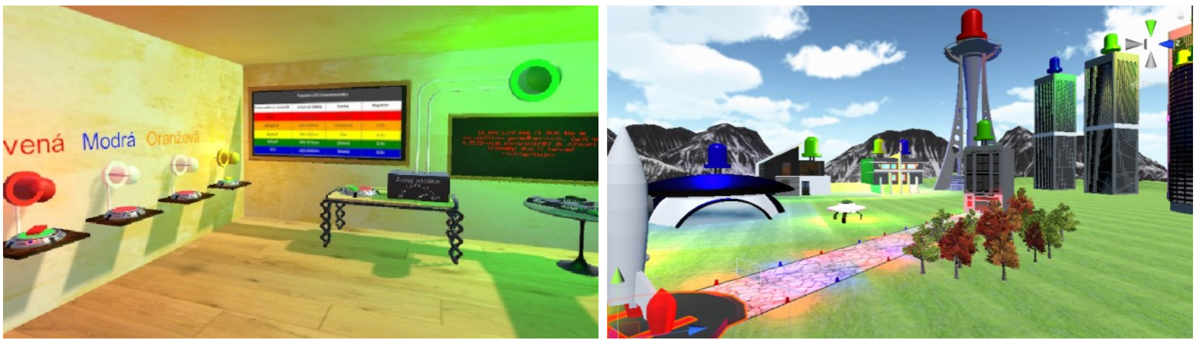


**Fig. 5.** VR educational game: Planet of LED.
Source: own

## CONCLUSION

As the teacher of nations J. A. Comenius wrote in his book Schola Ludus – "School by games encourages a playful approach to the knowledge transfer", we also need to be open-minded to use games in our educational process. Today's technology and the available means of visualisation can give us extraordinary opportunities to use games as a tool of knowledge transfer. In this paper, we have introduced original educational games developed by our team and students, which became part of the training of future engineers in the fields of Applied Informatics and Electronics and Photonics. At the same time, games were designed to increase the interest of young generations in Science and Technology. The presented games are or will be freely available on the educational portal eLearn central open (http://uef.fei.stuba.sk/moodleopen/).

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# TRANSFORMATION OF ELEMENTS OF THE BANACH SPACE BY SCHUR MATRIX

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**Abstract:** *The aim of this article is to show a necessary and sufficient condition that an infinite matrix with real number items transforms all bounded sequence of elements of Banach space to the convergent sequence with respect to the given norm.*

**Keywords:** matrix transformation, sequence, Banach spaces, convergence.

## INTRODUCTION

The matrix transformation of sequences of a metric space is a one way of generalization of convergence. In [6] assertions about the limitation of sequences of Banach space are mentioned. There are shown some properties for infinite matrix with real number items for convergence of a transformed sequence for each convergent sequence of Banach space. In this article we investigate convergence (with respect to norm) of the transformation of bounded sequences of Banach space. This type of matrices is called Schur matrices. Finally we show that the transformation matrix cannot be regular and a Schur matrix simultaneously (see [1], [4]).

## 1 PRELIMINARIES

Let $(X, \|\cdot\|)$ be a Banach space.The elements of Banach space we denote by $\alpha, \beta, \ldots$, the zero element by $\Theta$ and the unit element by $\epsilon$, $\|\epsilon\| = 1$. The sequence $\alpha = (\alpha_k)$, $\alpha_k \in X$, $k = 1, 2, \ldots$ converges to $\beta \in X$ if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ implies $\|\alpha_k - \beta\| < \varepsilon$. Denote by

$$B_\infty = \{\alpha = (\alpha_k), \alpha_k \in X, k = 1, 2, \ldots, \exists K_\alpha > 0, \forall k = 1, 2, \ldots, \|\alpha_k\| \leq K_\alpha\}$$

the set of all bounded sequences of $X$.

The sequence $\alpha = (\alpha_k)$, $\alpha_k \in X$, $k = 1, 2, \ldots$ is called a Cauchy-sequence if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $i, j > k_0$ the inequality $\|\alpha_i - \alpha_j\| < \varepsilon$ holds.

It is well known, if $(X, \|\cdot\|)$ is a Banach space that every Cauchy sequence is convergent in $X$ (see [2], [3] and [5]).

Let $A = (a_{nk})$ be an infinite matrix of real numbers. A sequence $\alpha = (\alpha_k)$, $\alpha_k \in X$, $k = 1, 2, \ldots$ is said to be $A$-limitable to the element $\gamma \in X$, if and only if $\lim_{n\to\infty} \beta_n = \gamma$, where

$\beta_n = \sum_{k=1}^{\infty} a_{nk}\alpha_k$.

If $\alpha = (\alpha_k)$ is $A$-limitable to the element $\gamma$, we write $A - \lim_{k\to\infty} \alpha_k = \gamma$. The matrix $A$ is said to be regular, if and only if $\lim_{k\to\infty} \alpha_k = \gamma$ implies $A - \lim_{k\to\infty} \alpha_k = \gamma$ (see [5], [6]).

Now we give a sufficient and necessary condition for a matrix $A = (a_{nk})$ to transform each bounded sequence of elements of Banach space into bounded sequence.

**Theorem 1.** *Let $A = (a_{nk})$ is an infinite matrix with real numbers. A sufficient and necessary condition for $A$ to transform all bounded sequences $\alpha = (\alpha_k)_{k=1}^{\infty}$ to bounded sequence $\beta = A\alpha$ is that there exists a constant $M$ such that*

$$\sum_{k=1}^{\infty} |a_{nk}| \leq M$$

*for all $n = 1, 2, \ldots$.*

*Proof.* See [6]. □

The following theorem was proved in [6] and it allows us to characterize the regular matrix.

**Theorem 2.** *Let $A = (a_{nk})$ is an infinite matrix with real numbers. The sequence $\beta_n = \sum_{k=1}^{\infty} a_{nk}\alpha_k$ converges to $\alpha$ for $n \to \infty$ and $\alpha_k \to \alpha$ if and only if the following conditions hold:*

*a)* $\exists M > 0$, $\forall n = 1, 2, \ldots$, $\sum_{k=1}^{\infty} |a_{nk}| \leq M$,

*b)* $\forall k = 1, 2, \ldots$ $\lim_{n\to\infty} a_{nk} = 0$,

*c)* $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

In this article we will use an another type of transformation of sequences of elements of Banach space.

**Definition 1.** *A matrix which transforms every bounded sequence into a convergent sequence of elements of Banach space is called* Schur matrix *(see [4]).*

Let $\alpha = (\alpha_k)$, $\alpha_k \in B_\infty$, $k = 1, 2, \ldots$. Then transformed sequence $\beta = (\beta_n)$ by the matrix $A = (a_{nk})$ is convergent in $X$,

$$\beta_n = \sum_{k=1}^{\infty} a_{nk}\alpha_k, \quad \alpha_k \in B_\infty,$$

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\alpha_k = \beta \in X.$$

## 2 MAIN RESULT

In the following theorem we shall give a necessary and sufficient condition for a matrix $A = (a_{nk})$ to be a Schur matrix for sequences of Banach space.

**Theorem 3.** *A matrix $A = (a_{nk})$ with real numbers is a Schur matrix if and only if*

*a)* $\lim_{n\to\infty} a_{nk}$ *exists for every $k$,*

*b)* $\sum_{k=1}^{\infty} |a_{nk}|$ *converges for every $n$ and the convergence is uniform in $n$, i.e.* $\sum_{k=1}^{\infty} |a_{nk}|$ *converges uniformly with respect to $n$.*

*Proof.* Suppose $A = (a_{nk})$ satisfies the two conditions of the theorem. If $\alpha = (\alpha_k)$ is a bounded sequence of elements of $X$, then by b), the transform given by

$$\beta_n = \sum_{k=1}^{\infty} a_{nk}\alpha_k$$

exists. Moreover the series converges uniformly in $n$. We wish to show that $\beta = (\beta_n)$ converges in $X$.

Given $\varepsilon > 0$ there is a $k_0$ such that

$$\left\| \beta_n - \sum_{k=1}^{k_0} a_{nk}\alpha_k \right\| < \varepsilon$$

for all $n$. Then since $\lim_{n\to\infty} a_{nk}$ exists for every $k$, there is an $n_0$ such that

$$\left\| \sum_{k=1}^{k_0} a_{n_1k}\alpha_k - \sum_{k=1}^{k_0} a_{n_2k}\alpha_k \right\| < \varepsilon$$

for all $n_1, n_2 > n_0$. Hence, for $n_1, n_2 > n_0$ we have

$$\|\beta_{n_1} - \beta_{n_2}\| \le \left\| \beta_{n_1} - \sum_{k=1}^{k_0} a_{n_1k}\alpha_k \right\| + \left\| \sum_{k=1}^{k_0} a_{n_1k}\alpha_k - \sum_{k=1}^{k_0} a_{n_2k}\alpha_k \right\| + \left\| \sum_{k=1}^{k_0} a_{n_2k}\alpha_k - \beta_{n_2} \right\| < 3\varepsilon.$$

Therefore $\beta = (\beta_n)$ is a Cauchy sequence in Banach space $(X, \|\cdot\|)$ and $\beta = (\beta_n)$ is a convergent sequence.

Suppose $A = (a_{nk})$is a Schur matrix.

a) Let $\beta_n = \sum_{k=1}^{\infty} a_{nk}\alpha_k$, $n = 1, 2, \ldots$ converges in $X$, where $\alpha = (\alpha_k)$, $\alpha_k \in B_\infty$, $k = 1, 2, \ldots$. For every $i = 1, 2, \ldots$ define the sequence $\varepsilon^{(i)} = \left(\varepsilon_k^{(i)}\right)$ by

$$\varepsilon_k^{(i)} = \begin{cases} \varepsilon, & k = i,\ \|\varepsilon\| = 1, \\ 0, & k \neq i. \end{cases}$$

Then $\beta_n^{(i)} = \sum_{k=1}^{\infty} a_{nk}\varepsilon_k^{(i)} = a_{ni}\varepsilon$. Since $\varepsilon^{(i)} = \left(\varepsilon_k^{(i)}\right)$ is bounded, $\lim_{n\to\infty} a_{ni}$ exists. Denote this limit by $L_i$, $i = 1, 2, \ldots$.

b) Since a Schur matrix is certainly a limitation method, $\sum_{k=1}^{\infty} |a_{nk}|$ converges for every $n$ and by Theorem 1 there is a $K$ such that $\sum_{k=1}^{\infty} |a_{nk}| \leq K$ for all $n = 1, 2, \ldots$. It follows that for every $k_1$, $\sum_{k=1}^{k_1} |L_k| \leq K$ and so $\sum_{k=1}^{\infty} |L_k|$ converges.

The matrix $B = (b_{nk})$ given by

$$b_{nk} = a_{nk} - L_k$$

is clearly a Schur matrix. We shall establish the uniform convergence of $\sum_{k=1}^{\infty} |a_{nk}|$ by showing that $\sum_{k=1}^{\infty} |b_{nk}|$ converges uniformly.

Suppose that $\sum_{k=1}^{\infty} |b_{nk}|$ does not converge uniformly. This means that there is a $\gamma > 0$ such that, for every integer $\nu$, there exists a sequence $\mu_j(\nu)$ of integers such that

$$\sum_{k=\nu}^{\infty} \left|b_{\mu_j(\nu)k}\right| > 5\gamma.$$

Using this as well as the fact that, for all $k$, $b_{nk} \to 0$ an $n \to \infty$ we now construct a sequence $n_i$ and an increasing sequence of integers $\nu_i$.

Take $\nu_1 = 1$ and $n_1$ be such that

$$\sum_{k=1}^{\infty} |b_{n_1 k}| > 5\gamma \quad \text{and} \quad |b_{n_1 1}| < \gamma.$$

In general, having chosen $\nu_i$ and let $n_i$ be such that

$$\sum_{k=\nu_i+1}^{\infty} |b_{n_i k}| > 5\gamma \quad \text{and} \quad |b_{n_i 1}| < \gamma.$$

Then, since $\sum_{k=1}^{\infty} |b_{n_i k}|$ converges, we may choose $\nu_{i+1} > \nu_i$ so that

$$\sum_{k=\nu_{i+1}+1}^{\infty} |b_{n_i k}| < \gamma$$

and therefore

$$\sum_{k=\nu_i+1}^{\nu_i+1} |b_{n_i k}| > 3\gamma.$$

Now let $\alpha = (\alpha_k)$ the sequence in $B_\infty$

$$\alpha_k = \begin{cases} (-1)^k \varepsilon \operatorname{sgn} b_{n_i k}, & \|\varepsilon\| = 1, \text{ for } \nu_i + 1 \leq k \leq \nu_i, \ i = 1, 2, \ldots, \\ \Theta, & \text{otherwise.} \end{cases}$$

Then $B - \lim \alpha_k$ is such that

$$\begin{aligned}
\|\beta_{n_i}\| = \left\|\sum_{k=1}^{\infty} b_{n_i k}\alpha_k\right\| &\geq \left(-\sum_{k=1}^{\nu_i} + \sum_{k=\nu_i+1}^{\nu_{i+1}} - \sum_{k=\nu_{i+1}+1}^{\infty}\right) |b_{n_i k}| \, \|\alpha_k\| \\
&\geq \sum_{k=\nu_i+1}^{\nu_{i+1}} |b_{n_i k}| \, \|\alpha_k\| \\
&= \|\varepsilon\| \sum_{k=\nu_i+1}^{\nu_{i+1}} |b_{n_i k}| \\
&> -\gamma + 3\gamma - \gamma = \gamma
\end{aligned}$$

when $k$ is even, while similarly

$$\|\beta_{n_i}\| < -\gamma$$

when $k$ is odd. Therefore $\beta = (\beta_{nk})$ diverges, which contradicts the fact that $B = (b_{nk})$ is a Schur matrix.

This completes the proof of theorem. □

In the example below we are going to illustrate a transformation by using specific Schur matrix. First of all we show that all conditions of Theorem 3 for matrix are satisfied. Next we take a bounded sequence of the space of all bounded sequences of real numbers with the supremum norm and transform it into a convergent sequence in the space $\ell^\infty$.

**Example 1.** *We define a matrix $A = (a_{nk})$ in the following way:*

$$a_{nk} = \frac{n}{n+1} . \frac{1}{k^2}, \quad n = 1, 2, \ldots, \quad k = 1, 2, \ldots .$$

*i.e.*

$$A = \begin{pmatrix} \frac{1}{2}\cdot\frac{1}{1^2} & \frac{1}{2}\cdot\frac{1}{2^2} & \cdots & \frac{1}{2}\cdot\frac{1}{k^2} & \cdots \\ \frac{2}{3}\cdot\frac{1}{1^2} & \frac{2}{3}\cdot\frac{1}{2^2} & \cdots & \frac{2}{3}\cdot\frac{1}{k^2} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ \frac{n}{n+1}\cdot\frac{1}{1^2} & \frac{n}{n+1}\cdot\frac{1}{2^2} & \cdots & \frac{n}{n+1}\cdot\frac{1}{k^2} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$

*It is easy to see that $A = (a_{nk})$ is a Schur matrix because*

*a)* $\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} \frac{n}{n+1} L_k = 1.L_k$, $k = 1, 2, \ldots$, *where* $(L_k) = \left(\frac{1}{k^2}\right)_{k=1}^{\infty}$,

*b)* $\sum_{k=1}^{\infty} |a_{nk}|$ *is an uniformly convergent series for all* $n = 1, 2, \ldots$. *We have for* $n \to \infty$

$$\sum_{k=1}^{\infty} |a_{nk}| = \sum_{k=1}^{\infty} L_k = \frac{\pi^2}{6}.$$

*Let $\alpha = (\alpha_k)$ is a sequence of elements of Banach space $\ell^\infty$ the set of all bounded sequences of real numbers endowed by norm $\|\alpha\| = \sup_{i=1,2,\dots} \left|\alpha^{(i)}\right|$, where $\alpha = \left(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(i)} \dots\right)$, $\alpha^{(i)} \in \mathbb{R}$, $i = 1, 2, \dots$.*

*Let*

$$\alpha_k = \begin{cases} (0, 0, \dots, 0, \dots), & k \text{ is odd}, \\ (1, 1, \dots, 1, \dots), & k \text{ is even}, \quad k = 1, 2, \dots . \end{cases}$$

*It is easy to see that $\alpha = (\alpha_k) \in B_\infty$, $\alpha_k \in \ell^\infty$, $k = 1, 2, \dots$. We transform $\alpha = (\alpha_k)$ by the matrix $A = (a_{nk})$ into $\beta = (\beta_n)$ i.e. $\beta_n = \sum_{k=1}^{\infty} a_{nk}\alpha_k$. Then*

$$\beta_1 = \left(\frac{1}{2}.\frac{\pi^2}{24}, \frac{1}{2}.\frac{\pi^2}{24}, \dots, \frac{1}{2}.\frac{\pi^2}{24}, \dots\right),$$
$$\beta_2 = \left(\frac{2}{3}.\frac{\pi^2}{24}, \frac{2}{3}.\frac{\pi^2}{24}, \dots, \frac{2}{3}.\frac{\pi^2}{24}, \dots\right),$$
$$\vdots$$
$$\beta_n = \left(\frac{n}{n+1}.\frac{\pi^2}{24}, \frac{n}{n+1}.\frac{\pi^2}{24}, \dots, \frac{n}{n+1}.\frac{\pi^2}{24}, \dots\right).$$

*The sequence $\beta = (\beta_n)$ converges in $\ell^\infty$ to $\beta = \left(\frac{\pi^2}{24}, \frac{\pi^2}{24}, \dots, \frac{\pi^2}{24}, \dots\right)$.*

About connection between regular and Schur matrix says the corollary below.

**Corollary 1.** *A matrix cannot be regular and a Schur matrix simultaneously. Hence, if $A = (a_{nk})$ is a regular, there is a bounded sequence which is transformed by $A$ into a divergent sequence (see [4]).*

## CONCLUSION

In this article we have shown some results about the method of matrix limitation of sequences of an arbitrary Banach space. We have also shown some properties of the infinite matrix with real number items that it transforms every bounded sequence of Banach space to the convergent sequence with respect to the given norm. Finally we give an example of transformation matrix. The results show that the infinite matrix theory can be extended to the sequences of elements of Banach space.

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